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Non Existence of Bounded-Energy Solutions for Some Semilinear Elliptic Equations with a Large Parameter.

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ABSTRACT - Following previous work of Druet-Hebey-Vaugon, we prove that the energy of positive solutions of the Dirichlet problem $-\Delta u + \lambda u = u^{\frac{N+2}{N-2}}$ in Ω, $u = 0$ in $\partial\Omega$ tends to infinity as $\lambda \to +\infty$. We also prove, extending and simplifying recent results, that bounded energy solutions to a mixed B.V.P. have at least one blow-up point on the Neumann component as $\lambda \to +\infty$.

1. Introduction.

Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$, $\lambda \geq 0$. We consider the problem

$$(D)_\lambda \begin{cases} -\Delta u + \lambda u = u^p & x \in \Omega \\ u > 0 & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases}$$

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where $p = \frac{N+2}{N-2}$. In case Ω is starshaped $(D)_\lambda$ has no solutions: an obstruction to existence is given by the well known Pohozaev identity. However, $(D)_\lambda$ might have solutions for any λ: this is the case if Ω is an annulus.

In [6], Druet-Hebey-Vaughon, investigating the role of Pohozaev type identities in a Riemannian context, discovered that such identities still provide some kind of obstruction to existence: if u_j solve $(D)_{\lambda_j}$ on some compact (conformally flat) Riemannian manifold M and $\lambda_j \to +\infty$, then $\int_M u_j^{p+1} \to +\infty$ (see [7] for extensions to fourth order elliptic PDE's and [10] for more questions). In other words, there are no positive solutions with energy below some given bound, if λ is too large.

This result does not carry over to manifolds with boundary under general boundary conditions (e.g. homogeneous Neumann boundary conditions, see [1], [2] for this non trivial fact.)

The main purpose of this note is to show that the result quoted above is indeed true for the homogeneous Dirichlet B.V.P. (D_λ); see Theorem 1 below.

Our approach is as in [6]: to show that a sequence of solutions cannot blow-up at a finite number of points (as it should be assuming a bound on the energy). The obstruction found in [6] is given by local L^2 $-$estimates. In turn, these estimates are based on inequalities obtained localizing the standard Pohozaev identity on balls centered at blow-up points (see (13) below). Now, differently from [6], where there is no boundary, we have to take into account possible blow-up at boundary points. Since Pohozaev type inequalities on balls centered at boundary points do not hold, in general, the main issue here is to get local L^2 estimates at boundary points.

Notice that a more or less straightforward application of arguments from [6] would only lead to the statement: bounded energy solutions have to blow up at least at one boundary point, which is the (quite interesting in itself) correct statement for the Neumann problem (see Th. 2 below) but which is not the result we are looking for the Dirichlet problem.

Since it does not seem easy to rule out blow up at boundary points, we stick to the approach in [6], but we have to deal with the new difficulty coming from possible blow up at boundary points.

To handle this difficulty, we will establish Pohozaev-type inequalities *suitably localized at interior points* (see Lemma 5 below), which, used in a clever way, will allow to obtain *estimates up to boundary points* (see Lemma 6, which also provides a simple adaptation and a self contained exposition of the main arguments in [6]).

This is the main technical contribution in this paper, and we believe that such estimates might be of interest by themselves. Our first result is

THEOREM 1. *Let $N \geqslant 3$. Let u_j be solutions of $(D)_{\lambda_j}$, with $\lambda_j \to +\infty$. Then $\int_\Omega u_j^{p+1} \to +\infty$.*

NOTE. Such a result is quite obvious if p is subcritical, i.e. $p < \frac{N+2}{N-2}$. In this case, it holds true, in contrast with the critical case, also for the homogeneous Neumann B.V.P. (see [11] for an explicite lower bound on the energy of ground state solutions).

A second question we address in this note is concerned with the mixed boundary value problem

$$(M)_\lambda \begin{cases} -\Delta u + \lambda u = u^{\frac{N+2}{N-2}} & x \in \Omega \\ u > 0 & x \in \Omega \\ u = 0 & x \in \Gamma_0 \\ \dfrac{\partial u}{\partial \nu} = 0 & x \in \Gamma_1 \end{cases}$$

Here, $\partial\Omega = \Gamma_0 \cup \Gamma_1$ with Γ_i disjoint components.

