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Brown-Peterson Spectra in Stable $\mathbb{A}^1$-Homotopy Theory.

GABRIELE VEZZOSI (*)

ABSTRACT - We characterize ring spectra morphisms from the algebraic cobordism spectrum $\mathbb{MGL}$ ([12]) to an oriented spectrum $\mathbb{E}$ (in the sense of Morel [4]) via formal group laws on the «topological» subring $E^* = \bigoplus_i E^{2i,\, i}$ of E^{**}. This result is then used to construct for any prime p a motivic Quillen idempotent on $\mathbb{MGL}_{(p)}$. This defines the BP-spectrum associated to the prime p as in Quillen's [6] for the complex-oriented topological case.

1. Introduction.

My interest in the subject of this paper originated from the idea to extend Totaro's construction ([11]) of a refined cycle map with values in a quotient of complex cobordism, into Voevodsky's algebraic cobordism setting ([12]). To construct such an extension, one need to prove, as Totaro did for complex cobordism, a slight refinement of Quillen theorem 5.1 of [7]. As Quillen's original proof rests on some finiteness results (Prop. 1.12, [7]) together with a geometric interpretation of complex cobordism groups and since these conditions are not currently (1) available in algebraic cobordi-

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This paper was written during a very pleasant visit at the Institute for Advanced Study, Princeton, NJ. The author wishes to thank everybody there for creating such an exceptional atmosphere.
(1) Actually, the first is genuinely false.

sm, we decided to follow Totaro's argument which uses BP and $BP\langle n\rangle$ spectra.

The first step, i.e. to define BP-spectra in $\mathbb{A}^1$-homotopy theory, is carried over in this paper. The other steps (construction of $BP\langle n\rangle$ spectra and Wilson's Theorem, [16], p. 413) will be considered in subsequent papers.

The intuition guiding the construction of BP-spectra obviously comes from topology although some extra care is required in $\mathbb{A}^1$-homotopy theory because of the lacking of some facts which are well known in topology. I am referring in particular to the fact that $\mathrm{MGL}^{**} = \mathrm{MGL}^{**}(\mathrm{Spec}\,(k))$ has not been computed yet, although what I call in this paper its «topological» subring $\mathrm{MGL}^* \doteq \bigoplus_i \mathrm{MGL}^{2i,\,i}$, is conjectured to be isomorphic to the complex cobordism ring MU^* and therefore, by Quillen theorem, to the Lazard ring ([12] 6.3).

Here is a brief description of the contents. The first section is devoted to explain the algebraic geometric motivation that lead us to the problem of defining Brown-Peterson spectra in $\mathbb{A}^1$-homotopy theory. In the second section we use some results of Morel ([4]), to establish the basic properties of oriented ring spectra in the stable $\mathbb{A}^1$-homotopy category $\mathcal{SH}(k)$ of smooth schemes over a ground field k ([12], [5]), especially the Thom isomorphism in its stable and unstable versions. In the third section, for an oriented spectrum $\mathbb{E}$, we prove the equivalence between giving an orientation on $\mathbb{E}$, giving a ring spectra map $\mathbb{MGL} \to E$ and giving a formal group law on $E^* = \bigoplus_i E^{2i,\,i}(\mathrm{Spec}\,(k))$ isomorphic to the one associated to the given orientation. This enables us to give the construction of BP-spectra in the fourth section through the construction of a «motivic» Quillen idempotent.

We remark that all our proofs work also in the topological case, with algebraic cobordism replaced by complex cobordism and that the safe path we have to follow in order to avoid any use of Quillen theorem on the Lazard ring, yields proofs which in the topological case are different from the standard ones.

Very similar results to the ones presented here in Section 4 and a sketchy version of Section 5 were obtained independently by Po Hu and Igor Kriz in their preprint «Some remarks on real and algebraic cobordism» available at the K-theory preprint archives (http://www.math.uiuc.edu/K-theory). I thank Prof. Peter May for this information.

This paper was «triggered» by a question Vladimir Voevodsky posed to

me during a visit at the Institute for Advanced Study, in March 2000. I wish to thank him for helpful discussions and advices on this and other topics. I am also indebted to Fabien Morel for allowing me to use the results in [4].

It is also a pleasure to thank Dan Christensen, Charles Rezk, Burt Totaro and Chuck Weibel for listening carefully and patiently to a non-topologist speaking about topology and for their useful advices.

2. A motivation.

In this section we briefly explain the motivations that lead us to the problem of finding a definition of Brown-Peterson spectra in $\mathbb{A}^1$-homotopy theory. This section is mainly conjectural, independent from the others and only meant to suggest one possible way to approach the problems described below.

In [11], Totaro defined a refined cycle class map

$$(1) \qquad A_*(X) \xrightarrow{\widetilde{\mathrm{cl}}_X} \widetilde{\mathrm{MU}}_{2*}(X^{\mathrm{an}}) \doteq (\mathrm{MU}_*(X^{\mathrm{an}}) \bigotimes_{\mathrm{MU}_*} \mathbb{Z})_{2*}$$

where X is an equidimensional smooth complex algebraic scheme, X^{an} the associated complex manifold, $A_*(X)$ is the Chow group of X ([2], Ch. 1), $MU_*(Y)$ denotes the complex bordism group of the topological space Y and

$$\mathrm{MU}_* = \mathrm{MU}_*(\mathrm{pt}) \simeq \mathbb{Z}[x_1, x_2, \dots, x_n, \dots]$$

with $\deg x_i = 2i$. Here $\mathbb{Z}$ is considered as an MU_*-module via the map sending each x_i to 0 i.e.

$$\widetilde{\mathrm{MU}}_*(X^{\mathrm{an}}) = \mathrm{MU}_*(X^{\mathrm{an}}) \bigotimes_{\mathrm{MU}_*} \mathbb{Z} \simeq \frac{\mathrm{MU}_*(X^{\mathrm{an}})}{\mathrm{MU}_{>0} \cdot \mathrm{MU}_*(X^{\mathrm{an}})}.$$

Totaro proved that the classical cycle map $\mathrm{cl}_X\colon A_*(X) \to H_{2*}^{\mathrm{BM}}(X^{\mathrm{an}}; \mathbb{Z})$ ([2], 19.1) to Borel-Moore homology factors as

$$A_*(X) \xrightarrow{\widetilde{\mathrm{cl}}_X} \widetilde{\mathrm{MU}}_{2*}(X^{\mathrm{an}}) \xrightarrow{\gamma_{X^{\mathrm{an}}}} \mathrm{H}_{2*}^{\mathrm{BM}}(X^{\mathrm{an}}; \mathbb{Z})$$

where $\gamma_{X^{\mathrm{an}}}$ is induced by the canonical map of homology theories

$$\mathrm{MU}_*(X^{\mathrm{an}}) \to \mathrm{H}_*^{\mathrm{BM}}(X^{\mathrm{an}}; \mathbb{Z})$$

$$[f: M \to X^{\mathrm{an}}] \mapsto f_*(\eta_M)$$

where we used the geometric interpretation of complex bordism classes as equivalence classes of proper maps from weakly complex manifolds ([9]) and η_M denotes the fundamental class of the weakly complex manifold M.

