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Stability, duality, 2-generated ideals and a canonical decomposition of modules

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Stability, Duality, 2-Generated Ideals and a Canonical Decomposition of Modules.

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Abstract - We study the relationship between stable domains, divisorial domains, the 2-generator property, Warfield duality and a canonical decomposition of modules. We classify Warfield domains (integral domains for which every overring is a reflexive domain), totally divisorial domains (integral domains for which every overring is a divisorial domain) and the class of integral domains that possess a canonical decomposition of torsion-free modules.

1. Introduction.

In his «Ubiquity» paper of 1963, Bass shows how a particular concept of algebraic geometry can be expressed in a surprising number of ways. With the unifying notion of a Gorenstein ring, he interprets this concept through algebraic geometry (locally free sheaves of differentials on a variety), homological algebra (finite injective dimension), module theory (decompositions and reflexivity) and ideal theory (divisoriality and the 2-generator property). The strength of his results depends on Noetherian hypotheses, and once the Noetherian assumption is relaxed (as it was by Matlis in the sixties and seventies), the «Gorenstein» properties begin to diverge.

For example, a Noetherian domain has injective dimension one if and only if every non-zero ideal is divisorial, but an integrally closed domain

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of injective dimension one can have many non-divisorial ideals. Examples are easy to find: take any almost maximal valuation domain R with non-principal maximal ideal. Then R has injective dimension one, but the maximal ideal of R is not divisorial. Conversely, if R is a valuation domain with divisorial maximal ideal, then every non-zero ideal of R is divisorial, but R need not have injective dimension one [8]. Thus the notions of divisoriality and injective dimension one are independent for non-Noetherian rings.

Over the past forty years, an interesting amount of commutative algebra and module theory has been devoted to the delineation of the different properties that appear in Bass's paper. Much of this work has been from a structural point of view, in the sense that the properties are studied in their own right and, at least as a starting point, in very general contexts (i. e. arbitrary integral domains, commutative rings, or, in another direction, non-commutative orders). Typically, one proceeds from general principles until Noetherian or integrally closed hypotheses are needed to obtain nice characterizations. The difficult classification, for example, of D-rings achieved by Matlis and others in the early seventies describes «only» integrally closed and Noetherian D-rings (see [11]). The general case remains open.

The rings studied in this article, all stemming in one way or another from properties that can be unified under the concept of a Gorenstein ring in the Noetherian case, are peculiar in that although they need not be Noetherian or integrally closed, they can be completely described over integral domains. One need not restrict to integrally closed or Noetherian rings to obtain the classification, although, in a sense, it is the integrally closed and Noetherian cases that determine the classification. In this one respect, the classification of the rings studied in this paper resembles Vasconcelos' classification of rings of global dimension two. A Warfield domain, for example, is either Noetherian, integrally closed or a kind of «umbrella ring» of Noetherian and integrally closed Warfield domains.

The defining criterion of a Warfield domain is that its torsion-free finite rank modules possess «Warfield duality.» This duality, first introduced for abelian groups in 1968 by R. B. Warfield, Jr., is a generalization of classical vector space duality to torsion-free modules where the «field of coefficients» is no longer a field but a submodule of the quotient field of an integral domain. Let R be an integral domain with quotient field K. If X is an R-submodule of K, then R is $X-$*reflexive* provided the canonical homomorphism

$$M \to Hom_R(Hom_R(M, X))$$

is an isomorphism whenever M is an $End_R(X)$ – submodule of a finite direct sum of copies of X. This generalizes the traditional notion of reflexivity: An integral domain R is R-reflexive if and only if it is reflexive in the sense of [11]. If R is X-reflexive for all submodules X of K, then R is a *Warfield domain.*

Bazzoni and Salce give an extensive treatment of Warfield domains in [2]. Their article is the culmination of a recent series of papers that explore Warfield's duality for torsion-free modules. They show in particular that every ideal of an integral domain is *2-generated,* i. e. can be generated by two elements, if and only if R is a Noetherian Warfield domain. Thus a Noetherian domain R is Warfield if and only if every integral overring of R is Gorenstein.

Bazzoni and Salce also give a complete classification of integrally closed Warfield domains (see Section 3), and they show that there exist non-Noetherian, non-integrally closed Warfield domains. Their approach is quite general in that they derive their Noetherian and integrally closed results as corollaries of general principles. They show that in order for a domain to be Warfield its rank one modules must possess a generalized form of divisoriality and have injective dimension one over their endomorphism rings. These two aspects of Warfield duality are, of course, intimately linked in the Noetherian context, and Matlis, in 1968, showed that reflexivity is decided for an arbitrary domain by the divisoriality of its ideals and the injective dimension of the ring (see [11]). In this way, Bazzoni and Salce use divisoriality and injective properties of rank one modules to link Warfield duality to the extensive literature on reflexive domains. They show that R is a Warfield domain if and only if every overring of R is reflexive [2, Theorem 6.6]. It is this characterization that we make much use of in the present article.

In addition to Warfield's duality paper and Matlis' study of reflexivity, a third paper of 1968, Heinzer's study of divisoriality [8], is crucial to the approach of Bazzoni and Salce. They generalize a number of Heinzer's results on *divisorial domains*, domains for which every non-zero ideal is divisorial, to a form of divisoriality for rank one modules. They show that Warfield duality holds for rank one modules if and only if R is *totally divisorial,* i.e. every overring of R is a divisorial domain, and they use the notion of total divisoriality to unify their approach to Warfield domains.