As for the Neumann problem, $(M)_\lambda$ possesses low energy solutions for any λ positive, at least if the mean curvature of Γ_1 is somewhere positive, and hence non existence of bounded energy solutions for λ large is false, in general. Indeed, several existence results for (bouded energy) solutions blowing up at (one or several) boundary points are known (see [12] for an extensive bibliography). Nothing is known, to our best knowledge, about existence of solutions blowing up both at interior and at boundary points. On the other hand, the extreme case of purely interior blow-up has been widely investigated:

(Q) Are there solutions which blow up only at interior points?

Let us review the known results, all of them actually concerning the homogeneous Neumann problem (i.e. $\Gamma_0 = \emptyset$).

A first, negative, answer has been given in [4] for $N \geqslant 5$: $(M)_\lambda$ has no solutions of the form

$$u_\lambda = w_\lambda + \sum_{j=1}^{k} U_{\mu_\lambda^j,\, y_\lambda^j}, \qquad w_\lambda \to 0 \ \text{ in } H^1(\Omega)$$

with $\mu_\lambda^j \to +\infty$, $y_\lambda^j \to y^j \in \Omega$ as λ goes to infinity and $y^i \neq y^j \ \forall i \neq j$. Here, $U(x) = \dfrac{[N(N-2)]^{\frac{N-2}{4}}}{(1+|x|^2)^{\frac{N-2}{2}}}$ and $U_{\mu, y} = \mu^{\frac{N-2}{2}} U(\mu(x-y))$.

Now, bounded energy solutions u_λ (with λ going to infinity) are known to be of the form given above, apart from the property $y^i \neq y^j$ $\forall i \neq j$, which has to be regarded as a " no multiple concentration at a single point " assumption. Under the even more restrictive assumption $k = 1$, a corresponding non existence result has been proven in [8] in any dimension $N \geqslant 3$.

At our best knowledge, the only result fully answering (Q), and due to Rey [12], is limited to the dimension $N = 3$. According to Rey, " the main difficulty is to eliminate the possibility of multiple interior peaks ", and he accomplishes this task through a very careful expansion of solutions blowing up at interior points: this is the basic tool to obtain a negative answer to (Q) in dimension $N = 3$. It is henceforth remarkable that, as a byproduct of our L^2 estimates, we can bypass this difficulty and easily prove

THEOREM 2. *Let $N \geqslant 3$. Let u_j be solutions of $(M)_{\lambda_j}$, with $\lambda_j \to +\infty$. If $\sup_j \int_\Omega u_j^{p+1} < \infty$, then u_j has at least one concentration point which lies on the Neumann component Γ_1.*

Actually, we expect that solutions for the mixed problem, with a uniform bound on the energy, should not even exist, for λ large, if the mean curvature of the Neumann component is strictly negative. This is fairly obvious in the case of one peak solutions, which, by the above, should blow-up at one boundary point. However, in such a point the mean curvature should be non negative (see [3], or [13]).

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2. L^2 global concentration.

To be self contained, we review in this section some essentially well known facts. Let u_n be solutions of $(D)_{\lambda_n}$ (but also homogeneous Neumann, or mixed, boundary conditions might be allowed, with minor changes, here). Multiplying the equation by u_n, integrating by parts and using Sobolev inequality, we see that

$$\int_\Omega |\nabla u_n|^2 \geqslant S^{\frac{N}{2}}, \qquad \int_\Omega u_n^{p+1} \geqslant S^{\frac{N}{2}} \tag{1}$$

where S denotes the best Sobolev constant. We start recalling concentration properties of u_n, assuming $u_n \rightharpoonup 0$ in H_0^1.

LEMMA 3. *Let u_n be solutions of $(D)_{\lambda_n}$. Assume $u_n \rightharpoonup 0$ in H_0^1. Then there is a finite set $C \subset \overline{\Omega}$ such that $u_n \to 0$ in $H^1_{\mathrm{loc}}(\overline{\Omega} \setminus C)$ and in $C^0_{\mathrm{loc}}(\overline{\Omega} \setminus C)$.*

PROOF. Let $C := \{x \in \overline{\Omega} : \limsup_{n\to\infty} \int_{B_r(x)\cap\Omega} |\nabla u_n|^2 > 0,\ \forall r > 0\}$.