The refined cycle map $\widetilde{\mathrm{cl}}_X$ sends the class of a cycle $Z \hookrightarrow X$ to the class of the composition

$$\tilde{Z} \to Z \hookrightarrow X$$

$\tilde{Z} \to Z$ being any resolution of singularities. The reason it is well defined is a combination of Hironaka's theorem, Poincaré duality and the following

THEOREM 2.1 (Quillen-Totaro theorem, [11] Thm. 2.2). *Let Y be a finite cell complex. Then the canonical map*

$$\widetilde{\mathrm{MU}}^*(Y) \xrightarrow{\gamma_Y} H^*(Y; \mathbb{Z})$$

is injective in degrees $\leqslant 2$.

In degrees $\leqslant 0$ this is a consequence of Quillen theorem ([7], 5.1) but Totaro's proofs uses a different approach, through BP-spectra $\mathbb{BP}$, truncated BP-spectra $\mathbb{BP}\langle n\rangle$ and Wilson theorem ([16], p. 118).

Our question is if there exists a generalization of Totaro's refined cycle map over an arbitrary field k admitting resolution of singularities, with complex (co)bordism replaced by algebraic (co)bordism. It turns out that even the formulation of the analog of Quillen theorem requires a little care.

Following [12], let us denote by $\mathbb{H}_{\mathbb{Z}}$ the motivic Eilenberg-Mac Lane spectrum and by $\mathbb{MGL} \doteq (\mathrm{MGL}(n))_n$ the algebraic cobordism spectrum ([12], 6.1 and 6.3). These are objects in the stable $\mathbb{A}^1$-homotopy category $\mathcal{SH}(k)$ of smooth schemes over k. First of all, by [14] Thm. 3.21

$$\mathrm{Hom}_{\mathcal{SH}(k)}(\mathbb{MGL}, \mathbb{H}_{\mathbb{Z}}) \doteq \mathrm{H}_{\mathbb{Z}}^{0,0}(\mathbb{MGL}) \simeq \mathbb{Z}$$

canonically and if we denote by τ a generator of $H_{\mathbb{Z}}^{0,0}(\mathbb{MGL})$, for any smooth scheme X over k we have an induced morphism of cohomology theories

$$\tau_X\colon \mathrm{MGL}^{**}(X) \to \mathrm{H}_{\mathbb{Z}}^{**}(X).$$

Let

$$\mathrm{MGL}^*(X) \doteq \bigoplus_i \mathrm{MGL}^{2i,i}(X)$$

and

$$H_{\mathbb{Z}}^*(X) \doteq \bigoplus_i \mathrm{H}_{\mathbb{Z}}^{2i,i}(X).$$

Since ([15], p. 293) $\mathrm{H}_{\mathbb{Z}}^p \doteq \mathrm{H}_{\mathbb{Z}}^p(\mathrm{Spec}\,k) \simeq A^p(\mathrm{Spec}\,k) = 0$ if $p<0$, the restriction of τ_X to $\mathrm{MGL}^*(X)$ factors through

$$(2)\qquad \widetilde{\tau_X}\colon \widetilde{\mathrm{MGL}}^*(X) \doteq \frac{\mathrm{MGL}^*(X)}{\mathrm{MGL}^{<0}\cdot \mathrm{MGL}^*(X)} \to H_{\mathbb{Z}}^*(X)$$

as in the complex (oriented) case.

Remark 2.2. The ring $\mathrm{MGL}^{**} = \mathrm{MGL}^{**}(\mathrm{Spec}\,k)$ is not known but it is conjectured that its subring MGL^* is isomorphic to MU^* ([12], 6.3). A part of this conjecture, i.e. that MGL^* is zero in positive degrees, follows immediately from the $\mathcal{SH}(k)$-version of the Connectivity Theorem 4.14 in [12]. The rest of the conjecture is, as far as we know, still open. This is the main reason why we will need to avoid the relation of MGL^* to the Lazard ring in the following sections.

Now, if $\eta_Y \in \mathrm{MGL}_{2n,n}(Y)$ denotes the fundamental class of an n-dimensional smooth k-scheme Y in algebraic cobordism ([14]), the motivic analog of Totaro's refined cycle map should be the map

$$(3)\qquad \widetilde{\mathrm{CL}}_X\colon A_*(X) \to \widetilde{\mathrm{MGL}}_*(X) = \frac{\mathrm{MGL}_*(X)}{\mathrm{MGL}_{>0}\cdot \mathrm{MGL}_*(X)}$$

sending the class $[Z \hookrightarrow X]$ of a cycle of dimension i in X to the class modulo $\mathrm{MGL}_{>0}\cdot\mathrm{MGL}_*(X)$ of

$$((\mathbb{P}^1,\infty)^{\wedge i} \xrightarrow{\eta_{\tilde{Z}}} \Sigma^\infty \tilde{Z}_+ \wedge \mathbb{MGL} \xrightarrow{\Sigma^\infty f\wedge \mathrm{id}} \Sigma^\infty X_+ \wedge \mathbb{MGL}) \in \mathrm{MGL}_{2i,i}(X)$$

where

$$f\colon \tilde{Z} \to Z \hookrightarrow X$$

$\tilde{Z} \to Z$ being any resolution of singularities. Assuming we have Poincaré duality for algebraic (co)bordism and motivic (co)homology, well definiteness of (3) would be a consequence of the following

CONJECTURE 2.3 (Motivic Quillen-Totaro Theorem). *For any smooth scheme over k, the map* (2)

$$\widetilde{\tau_X}: \widetilde{\mathrm{MGL}}^*(X) \to \mathrm{H}^*_{\mathbb{Z}}(X)$$

is injective in degrees ≤ 2.