Because of (1) and compactness of $\overline{\Omega}$, C cannot be empty. We claim that

$$\forall x \in C, \quad \forall r > 0, \qquad \limsup_{n\to\infty} \int_{B_r(x)\cap\Omega} u_n^{p+1} \geqslant S^{\frac{N}{2}}. \tag{2}$$

To prove the claim, let $\varphi \in C_0^\infty(B_{2r}(x))$, $\varphi \equiv 1$ on $B_r(x)$, $0 \leqslant \varphi \leqslant 1$. Notice that $-\int_\Omega u_n \varphi^2 \Delta u_n = \int_\Omega |\nabla u_n|^2 \varphi^2 + \circ(1) = \int_\Omega |\nabla u_n \varphi|^2 + \circ(1)$ because $u_n \rightharpoonup 0$.

From the equation, and using Holder and Sobolev inequalities, we get

$$\int |\nabla u_n \varphi|^2 + \circ(1) \leqslant \int u_n^{\frac{4}{N-2}} (u_n \varphi)^2 \leqslant \frac{1}{S} \left(\int_{B_{2r}(x)} u_n^{\frac{2N}{N-2}} \right)^{\frac{2}{N}} \int |\nabla u_n \varphi|^2 \tag{3}$$

For $x \in C$, $\int |\nabla u_n \varphi|^2$ is bounded away from zero along some subsequence, and then (2) follows by (3). Also, if $x_1, \dots, x_k \in C$, choosing $B_r(x_j)$ disjoint balls, and eventually passing to a subsequence, we get by (2)

$$kS^{\frac{N}{2}} \leqslant \sum_j \int_{B_r(x_j)\cap\Omega} u_n^{p+1} \leqslant \sup_n \int_\Omega u_n^{p+1} < +\infty$$

Thus C is finite. Also, from the very definition of C, it follows that $\nabla u_n \to \to 0$ in $L^2_{\rm loc}(\Omega \setminus C)$, and, by Sobolev inequality, $u_n \to 0$ in $L^{p+1}_{\rm loc}(\Omega \setminus C)$ as well.

Finally, $C^0_{\rm loc}(\overline{\Omega} \setminus C)$ convergence will follow by standard elliptic theory once one has proved that $u_n \to 0$ in $L^q_{\rm loc}(\Omega \setminus C)$ $\forall q$. In turn, this fact readily follows iterating the (Moser type) scheme

$$(4) \qquad q \geqslant 2, \int_{B_{2r}(x)} u^{p+1} \leqslant \left(\frac{S}{q}\right)^{\frac{N}{2}} \Rightarrow \left(\int_{B_r(x)} u^{sq}\right)^{\frac{1}{s}} \leqslant \frac{8}{Sr^2} \int_{B_{2r}(x)} u^q, \quad s := \frac{p+1}{2}$$

To prove (4), we can procede as for (3), choosing now as test function $\varphi^2 u^{q-1}$, $\|\nabla \varphi\|_\infty \leqslant \frac{2}{r}$. We now obtain

$$(5) \qquad \int \nabla u \nabla (u^{q-1} \varphi^2) \leqslant \int u^{p-1} (\varphi u^{\frac{q}{2}})^2 \leqslant \frac{1}{S} \left(\int_{B_{2r}(x)} u^{p+1}\right)^{\frac{2}{N}} \int |\nabla \varphi u^{\frac{q}{2}}|^2$$

On the other hand

$$(6) \qquad \int \nabla u \nabla (u^{q-1} \varphi^2) = \int (q-1)\, \varphi^2 u^{q-2} |\nabla u|^2 + 2 u^{q-1} \varphi \nabla u \nabla \varphi$$

$$(7) \qquad \frac{2}{q} \int |\nabla (\varphi u^{\frac{q}{2}})|^2 = \int \frac{q}{2} \varphi^2 u^{q-2} |\nabla u|^2 + 2 u^{q-1} \varphi \nabla u \nabla \varphi + \frac{2}{q} |\nabla \varphi|^2 u^q$$

Substracting (6) from (7) and then using (5), we obtain

$$(8) \qquad \frac{2}{q} \int |\nabla (\varphi u^{\frac{q}{2}})|^2 \leqslant \int \nabla u \nabla (u^{q-1} \varphi^2) + \frac{2}{q} \int |\nabla \varphi|^2 u^q \leqslant$$

$$\leqslant \frac{1}{S} \left(\int_{B_{2r}(x)} u^{p+1}\right)^{\frac{2}{N}} \int |\nabla (\varphi u^{\frac{q}{2}})|^2 + \frac{8}{qr^2} \int_{B_{2r}(x)} u^q$$

Hence, using the assumption $\int_{B_{2r}(x)} u^{p+1} \leqslant \left(\frac{S}{q}\right)^{\frac{N}{2}}$ and Sobolev inequality, we get (4). Now, iterating (4) with $x \in \overline{\Omega} \setminus C_\delta$, $2r < \delta$, $u = u_n$ and using elliptic estimates ([9], page 194), we obtain

$$\sup_{\Omega \setminus C_\delta} u_n \leqslant \frac{c_N}{\delta^{\frac{N}{2}}} \left(\int_{\Omega \setminus C_{\delta/2}} u_n^2 \right)^{\frac{1}{2}} \to 0 \tag{9}$$

Points in C are called " geometric " concentration points and C is the concentration set. A crucial observation is that L^2 norm concentrates around C (see [6]):

Lemma 4. *Let u_n be solutions of $(D)_{\lambda_n}$. Assume $u_n \rightharpoonup 0$, and let $C := \{x_1, \dots, x_m\}$ be its concentration set. Let $C_\delta := \bigcup_{j=1}^{m} B_\delta(x_j)$, $B_\delta(x_j)$ disjoint closed balls. Then, for n large,*

$$\int_{\Omega \setminus C_\delta} u_n^2 \leqslant \frac{16}{\delta^2 \lambda_n} \int_\Omega u_n^2$$

In particular, if $\lambda_n \to +\infty$, then

$$\frac{\int_{\Omega \setminus C_\delta} u_n^2}{\int_{C_\delta} u_n^2} \to 0 . \tag{10}$$

Proof. Let $\varphi \in C^\infty(\mathbb{R}^N)$, $0 \leqslant \varphi \leqslant 1$, $\varphi \equiv 0$ in $C_{\frac{\delta}{2}}$, $\varphi \equiv 1$ in $\mathbb{R}^N \setminus C_\delta$, $\|\nabla \varphi\|_\infty < \frac{4}{\delta}$. Multiplying the equation by $u_n \varphi^2$ and integrating, we get

$$\int |\nabla u_n|^2 \varphi^2 + 2 \int u_n \varphi \nabla u_n \nabla \varphi + \lambda_n \int u_n^2 \varphi^2 = \circ(1) \int u_n^2 \varphi^2 \tag{11}$$

because $u_n^{p-1}(x) \to 0$ uniformly in $\overline{\Omega} \setminus C_{\delta/2}$ by Lemma. After setting $\gamma_n^2 := \frac{\int |\nabla u_n|^2 \varphi^2}{\int_{\Omega \setminus C_{\delta/2}} u_n^2}$ we get from (11) the desired inequality

$$\lambda_n \frac{\int_{\Omega \setminus C_\delta} u_n^2}{\int_{\Omega \setminus C_{\delta/2}} u_n^2} \leqslant \circ(1) + 2\|\nabla \varphi\|_\infty \gamma_n - \gamma_n^2 \leqslant \circ(1) + \|\nabla \varphi\|_\infty^2 \leqslant \frac{16}{\delta^2}$$

3. Pohozaev identity and «reverse» L^2 concentration.

In this Section, after briefly recalling the Pohozaev identity, we first derive suitably localized Pohozaev inequalities which will allow to get uniform L^2-local estimates up to boundary points. These up-to-the-boundary estimates are the main novelty with respect to [6]: combined with the L^2 global concentration reviwed in Section 2, they will readily imply Theorems 1 and 2.