Note that in this case, the «classical» cycle map should be the identity

$$\mathrm{CL}_X = \mathrm{id} : A_*(X) \to \mathrm{H}^{\mathbb{Z}}_*(X) \simeq A_*(X)$$

and the factorization $\mathrm{id} = \widetilde{\tau^X} \circ \widetilde{\mathrm{CL}}_X$ would imply that $\widetilde{\tau^X}$ is surjective and $\widetilde{\mathrm{CL}}_X$ injective.

Following the idea in Totaro's proof of [11], Thm. 2.2, the first thing to know is how to construct BP-spectra in $\mathcal{SH}(k)$. This is done in the following sections.

3. Oriented spectra and Thom isomorphism.

Throughout the paper we fix a base field k and work in the stable $\mathbb{A}^1$-homotopy category $\mathcal{SH}(k)$ of smooth schemes over k, as described in [12], whose notations we follow closely. In particular, Σ^∞ will denote the infinite $(\mathbb{P}^1, \infty)$-suspension, and for any space X over k, we write X_+ for the space $X \coprod \mathrm{Spec}(k)$ pointed by $\mathrm{Spec}(k)$. S^0 will denote $\mathrm{Spec}(k)_+$ and $\mathbb{S}^{p,q}$ will denote the spectrum obtained from the smash product of the mixed spheres S_s^{p-q} and S_t^q as described in [5] 3.2.2; recall that for any spectrum $\mathbb{E}$ in $\mathcal{SH}(k)$ and any space X (respectively, spectrum $\mathbb{F}$) we have a cohomology theory

$$E^{p,q}(X) \doteq \mathrm{Hom}_{\mathcal{SH}(k)}(\Sigma^\infty(X_+), \mathbb{S}^{p,q} \wedge \mathbb{E})$$

(respectively,

$$E^{p,q}(\mathbb{F}) \doteq \mathrm{Hom}_{\mathcal{SH}(k)}(\mathbb{F}, \mathbb{S}^{p,q} \wedge \mathbb{E})).$$

When no reasonable ambiguity seems to take place, we also write simply $\mathbb{P}^1$ for the pointed space $(\mathbb{P}^1, \infty)$.

In this section we mainly follow [4] and draw some consequences thereof.

If $\mathbb{E}$ is a ring spectrum in $\mathcal{SH}(k)$ (in the weak sense), there is a canonical element $x_E^0 \in E^{2,1}(\mathbb{P}^1)$ given by the composition

$$\Sigma^\infty \mathbb{P}^1_+ \to \Sigma^\infty(\mathbb{P}^1, \infty) \xrightarrow{\sim} \Sigma^\infty(\mathbb{P}^1, \infty) \wedge S^0 \xrightarrow{\mathrm{id}\wedge\eta} \Sigma^\infty(\mathbb{P}^1, \infty) \wedge \mathbb{E}$$

where $\eta : \Sigma^\infty(\mathbb{P}^1, \infty) \to \mathbb{E}$ denotes the unit morphism of the ring spectrum $\mathbb{E}$.

DEFINITION 3.1 ([4], 3.2.3). *An* oriented ring spectrum *in $\mathcal{SH}(k)$ is a pair $(\mathbb{E}, x_E)$ where $\mathbb{E}$ is a commutative ring spectrum and x_E is an element in $E^{2,1}(\mathbb{P}^\infty)$ restricting to the canonical element x_E^0 along the canonical inclusion $\mathbb{P}^1 \to \mathbb{P}^\infty$.*

Here, $\mathbb{P}^\infty \doteq \operatorname{colim}_n \mathbb{P}^n$.

If $\mathrm{Gr}_{n,N}$ denote the Grassmannian of n-planes in $\mathbb{A}^N$, $N > n$, let us denote by BGL_n (respectively, BGL) the infinite Grassmannian $\operatorname{colim}_N(\mathrm{Gr}_{n,N})$ of n-planes (respectively, $\operatorname{colim}_n(\mathrm{BGL}_n)$). Moreover, let us denote by $\mathbb{MGL} \doteq (\mathrm{MGL}(n))_n$ the algebraic cobordism spectrum ([12], 6.3).

LEMMA 3.2. *The zero section map*

$$s_0 : \mathbb{P}^\infty = \mathrm{BGL}_1 \to \mathrm{MGL}(1)$$

is a weak equivalence.

PROOF. For any $n > 0$, the closed immersion $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$ has normal bundle the canonical line bundle $\mathcal{L}_{n-1}$ on $\mathbb{P}^{n-1}$ and $\mathbb{P}^n - \mathbb{P}^{n-1}$ is isomorphic to $\mathbb{A}^n$; hence ([5], Lemma 2.2.10 and Theorem 3.2.23) the Thom space $\mathrm{Th}(\mathcal{L}_{n-1})$ ([12], p. 422) is weakly equivalent to $\mathbb{P}^n$ and these weak equivalences are compatible with respect to the maps in the direct system $\{\ldots \hookrightarrow \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n \hookrightarrow \mathbb{P}^{n+1} \hookrightarrow \ldots\}$. The result follows by passing to the colimit. ■

REMARK 3.3. Note that the algebraic cobordism spectrum $\mathbb{MGL}$ has a canonical orientation x_{MGL} given by the composition

$$\Sigma^\infty \mathbb{P}^\infty_+ \to \Sigma^\infty \mathbb{P}^\infty \xrightarrow{\Sigma^\infty(s_0)} \Sigma^\infty \mathrm{MGL}(1) \xrightarrow{\nu} \Sigma^\infty \mathbb{P}^1 \wedge \mathbb{MGL}$$

where ν is defined using the structural maps of the MGL-spectrum as

$$(\mathbb{P}^1)^{\wedge n} \wedge \mathrm{MGL}(1) \to (\mathbb{P}^1)^{\wedge n-1} \wedge \mathrm{MGL}(2) \to \ldots \to \mathbb{P}^1 \wedge \mathrm{MGL}(n).$$

PROPOSITION 3.4 ([4], 3.2.9). (i) *If* $(\mathbb{E}, x_E)$ *is an oriented ring spectrum, for any space* X *over* k, *we have*

$$\alpha \cup \beta = (-1)^{pp'} \beta \cup \alpha$$

for any $\alpha \in E^{p,q}(X)$ *and* $\beta \in E^{p',q'}(X)$. *In particular, the subring* $E^* \doteq \bigoplus_i E^{2i,i}(\mathrm{Spec}(k))$ *is commutative.*

(ii) *For any* $n > 0$, *there is a canonical isomorphism*

$$E^{**}(\mathbb{P}^n) \simeq E^{**}[x_E]/(x_E^{n+1})$$

where we still denote by x_E *its pullback along* $\mathbb{P}^n \hookrightarrow \mathbb{P}^\infty$.