Given any $v \in C^2(\Omega)$, $x_0 \in \mathbb{R}^N$, an elementary computation (see [14]) gives

$$\langle x - x_0, \nabla v \rangle \Delta v - \frac{N-2}{2} |\nabla v|^2 = \operatorname{div}\left(\langle x - x_0, \nabla v \rangle \nabla v - \frac{|\nabla v|^2}{2} (x - x_0) \right)$$

If in addition $v \in C_0^1(\overline{\Omega})$, so that $\nabla v(x) = \frac{\partial v}{\partial \nu} \nu(x)$ for any $x \in \partial\Omega$, where $\nu(x)$ is the exterior unit normal at $x \in \partial\Omega$, an integration by parts yields

$$(12) \qquad \int_\Omega \langle x - x_0, \nabla v \rangle \Delta v + \frac{N-2}{2} v \Delta v = \frac{1}{2} \int_{\partial\Omega} \langle x - x_0, \nu(x) \rangle |\nabla v|^2 d\sigma$$

If furthermore $-\Delta v = g(x, v)$, $g(x, t) \equiv b(x) |t|^{p-1} t - \lambda a(x) t$, through another integration by parts we obtain

$$\frac{1}{N} \int_\Omega \left\langle x - x_0, v^{p+1} \frac{\nabla b}{p+1} - \frac{\lambda}{2} v^2 \nabla a \right\rangle - \frac{\lambda}{N} \int_\Omega a v^2 = \frac{1}{2} \int_{\partial\Omega} \langle x - x_0, \nu \rangle |\nabla v|^2 d\sigma$$

A straightforward and well known consequence of this identity is that v has to be identically zero if Ω is starshaped with respect to some x_0 and $\langle x - x_0, \nabla b \rangle \leqslant 0 \leqslant \langle x - x_0, \nabla a \rangle$. However, this conclusion is false, in general. The idea, following [6], would be to localize (12) to obtain, for every given $x_0 \in \overline{\Omega}$ and some $\delta = \delta_{x_0} > 0$ and for all $\varphi \in C_0^\infty(B_{4\delta}(x_0))$, inequalities of the form

$$(13) \qquad \int_{B_{4\delta}(x_0) \cap \Omega} \langle x - x_0, \nabla(\varphi v) \rangle \Delta(\varphi v) + \frac{N-2}{2} \varphi v \Delta(\varphi v) \geqslant 0 ,$$

However, while this can be done at interior points (e.g. with $4\delta = d(x_0, \partial\Omega)$), (13) is in general false, $\forall\, \delta$ small, if $x_0 \in \partial\Omega$. So, we have to localize (12) at interior points but in a carefull way, to cover, in some sense, also boundary points. Our basic observation is that (13) holds true for x_0 as long as $d(x_0, \partial\Omega) \geqslant \delta$ if δ is sufficiently small, and this will be enough to get control up to the boundary. The first statement is the content of the following simple but crucial lemma.

LEMMA 5. *There is* $\overline{\delta} = \overline{\delta}(\partial\Omega)$ *such that, if* $0 < \delta \leqslant \overline{\delta}$, $v \in C^2(\Omega) \cap \cap C^1(\overline{\Omega})$, $v \equiv 0$ *on* $\partial\Omega$ *and* $x_0 \in \Omega$ *with* $d(x_0, \partial\Omega) \geqslant \delta$, *then* (3.13) *holds true.*

PROOF. Let $\overline{\delta}$ be such that, for every $z \in \partial\Omega$, any $x \in \partial\Omega \cap B_{8\overline{\delta}}(z)$ can be uniquely written in the form

(*i*) $x = z + \eta + \gamma^z(\eta)\,\nu(z)$, $\langle \eta, \nu(z)\rangle = 0$, with $|\gamma^z(\eta)| \leqslant c(\partial\Omega)\,|\eta|^2$, for some smooth γ^z, with $\gamma^z(0) = 0$, $\nabla\gamma^z(0) = 0$, $|\eta| \leqslant 8\,\overline{\delta}$ and some constant c only depending on $\partial\Omega$. We will also require

(*ii*) $|\nu(z') - \nu(z'')| \leqslant \frac{1}{8}$ for any $z', z'' \in \partial\Omega$ with $|z' - z''| \leqslant 8\,\overline{\delta}$

(*iii*) $\overline{\delta} < \frac{1}{128c}$, $c = c(\partial\Omega)$.