(iii) *Moreover, for any* $n > 0$, *there is a canonical isomorphism*

$$E^{**}\Bigg(\underbrace{\mathbb{P}^\infty \times \ldots \times \mathbb{P}^\infty}_{n \text{ times}}\Bigg) \simeq E^{**}[\![x_1, \ldots x_n]\!]$$

where x_i *denote the pullback of* x_E *along the i-th projection* $\mathrm{pr}_i \colon \mathbb{P}^\infty \times \ldots \times \mathbb{P}^\infty \to \mathbb{P}^\infty$, $i = 1, \ldots, n$.

Note that (iii), follows from (ii) (which is [4], 3.2.9, (2)), by passing to the limit after recognizing the Mittag-Leffler condition is satisfied.

PROPOSITION 3.5. *If* $(\mathbb{E}, x_E)$ *is an oriented ring spectrum, for any* $n > 0$, *the pullback along the canonical map*

$$\underbrace{\mathbb{P}^\infty \times \ldots \times \mathbb{P}^\infty}_{n \text{ times}} \to \mathrm{BGL}_n \tag{4}$$

classifying the product of canonical line bundles (respectively, the pullback along the colimit (²) *of maps* (4)

$$\theta : (\mathbb{P}^\infty)^\infty \doteq \operatorname*{colim}_n (\mathbb{P}^\infty) \to \mathrm{BGL})$$

(²) The colimit over the inclusion of any rational point in $\mathbb{P}^\infty$. Any choice will yield the same direct system up to weak equivalences since any two such points belong to an affine line over k.

induces a monomorphism

$$E^{**}(\mathrm{BGL}_n) \simeq E^{**}[\![c_1, \ldots, c_n]\!] \hookrightarrow E^{**}[\![x_1, \ldots, x_n]\!] \simeq E^{**}\Big(\underbrace{\mathbb{P}^\infty \times \ldots \times \mathbb{P}^\infty}_{n \text{ times}}\Big)$$

(*respectively,*

$$E^{**}(\mathrm{BGL}) \simeq E^{**}[\![c_1, \ldots, c_n, \ldots]\!] \hookrightarrow E^{**}[\![x_1, \ldots, x_n, \ldots]\!] \simeq E^{**}((\mathbb{P}^\infty)^\infty))$$

where c_i denotes the i-th elementary symmetric function on the x_j's.

PROOF. Both assertions follows from [4] 3.2.10 (2), by induction on n. ■

COROLLARY 3.6. *If $(\mathbb{E}, x_E)$ is an oriented ring spectrum, there exists a canonical «family of universal Thom classes» ([1]) i.e. a family $(\tau_n^E)_{n>0}$ with $\tau_n^E \in E^{2n,n}(\mathrm{MGL}(n))$, such that:*

(i) $\tau_1^E = x_E$ (*this makes sense because of Lemma* 3.2);

(ii) *the family is multiplicative in the sense that τ_{n+m}^E pulls back to $\tau_n^E \wedge \tau_m^E$ along the map* $\mathrm{MGL}(n) \wedge \mathrm{MGL}(m) \to \mathrm{MGL}(n+m)$ (*induced by the canonical map* $\mathrm{BGL}_n \times \mathrm{BGL}_m \to \mathrm{BGL}_{n+m}$ *making* BGL *into an H-space*).

PROOF. If $\xi_n \to \mathrm{BGL}_n$ denotes the universal n-plane bundle, we have a cofiber sequence ([5], 3.2.17)

$$\mathbb{P}(\xi_n) \to \mathbb{P}(\xi_n \oplus \mathbf{1}) \to \mathrm{MGL}(n)$$

whose associated E-cohomology long exact sequence yields, by the projective bundle theorem ([4], 3.2.10, (1)), a short exact sequence

$$0 \to E^{**}(\mathrm{MGL}(n)) \to E^{**}(\mathrm{BGL}_n)[t]/(t \cdot f(t)) \xrightarrow{\pi} E^{**}(\mathrm{BGL}_n)[s]/(f(s)) \to 0$$

where $f(t) \doteq t^n + c_1(\xi_n)\, t^{n-1} + \ldots + c_n(\xi_n)$, t corresponding to the first Chern class of the canonical line bundle on $\mathbb{P}(\xi_n)$, $\mathbf{1}$ denotes the trivial line bundle over BGL_n, π maps t to s and $c_i(\mathcal{E}) \in E^{2i,i}(\mathrm{BGL}_n)$ denotes here the i-th Chern class of a vector bundle $\mathcal{E}$ (we have used that $c_i(\xi_n \oplus 1) = c_i(\xi_n)$). Then $f(t)$ has bidegree $(2n, n)$ and is in the kernel of π. Then, τ_n^E is the unique element in $E^{2n,n}(\mathrm{MGL}(n))$ mapping to $f(t)$ and it is easy to verify that the family $(\tau_n^E)_{n>0}$ defined in this way is multiplicative. ■

REMARK 3.7. The universal Thom classes $(\tau_n^E)_{n>0}$ admit the following equivalent characterization. For any $n>0$, consider the product $c_n = x_1 \ldots x_n \in E^{2n,n}(\mathrm{BGL}_n)$ (Prop. 3.5). By Prop. 3.5, the E-cohomology long exact sequence associated to the cofiber sequence

$$\mathrm{BGL}_{n-1} \to \mathrm{BGL}_n \to \mathrm{MGL}(n)$$

yields a short exact sequence

$$0 \to E^{2n,n}(\mathrm{MGL}(n)) \xrightarrow{\varphi} E^{2n,n}(\mathrm{BGL}_n) \xrightarrow{\psi} E^{2n,n}(\mathrm{BGL}_{n-1}) \to 0$$

where ψ maps $g(c_1, \ldots, c_n)$ to $g(c_1, \ldots, c_{n-1}, 0)$. Therefore there exists a unique class

$$\tau_n^E \in E^{2n,n}(\mathrm{MGL}(n))$$

such that $\varphi(\tau_n^E) = c_n$.

As in the topological case, the projective bundle structure theorem for E-cohomology ([4], 3.2.10, (1)) implies the Thom isomorphism.

Let $(\mathbb{E}, x_E)$ be an oriented ring spectrum, X a smooth scheme over k and $\mathcal{E} \to X$ be a vector bundle of rank r. If $Th(\mathcal{E}/X)$ denotes the Thom space of $\mathcal{E}$ ([12], p. 422), the diagonal map $\delta : X \to X \times X$ induces a Thom diagonal

$$\Delta_{\mathcal{E}} \colon \mathrm{Th}(\mathcal{E}/X) \to \mathrm{Th}(\mathcal{E}/X) \wedge X_+ . \tag{5}$$

Since $\mathcal{E}$ has rank r, there is a canonical map

$$\lambda_{\mathcal{E}} \colon \mathrm{Th}(\mathcal{E}/X) \to \mathrm{MGL}(r).$$

Therefore we have a Thom map

$$\Phi_{\mathcal{E}} \colon E^{**}(X) \to E^{*+2r,*+r}(\mathrm{Th}(\mathcal{E}/X)) \tag{6}$$

which assigns to an element $\alpha \in E^{p,q}(X) = \mathrm{Hom}_{\mathcal{SH}(k)}(\Sigma^\infty(X_+), \mathbb{S}^{p,q} \wedge \mathbb{E})$ the element $\Phi_{\mathcal{E}}(\alpha)$ in $E^{p+2r,q+r}(\mathrm{Th}(\mathcal{E}/X))$ given by the composition

$$\Sigma^\infty \mathrm{Th}(\mathcal{E}/X) \xrightarrow{\Sigma^\infty \Delta_{\mathcal{E}}} \Sigma^\infty(\mathrm{Th}(\mathcal{E}/X) \wedge X_+) \xrightarrow{\Sigma^\infty(\lambda_{\mathcal{E}} \wedge \alpha)} \Sigma^\infty(\mathrm{MGL}(r) \wedge \mathbb{S}^{p,q} \wedge \mathbb{E}) \to$$

$$\xrightarrow{\tau_r^E \wedge \mathrm{id}} \mathbb{S}^{2r,r} \wedge \mathbb{E} \wedge S^{p,q} \wedge \mathbb{E} \to \mathbb{S}^{p+2r,q+r} \wedge \mathbb{E}$$

where $\tau_r^E \in E^{2r,r}(\mathrm{MGL}(r))$ is the universal Thom class of Prop. 3.6 and the last map is induced by the ring structure on $\mathbb{E}$.

THEOREM 3.8. *Let* $(\mathbb{E}, x_E)$ *be an oriented ring spectrum, X a smooth scheme over k and* $\mathcal{E} \to X$ *be a vector bundle of rank r. Then the Thom map* (6)

$$\Phi_{\mathcal{E}}: E^{**}(X) \to E^{*+2r, *+r}(\mathrm{Th}(\mathcal{E}/X))$$

is an isomorphism.

PROOF. By [5], 3.2.17, we have a canonical cofiber sequence

$$\mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{E} \oplus \mathbf{1}) \to \mathrm{Th}(\mathcal{E}/X). \tag{7}$$

The projective bundle structure theorem for E-cohomology ([4], 3.2.10, (1)) together with the E-cohomology long exact sequence associated to (7) yield a short exact sequence

$$0 \to E^{**}(\mathrm{Th}(\mathcal{E}/X)) \to E^{**}(X)[t]/(t \cdot f(t)) \xrightarrow{\pi} E^{**}(X)[s]/(f(s)) \to 0 \tag{8}$$

where $f(t) \doteq t^r + c_1(\mathcal{E})\, t^{r-1} + \ldots + c_r(\mathcal{E})$ (t corresponding to the first Chern class of the canonical line bundle on $\mathbb{P}(\mathcal{E})$), $\mathbf{1}$ denotes the trivial line bundle over X, π maps t to s and $c_i(\mathcal{E}) \in E^{2i, i}(X)$ denotes the i-th Chern class of the vector bundle $\mathcal{E}$. We have used that $c_i(\mathcal{E} \oplus 1) = c_i(\mathcal{E})$. Then $f(t)$ has bidegree $(2r, r)$ and is in the kernel of π. Then, there is a unique Thom class $\tau_r^E(\mathcal{E})$ for $\mathcal{E}$ in $E^{2r, r}(Th(\mathcal{E}/X))$ mapping to $f(t)$. Obviously we have $\tau_r^E(\mathcal{E}) = \tau_r^E \circ \Sigma^\infty(\lambda_{\mathcal{E}})$ and the theorem follows from the exactness of (8). ■

REMARK 3.9. It is clear from the proof above that to prove the Thom isomorphism for oriented spectra it is enough to know the projective bundle theorem together with the fact that «orientability» of the spectrum implies the existence of «universal Thom classes» which in its turn implies «orientability» of any vector bundle.

For the construction of BP-spectra we will only need the Thom isomorphism for the localization $\mathbb{MGL}_{(p)}$ of $\mathbb{MGL}$, at a prime p and this actually follows from the Thom isomorphism for $\mathbb{MGL}$. A proof of this case can be found in [13], Lecture 3.

COROLLARY 3.10. *If* $(\mathbb{E}, x_E)$ *is an oriented ring spectrum, there is a canonical Thom isomorphism*

$$\Phi : E^{**}(\mathrm{BGL}) \xrightarrow{\sim} E^{**}(\mathbb{MGL}).$$

Moreover Φ *restricts, in bidegree* $(0, 0)$, *to a bijection between ring spectra maps* $\Sigma^\infty \mathrm{BGL}_+ \to \mathbb{E}$ *and ring spectra maps* $\mathbb{MGL} \to \mathbb{E}$.