Let $0 < \delta \leqslant \overline{\delta}$. We are going to apply (12) with Ω replaced by $B_{4\delta}(x_0) \cap \Omega$ and v by φv, $\varphi \in C_0^\infty(B_{4\delta}(x_0))$. If $d(x_0, \partial\Omega) \geqslant 4\delta$, then equality holds in (13), so, let us assume $0 < \delta \leqslant d(x_0, \partial\Omega) \leqslant 4\delta$. We can write $x_0 = z - \tau\nu(z)$ for some $z \in \partial\Omega$ and $\delta \leqslant \tau \leqslant 4\delta$. For $x \in B_{4\delta}(x_0)$ we have $|x - z| \leqslant 8\delta$ and hence, (*i*) holds: $x = z + \eta + \gamma^z(\eta)\nu(z)$, $|\eta| \leqslant 8\delta$. Now, using (*ii*)-(*iii*), we see that

$$\langle x - x_0, \nu(x)\rangle = \langle x - x_0, \nu(x) - \nu(z)\rangle +$$

$$+ \langle z + \eta + \gamma^z(\eta)\,\nu(z) - (z - \tau\nu(z)), \nu(z)\rangle \geqslant \frac{\delta}{2} - 64c\delta^2 \geqslant 0\,.$$

Hence the r.h.s. in (12) (with Ω replaced by $\Omega \cap B_{4\delta}(x_0)$ and v by φv) is nonnegative and the Lemma is proved. ■

In the Lemma below we will show how Pohozaev inequalities lead to «reverse L^2-concentration» of solutions at any blow up point. We will adapt arguments from [6], where, however, it is made a crucial use of the validity of (13) at any point, which is not the case here. Still, a clever use of Lemma 5, i.e. of (13) limited to points which are δ-away from the boundary, will unable to get the desired estimates up to boundary points.

LEMMA 6. *There is a constant* $c = c_N$, *only depending on* N, *such that if* u_λ *is a solution of* $(D)_\lambda$ *and* $0 < \delta \leqslant \overline{\delta}(\partial\Omega) \leqslant 1$, *then*

$$\lambda \int_{B_\delta(x)} u_\lambda^2 \leqslant \frac{c}{\delta^3} \int_{B_{4\delta}(x)\setminus B_\delta(x)} (u_\lambda^2 + u_\lambda^{p+1}), \quad \forall x \in \overline{\Omega} \tag{14}$$

PROOF. We are going to apply (13) with $x_0 = x$ if $d(x, \partial\Omega) \geqslant \delta$, while, if $d(x, \partial\Omega) < \delta$, $x = z - \tau\nu(z)$ for some $z \in \partial\Omega$ and $0 \leqslant \tau < \delta$, we will choose $x_0 = z - \delta\nu(z)$. Let, without loss of generality, $x_0 = 0$. By Lemma 5 we have

$$\int_{B_{4\delta}\cap\Omega} \langle x, \nabla(\varphi u_\lambda)\rangle\, \Delta(\varphi u_\lambda) + \frac{N-2}{2}\varphi u_\lambda \Delta(\varphi u_\lambda) \geqslant 0$$

i.e. (dropping subscript $B_{4\delta}\cap\Omega$)

$$\int \left[\langle x,\nabla(\varphi u_\lambda)\rangle + \frac{N-2}{2}\varphi u_\lambda\right]\varphi\Delta u_\lambda + 2\varphi\langle x,\nabla u_\lambda\rangle\langle\nabla\varphi,\nabla u_\lambda\rangle + R(\lambda) \geqslant 0\,, \tag{15}$$

where $R(\lambda) := R_1(\lambda) + R_2(\lambda)$, R_1, R_2 given by

$$R_1(\lambda) := \int 2u_\lambda\langle x, \nabla\varphi\rangle\langle\nabla\varphi, \nabla u_\lambda\rangle + \langle x, \nabla(\varphi u_\lambda)\rangle\, u_\lambda \Delta\varphi$$

$$R_2(\lambda) := \frac{N-2}{2}\int \varphi u_\lambda[u_\lambda\Delta\varphi + 2\langle\nabla u_\lambda, \nabla\varphi\rangle]$$