PROOF. The first assertion is just the stable version of Theorem 3.8 applied to the canonical n-plane bundles $\xi_n \to \mathrm{BGL}_n$. In fact, the naturality of the Thom diagonal (5) implies the commutativity of

$$\begin{tikzcd}
\mathbb{P}^1 \wedge \mathrm{MGL}(n) \arrow[r, "\Delta_{\xi_n \oplus 1}"] \arrow[d, "\sigma_n"'] & (\mathrm{BGL}_n)_+ \wedge \mathbb{P}^1 \wedge \mathrm{MGL}(n) \arrow[d, "(i_n)_+ \wedge \sigma_n"] \\
\mathrm{MGL}(n+1) \arrow[r, "\Delta_{\xi_n}"'] & (\mathrm{BGL}_{n+1})_+ \wedge \mathrm{MGL}(n+1)
\end{tikzcd}$$

for any $n > 0$, where the σ_n's are the structural maps of algebraic cobordism, the i_n's are the natural inclusions $\mathrm{BGL}_n \hookrightarrow \mathrm{BGL}_{n+1}$ and we used the multiplicativity property of Thom spaces

$$\mathrm{Th}((\mathcal{E} \oplus \mathbf{1})/X) \simeq \mathbb{P}^1 \wedge \mathrm{Th}(\mathcal{E}/X)$$

which holds for any vector bundle $\mathcal{E}$ over X. Therefore, for any (p, q), the diagram

$$\begin{tikzcd}
E^{p,q}(\mathrm{BGL}_{n+1}) \arrow[rr, "(i_n)_+^*"] \arrow[d, "\Phi_{\xi_{n+1}}"'] & & E^{p,q}(\mathrm{BGL}_n) \arrow[d, "\Phi_{\xi_n}"] \\
E^{p+2n+2, q+n+1}(\mathrm{MGL}(n+1)) \arrow[dr, "\sigma_n^*"'] & & E^{p+2n, q+n}(\mathrm{MGL}(n)) \\
& E^{p+2n+2, q+n+1}(\mathbb{P}^1 \wedge \mathrm{MGL}(n)) \arrow[ur, "(\mathbb{P}^1)^{\wedge -1}"'] &
\end{tikzcd}$$

is commutative and so the family of unstable Thom isomorphism $(\Phi_{\xi_n})_{n>0}$ stabilizes to an isomorphism $\Phi : E^{**}(\mathrm{BGL}) \xrightarrow{\sim} E^{**}(\mathbb{MGL})$.

The second assertion is a long but straightforward verification using the commutativity of the diagram

$$\begin{tikzcd}
E^{**}(\mathbb{MGL}) \arrow[r, "\mu^*"] & E^{**}(\mathbb{MGL} \wedge \mathbb{MGL}) \\
E^{**}(\mathrm{BGL}) \arrow[r, "m^*"'] \arrow[u, "\Phi"] & E^{**}(\mathrm{BGL} \times \mathrm{BGL}) \arrow[u, "\Phi'"']
\end{tikzcd}$$

where $\mu : \mathbb{MGL} \wedge \mathbb{MGL} \to \mathbb{MGL}$ is the product, $m : \mathrm{BGL} \times \mathrm{BGL} \to \mathrm{BGL}$ is the canonical map induced by the map $BGL_n \times \mathrm{BGL}_m \to \mathrm{BGL}_{n+m}$ (and making BGL into an H-space) and Φ' are stable Thom isomorphisms. ■

4. Orientations, ring spectra maps and formal group laws.

In this section we establishes the basic correspondence, well known in the topological complex oriented case, between orientations, maps of ring spectra and formal group laws. We notice that our proof works also in the topological case and avoids the use of Quillen result that the complex cobordism ring MU* is isomorphic to the Lazard ring, a result that in fact is not known in the case of algebraic cobordism.

LEMMA 4.1. *If* $(\mathbb{E}, x_E)$ *is an oriented ring spectrum, an element*

$$\varphi \in E^{0,0}(\mathrm{BGL}) = \mathrm{Hom}_{\mathcal{SH}(k)}(\Sigma^\infty \mathrm{BGL}_+, \mathbb{E})$$

is a map of ring spectra iff it corresponds via the isomorphism of Proposition 3.5 *to a power series* $\widehat{\varphi}$ *of the form*

$$\widehat{\varphi}(x_1, \ldots, x_n, \ldots) = \prod_{i=1}^{\infty} h(x_i)$$

with $h(t)$ *a degree zero homogeneous power series of the form* $1 + a_1 t + a_2 t^2 + \ldots$.

PROOF. A map $\varphi \in \mathrm{Hom}_{\mathcal{SH}(k)}(\Sigma^\infty \mathrm{BGL}_+, \mathbb{E})$ is a ring map iff the following diagram commutes

$$\begin{tikzcd} \Sigma^\infty \mathrm{BGL}_+ \wedge \Sigma^\infty \mathrm{BGL}_+ \arrow[r] \arrow[d, "\varphi\wedge\varphi"'] & \Sigma^\infty \mathrm{BGL}_+ \arrow[d, "\varphi"] \\ \mathbb{E}\wedge\mathbb{E} \arrow[r] & \mathbb{E} \end{tikzcd}$$

Arguing as in [10], 16.47 pp. 404-406, we see that if g is the map

$$g : (\mathbb{P}^\infty)^\infty \times (\mathbb{P}^\infty)^\infty \to (\mathbb{P}^\infty)^\infty$$

$$(u_1, \ldots, u_n, \ldots; v_1, \ldots, v_n, \ldots) \mapsto (u_1, v_1, u_2, v_2, \ldots, u_n, v_n, \ldots),$$

the diagram

$$\begin{tikzcd} (\mathbb{P}^\infty)^\infty \times (\mathbb{P}^\infty)^\infty \arrow[r, "g"] \arrow[d, "\theta\times\theta"'] & (\mathbb{P}^\infty)^\infty \arrow[d, "\theta"] \\ \mathrm{BGL}\times\mathrm{BGL} \arrow[r, "m"'] & \mathrm{BGL} \end{tikzcd}$$

is (homotopy) commutative, where θ is the map defined in Proposition 3.5. Therefore, φ is a ring map iff its power series $\widehat{\varphi}$ satisfies the relation

$$\widehat{\varphi}(x_1, \ldots, x_n, \ldots)\cdot\widehat{\varphi}(y_1, \ldots, y_n, \ldots) = \widehat{\varphi}(x_1, y_1, x_2, y_2, \ldots, x_n, y_n, \ldots)$$

and we easily conclude by defining $h(t) \doteq \widehat{\varphi}(t, 0, \ldots, 0, \ldots)$. ∎

Let $(\mathbb{E}, x_E)$ be an oriented ring spectrum and $w : \mathbb{P}^\infty \times \mathbb{P}^\infty \to \mathbb{P}^\infty$ be the canonical map making $\mathbb{P}^\infty$ into an H-space. By Proposition 3.4 (iii), $w^*(x_E)$ defines a power series F_{x_E} in $E^{**}[\![x_1, x_2]\!]$. But since x_E has bidegree $(2, 1)$, F_{x_E} is actually an element of the subring $E^*[\![x_1, x_2]\!]$, where $E^* \doteq \bigoplus_i E^{2i, i}$. Recall that E^* is a commutative ring (3.4 (i)).