In what follows we properly adapt and simplify arguments from [6]. Taking φ radially symmetric and radially decreasing, we have $\nabla\varphi = \left\langle\nabla\varphi, \frac{x}{|x|}\right\rangle\frac{x}{|x|}$, with $\langle\nabla\varphi(x), x\rangle \leqslant 0$. In particular, $\langle\nabla\varphi(x), \nabla u_\lambda(x)\rangle\langle x, \nabla u_\lambda(x)\rangle = \left\langle\nabla\varphi(x), \frac{x}{|x|^2}\right\rangle\ \langle x, \nabla u_\lambda(x)\rangle^2 \leqslant 0$ and hence (15) yields

$$R(\lambda) + \int \varphi\langle x, \nabla(\varphi u_\lambda)\rangle\, \Delta u_\lambda + \frac{N-2}{2}\int \varphi^2 u_\lambda \Delta u_\lambda \geqslant 0 \tag{16}$$

Now, let us write $g(t) := \lambda t - |t|^{p-1}t$, $G(t) := \frac{\lambda}{2}u^2 - \frac{u^{p+1}}{p+1}$. We first rewrite, integrating by parts, the second term in (16) as follows:

$$\text{(17)} \qquad \int \varphi\langle x, \nabla(\varphi u_\lambda)\rangle \Delta u_\lambda = \int \varphi\langle x, \nabla\varphi\rangle u_\lambda g(u_\lambda) + \sum_{j=1}^{N} \int \varphi^2 x_j g(u_\lambda) \frac{\partial u_\lambda}{\partial x_j} =$$

$$= \int \varphi\langle x, \nabla(\varphi)\rangle [u_\lambda g(u_\lambda) - 2G(u_\lambda)] - N \int \varphi^2 G(u_\lambda) =$$

$$= -\frac{2}{N} \int \varphi u_\lambda^{p+1} - N \int \varphi^2 G(u_\lambda)$$

because $2G(u) - ug(u) = -\frac{2}{N} u^{p+1}$. Since $NG(u_\lambda) - \frac{N-2}{2} u_\lambda g(u_\lambda) = -$ $-\lambda u_\lambda^2$, (16) gives

$$\text{(18)} \qquad R(\lambda) - \frac{2}{N} \int \langle x, \nabla\varphi\rangle \varphi u_\lambda^{p+1} \geqslant -\lambda \int u_\lambda^2$$

Let us now transform, integrating by parts, $R(\lambda)$ as an integral against $u_\lambda^2\, dx$.

$$R_1(\lambda) = \int \sum_{j=1}^{N} \left[\langle x, \nabla\varphi\rangle \frac{\partial\varphi}{\partial x_j} + \frac{1}{2} x_j \varphi \Delta\varphi \right] \frac{\partial^2 u_\lambda^2}{\partial x_j^2} + \langle x, \nabla\varphi\rangle u_\lambda^2 \Delta\varphi =$$

$$= -\int u_\lambda^2 \sum_{j=1}^{N} \left[\langle x, \nabla\varphi\rangle \frac{\partial^2\varphi}{\partial x_j^2} + \frac{\partial}{\partial x_j} \langle x, \nabla\varphi\rangle \frac{\partial\varphi}{\partial x_j} + \frac{1}{2}\varphi\Delta\varphi + \frac{1}{2} x_j \frac{\partial}{\partial x_j} (\varphi\Delta\varphi) \right] +$$

$$+ u_\lambda^2 \langle x, \nabla\varphi\rangle \Delta\varphi = -\int u_\lambda^2 \left[\langle \nabla\varphi, \nabla(\langle x, \nabla\varphi\rangle)\rangle + \frac{N}{2}\varphi\Delta\varphi + \frac{1}{2}\langle x, \nabla(\varphi\Delta\varphi)\rangle \right].$$

$$R_2(\lambda) = \frac{N-2}{2} \left[\int u_\lambda^2 \varphi\Delta\varphi - \frac{1}{2} \int u_\lambda^2 \Delta\varphi^2 \right] = -\frac{N-2}{2} \int u_\lambda^2 |\nabla\varphi|^2$$

and thus

$$R(\lambda) = -\int u_\lambda^2 \left[\langle \nabla\varphi, \nabla\langle x, \nabla\varphi\rangle\rangle + \frac{N}{2}\varphi\Delta\varphi + \frac{1}{2}\langle x, \nabla(\varphi\Delta\varphi)\rangle + \frac{N-2}{2} |\nabla\varphi|^2 \right].$$

Now, assuming $\varphi \equiv 1$ on $B_{2\delta}$, $\varphi \equiv 0$ outside $B_{3\delta}$, we obtain

$$R(\lambda) \leqslant \frac{c}{\delta^3} \int_{B_{3\delta} \setminus B_{2\delta}} u_\lambda^2$$

for some $c = c(N)$, and hence, by (18),

$$\lambda \int_{B_{2\delta}} u_\lambda^2 \leqslant \frac{c}{\delta^3} \int_{B_{3\delta}\setminus B_{2\delta}} u_\lambda^2 + u_\lambda^{p+1}$$

Since $B_\delta(x) \subset B_{2\delta}$ and $B_{3\delta} \setminus B_{2\delta} \subset B_{4\delta}(x) \setminus B_\delta(x)$, the Lemma is proved.

PROOF OF THEOREM 1. Lemma 4 and 6 provide the tools for the proof, which, at this stage, goes like in [6]. We briefly sketch the argument.

We have to prove that if u_n are solutions of $(D)_{\lambda_n}$ with $\sup_n \int u_n^{p+1} < +\infty$, then $\sup_n \lambda_n < +\infty$. This is clear if u_n has a non zero weak limit, so we can assume $u_n \rightharpoonup 0$.

According to Lemma 3, there are $x_1, \ldots, x_k \in \overline{\Omega}$ such that $u_n \to 0$ in $C^0_{\mathrm{loc}}(\mathbb{R}^N \setminus C)$ with $C \equiv \{x_1, \ldots, x_k\}$. Let $0 < \delta < \min\{\overline{\delta}(\partial\Omega), \frac{1}{8} d(x_i, x_j), i \neq j\}$, so that (3.14) in Lemma 6 holds for all $x_j \in C$:

$$\lambda_n \int_{B_\delta(x_j)} u_n^2 \leqslant \frac{2c_N}{\delta^3} \int_{B_{4\delta}(x_j)\setminus B_\delta(x_j)} u_n^2 \ \forall x_j \in C$$

Since the balls $B_{4\delta}(x_j)$ are taken disjoint, we get

$$\lambda_n \int_{C_\delta} u_n^2 \leqslant \frac{2c}{\delta^3} \int_{\Omega\setminus C_\delta} u_n^2 \tag{19}$$

which, jointly with (10), implies λ_n remains bounded.

PROOF OF THEOREM 2. The proof is by contradiction: we assume that there is a sequence u_λ of bounded energy solutions with $\lambda \to +\infty$, and no blow-up points on Γ_1. Hence, for this sequence, (14) holds true at any blow-up point. In addition, Lemma 4 holds true for the problem $(M)_\lambda$. In fact, arguments in the proof of Lemma 4 are not affected by the presence of Neumann boundary conditions, and the concentration behaviour assumed therein follows by a simple adjustment in the proof of Lemma 3: in (2) the term $S^{\frac{N}{2}}$ becomes $\frac{S^{\frac{N}{2}}}{2}$ as it follows by replacing in (3), the Sobolev inequality with the Cherrier inequality (see [5]) for any

$\delta > 0$ there exists $C(\delta) > 0$ such that for any $u \in H^1(\Omega)$

$$\left(\frac{S}{2^{\frac{2}{N}}} - \delta\right)\left(\int_\Omega |u|^{p+1}\right)^{\frac{2}{p+1}} \leqslant \int_\Omega |\nabla u|^2 + C(\delta)\int_\Omega u^2$$

and hence Lemma 3 holds for $(M)_\lambda$ as well, thanks to this inequality, to the fact that $\int_\Omega u_n^2 \to 0$ (so that $C(\delta)|u_n|_2^2 = \circ(1)$) and since, because of the null boundary conditions, there are no boundary contributions in the estimates. We use Cherrier inequality also in the Moser-type scheme to obtain $C^0_{\mathrm{loc}}(\overline{\Omega}\backslash C)$ convergence.

Since (10) and (14) are satisfied, the same argument as in the proof of Theorem 1 applies, giving a contradiction.

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