PROPOSITION 4.2. *If $(\mathbb{E}, x_E)$ is an oriented ring spectrum and $w : \mathbb{P}^\infty \times \mathbb{P}^\infty \to \mathbb{P}^\infty$ the canonical map making $\mathbb{P}^\infty$ into an H-space, the power series $F_{x_E} \doteq w^*(x_E)$ is a formal group law* ([3]) *on the «topological» subring E^*.*

PROOF. This is a standard consequence of the fact that w defines an H-structure on $\mathbb{P}^\infty$ (see for example [8], VII.6.2). ■

THEOREM 4.3. *Let $(\mathbb{E}, x_E)$ be an oriented ring spectrum and F_{x_E} the formal group law on E^* associated to the given orientation x_E. Then the following sets correspond bijectively:*

(i) *orientations on $\mathbb{E}$;*

(ii) *maps of ring spectra $\mathrm{MGL} \to \mathbb{E}$;*

(iii) *pairs (F, ε) where F is a formal group law on E^* and $\varepsilon : F \xrightarrow{\sim} F_{x_E}$ is an isomorphism of formal group laws.*

PROOF. By Proposition 3.4 (ii) an orientation x on $\mathbb{E}$ is of the form $f(x_E)$ where $f(t) = t + a_2 t^2 + a_3 t^3 + \ldots$. Such an f gives an isomorphism of formal group laws $F_x \xrightarrow{\sim} F_{x_E}$ and viceversa. Hence (i) and (iii) are in bijection.

With the same notations, the power series

$$\widehat{\varphi}(x_1, \ldots, x_n, \ldots) \doteq \prod_{i=1}^{\infty} \frac{f(x_i)}{x_i}$$

defines a map of ring spectra $\varphi : \mathrm{MGL} \to \mathbb{E}$, by Lemma 4.1 and Corollary 3.10, and this construction can be inverted. ■

REMARK 4.4. Replacing $\mathcal{SH}(k)$ with the topological stable homotopy category and MGL with the complex cobordism spectrum MU, the proofs of Lemma 4.1 and Theorem 4.3 carry over without modifications. This is a slightly different approach from the usual one in topology where one uses Quillen Theorem (i.e. the isomorphism of MU* with the universal Lazard ring) to prove the equivalence between (ii) and (iii) in Theorem 4.3. Actually, our proof is forced to be different from that since neither MGL** nor MGL* are known. Moreover, note that Quillen Theorem is actually stronger than Theorem 4.3, in the topological case, in the sense that Quillen's result cannot be deduced from (the topological version of) Theorem 4.3.

5. The motivic Quillen idempotent and Brown-Peterson spectra in $\mathcal{SH}(k)$.

We are going to apply Theorem 4.3 to the localization of the algebraic cobordism spectrum at a prime p.

Throughout this section, we fix a prime p. Let $\mathrm{MGL}_{(p)}$ be the (Bousfield) localization of MGL at the prime p. Since the localization map

$$l: \mathrm{MGL} \to \mathrm{MGL}_{(p)}$$

is a map of ring spectra, it maps the canonical orientation x_{MGL} of MGL (Remark 3.3) to an orientation $x_{(p)}$ of $\mathrm{MGL}_{(p)}$. Hence, $\mathrm{MGL}_{(p)}$ is canonically oriented by $x_{(p)}$. Let us denote by $\mathrm{F}_{x_{(p)}}$ the corresponding formal group law on $\mathrm{MGL}^*_{(p)} \doteq \bigoplus_i \mathrm{MGL}^{2i,i}_{(p)}$ (Proposition 4.2). Since $\mathrm{MGL}^*_{(p)}$ is a commutative (Proposition 3.4) $\mathbb{Z}_{(p)}$-algebra, by Cartier theorem, there exists a canonical strict isomorphism of formal group laws on $\mathrm{MGL}^*_{(p)}$

$$\varepsilon: F_{x^0_{(p)}} \xrightarrow{\sim} F_{x_{(p)}}$$

with $F_{x^0_{(p)}}$ a p-typical formal group law ([3], 16.4.14). Therefore, by Theorem 4.3 applied to the oriented ring spectrum $(\mathrm{MGL}_{(p)}, x_{(p)})$, to such an ε is uniquely associated a ring spectra map

$$\mathrm{e}: \mathrm{MGL} \to \mathrm{MGL}_{(p)}. \tag{9}$$

If C denotes the cofiber of the localization map $l: \mathrm{MGL} \to \mathrm{MGL}_{(p)}$, clearly one has

$$\mathrm{MGL}^{**}_{(p)}(C) = 0 \tag{10}$$

and therefore the natural map

$$l^*: \mathrm{MGL}^{**}_{(p)}(\mathrm{MGL}_{(p)}) \to \mathrm{MGL}^{**}_{(p)}(\mathrm{MGL})$$

is an isomorphism.

PROPOSITION 5.1. *The isomorphism l^* establishes, in bidegree* $(0, 0)$, *a bijection between ring spectra maps* $\mathrm{MGL}_{(p)} \to \mathrm{MGL}_{(p)}$ *and ring spectra maps* $\mathrm{MGL} \to \mathrm{MGL}_{(p)}$.

PROOF. One direction is clear since l is a map of ring spectra. On the other hand, let us consider a map of ring spectra $\alpha : \mathrm{MGL} \to \mathrm{MGL}_{(p)}$ and let $\beta : \mathrm{MGL}_{(p)} \to \mathrm{MGL}_{(p)}$ the unique map such that $l^*(\beta) = \alpha$. We must prove that β is a map of ring spectra.

In the following diagram

$$\begin{array}{ccc}
\mathrm{MGL} \wedge \mathrm{MGL} & \xrightarrow{\mu} & \mathrm{MGL} \\
{\scriptstyle l \wedge l}\downarrow & & \downarrow{\scriptstyle l} \\
\mathrm{MGL}_{(p)} \wedge \mathrm{MGL}_{(p)} & \xrightarrow{\mu_{(p)}} & \mathrm{MGL}_{(p)} \\
{\scriptstyle \beta \wedge \beta}\downarrow & & \downarrow{\scriptstyle \beta} \\
\mathrm{MGL}_{(p)} \wedge \mathrm{MGL}_{(p)} & \xrightarrow[\mu_{(p)}]{} & \mathrm{MGL}_{(p)}
\end{array}$$

(where the horizontal arrows are product maps) the upper square is commutative and the outer square too since α is a map of ring spectra. If d denotes the difference

$$\beta \circ \mu_{(p)} - \mu_{(p)} \circ (\beta \wedge \beta),$$

we know that $d \circ (l \wedge l)$ is zero. But since

$$(\mathrm{MGL} \wedge \mathrm{MGL})_{(p)} \simeq \mathrm{MGL}_{(p)} \wedge \mathrm{MGL}_{(p)}$$

and

$$\mathrm{MGL}^{**}_{(p)}(\mathrm{MGL} \wedge \mathrm{MGL}) \to \mathrm{MGL}^{**}_{(p)}((\mathrm{MGL} \wedge \mathrm{MGL})_{(p)})$$

is an isomorphism by the same argument used in (10), we conclude that also d is zero i.e. that β is indeed a map of ring spectra. ■

COROLLARY 5.2. *The unique map* $e_{(p)} : \mathrm{MGL}_{(p)} \to \mathrm{MGL}_{(p)}$ *such that* $l^*(e_{(p)}) = \mathrm{e}$, *is a map of ring spectra.*

Since the canonical procedure to make a given formal group law p-typical, is trivial when applied to a formal group law which is already p-typical ([3], 31.1.9, p. 429), the ring map $\mathrm{e}_{(p)}$ is idempotent. We call $\mathrm{e}_{(p)}$ the *motivic Quillen idempotent.*

DEFINITION 5.3. *The* Brown-Peterson spectrum *in* $\mathcal{SH}(k)$ *associated to the prime* p *is the spectrum* BP *colimit of the diagram of ring spectra* *and ring spectra maps in* $\mathcal{SH}(k)$

$$\dots \to \mathrm{MGL}_{(p)} \xrightarrow{e_{(p)}} \mathrm{MGL}_{(p)} \xrightarrow{e_{(p)}} \mathrm{MGL}_{(p)} \xrightarrow{e_{(p)}} \mathrm{MGL}_{(p)} \to \dots$$

Therefore, $\mathbb{BP}$ is a commutative ring spectrum and there are canonical maps of ring spectra $u : \mathbb{BP} \to \mathrm{MGL}_{(p)}$ and $\tilde{e}: \mathrm{MGL}_{(p)} \to \mathbb{BP}$ such that $\tilde{e} \circ u = \mathrm{id}_{\mathrm{BP}}$ and $u \circ \tilde{e} = e_{(p)}$. In particular, $\mathbb{BP}$ is a direct summand of $\mathrm{MGL}_{(p)}$.

REMARK 5.4. Note that we were forced (unlike in the topological case) to prove the existence of the Quillen idempotent without resorting to Quillen theorem which is not known to hold in the algebraic case. However, the construction of the idempotent given above works in the topological case too, hence yielding a different construction from the usual one that uses the isomorphism between MU* and the Lazard ring.

REFERENCES

[1] M. COLE - J. P. C. GREENLESS - I. KRIZ, *The universality of equivariant complex bordism*, preprint, 1999, http://hopf.math.purdue.edu/pub/Cole-Greenlees-Kriz/AThom

[2] W. FULTON, *Intersection theory*, Second edition, Springer Verlag, New York, 1998.

[3] M. HAZEWINKEL, *Formal groups and applications*, Academic Press, New York, 1978.

[4] F. MOREL, *Basic properties of the stable homotopy categories of smooth schemes*, Preprint, http://www.math.jussieu.fr/ morel/Stable.ps

[5] F. MOREL - V. VOEVODSKY, *$\mathbb{A}^1$-homotopy of schemes*, to appear in Publ. Math. IHES.

[6] D. QUILLEN, *On the formal group laws of unoriented and complex cobordism theory*, Bull. A.M.S., **75** (1969), pp. 1293-1298.

[7] D. QUILLEN, *Elementary proofs of some results of cobordism theory using Steenrod operations*, Adv. in Math., **7** (1971), pp. 29-56.

[8] Y. B. RUDYAK, *On Thom spectra, orientability and cobordism*, Springer Monographs in Mathematics, Springer Verlag, New York, 1998.

[9] R. STONG, *Notes on cobordism theory*, Princeton University Press, Princeton, 1968.

[10] R. M. SWITZER, *Algebraic topology-Homotopy and Homology,* Springer Verlag, New York, 1975.

[11] B. TOTARO, *Torsion algebraic cycles and complex cobordism*, Journal A.M.S., **10** (1992), pp. 467-493.

[12] V. VOEVODSKY, *$\mathbb{A}^1$-homotopy theory*, Doc. Math. J. DMV, Extra volume ICM 1998, I, 417-442.

[13] V. VOEVODSKY, *Lecture 3: Algebraic cobordism*, Workshop on homotopy theory of algebraic varieties, MSRI, Berkeley, May 1998, http://www.msri.org/publications/ln/msri/ 1998/homotopy/voevodsky/3.
[14] V. VOEVODSKY, *The Milnor conjecture*, preprint MPI, 1996.
[15] C. WEIBEL, *Voevodsky's Seattle Lectures: K-theory and motivic cohomology*, Proc. symp. Pure Math., **67** (1999).
[16] W. S. WILSON, *The W-spetrum for Brown-Peterson cohomology, part II*, Am. J. Math., **97** (1975), pp. 101-123.

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