In the present article we study Warfield (and totally divisorial) domains with a notion that is more general than total divisoriality, that of stability. A non-zero ideal of an integral domain R is *stable* if it is projective over its ring of endomorphisms. If every non-zero ideal of R is stable, then R is said to be *stable*. Stable rings and ideals were first systematically studied by Sally and Vasconcelos in [21] and Lipman in [10], although the concept of a stable ideal predates both these articles and can be found in Bass's «Ubiquity» paper [1]. In fact, Bass shows that if every ideal of a Macaulay ring is 2-generated (i. e. can be generated by two elements), then R is stable. The converse fails, and consequently, even for Noetherian integral domains, stability is weaker than the property that every ideal be 2-generated [21, Example 5.4].

Bazzoni and Salce leave open the problem of the classification of arbitrary totally divisorial and Warfield domains. Using techniques from the study of stable domains, we address this problem in Section 4 and show there is a sense in which Bazzoni and Salce's characterization of the Noetherian and integrally closed cases is complete. The classification of Warfield domains reduces to the quasilocal case, and we prove that a quasilocal Warfield domain is Noetherian, integrally closed or a pullback of Noetherian and integrally closed Warfield domains.

In Section 3 we frame the concept of divisoriality in terms of stablity. Every totally divisorial domain is stable but not vice versa. Thus the class of totally divisorial domains is properly situated between the classes of stable and Warfield domains. We use this observation to bridge the two classes of integral domains.

Since the notion of stability is central to the present approach, we use Section 2 to review some results on stable domains from [14, 18] that are needed in subsequent sections. Section 3 treats totally divisorial domains, while Section 4 focuses on Warfield domains. For both classes, we prove «ascent» and «descent» theorems. By an «ascent theorem» we mean an assertion made about every overring of an integral domain R which *a priori* depends only on ideal-theoretic properties of R (rather than the overrings of R). Similarly, by «descent» we mean an assertion about the integral closure $\overline{R}$ of an integral domain R which, along with hypotheses about the lattice of submodules of $\overline{R}/R$, implies a similar assertion about R.

A number of our results are motivated by Rush's treatement of stability and the 2-generator property for Noetherian rings [20]. Several of our results are generalizations of Rush's Noetherian theorems to arbitrary integral domains, and of particular relevance is Rush's recent solution to a problem that originates with Bass. In the «Ubiquity» paper, Bass shows that if R is a Noetherian ring and every torsionless R-module is isomorphic to a direct sum of ideals, then every ideal of R can be generated by two elements. Although Bass established the converse under the assumption that the integral closure of R be module-finite, the case in which the integral closure is not necessarily module-finite remained open until 1991, when Rush proved the converse in full generality. Extending structure results of Borevic and Fadeev [3] and Levy and R. Wiegand [9], he showed in particular that every ideal of a Noethertan domain is 2-generated if and only if every torsionless R-module G is isomorphic to a direct sum of the form $S_1 \oplus S_2 \oplus \ldots \oplus S_{n-1} \oplus I$, where $R \subseteq S_1 \subseteq \ldots \subseteq S_{n-1} \subseteq S_n$ are overrings of R and I is an invertible ideal of S_n. In Section 5, we make use of the results of Section 4 to show that the class of integral domains whose torsionless modules possess this decomposition property is precisely the class of Warfield domains.

Notation and terminology. Let R be an integral domain with quotient field F. Then $\overline{R}$ will denote the integral closure of R in F. If X and Y are R-submodules of F, then $[Y : X]$ will denote the R-module $\{q \in F : qX \subseteq Y\}$. Note that $[X : X]$ can be identified with $End_R(X)$ and hence we often write $E(X)$ for $[X : X]$.

2. Stable domains.

This section reviews properties of stable domains developed in [18] and [14]. Several of these results and constructions are well-known for order-divisorial domains (see [2]). However, to frame a number of our results we need the validity of these constructions in the greater generality of stable domains.

One may reduce consideration of stable domains to the quasilocal case, namely, an integral domain R is stable if and only if every non-zero ideal of R is contained in at most finitely many maximal ideals of R and R_M is stable for all maximal ideals M of R [18, Theorem 3.3].

A stable ideal of a quasilocal domain is principal over its ring of endomorphisms [18, Lemma 3.1]. Using this fact, it is not hard to check that if R is a quasilocal domain and I is an ideal of R, then I is stable if and only if $I^2 = Ii$ for some $i \in I$.

Let R be a quasilocal stable domain with maximal ideal M. Then there exists $m \in M$ such that $M = E(M)m$. It is shown in [18, Theorem 4.8] that one may express $\overline{R}$ as the union of a countable chain of integral extensions of R. We recall the properties of this construction. Define $R_0 = R$ and $R_1 = E(M)$. If $R_1 \neq R$, then $R_1 = [R : M]$. Inductively define, for $i > 1$, $R_i = R_{i-1}$ if R_{i-1} is not quasilocal, and $R_i = E(M_{i-1})$ if R_{i-1} is quasilocal with maximal ideal M_{i-1}. For all $i > 0$, there exists $k > 0$ such that $M_i^k \subseteq M$ [18, Corollary 4.4].

For $i > 0$, if $R_{i-1} \neq R_i$, then Corollary 4.3 of [18] implies that for all $i \geqslant 1$, R_i is an integral extension of R, R_i is a divisorial fractional ideal of R and R_i must satisfy one of the following conditions:

(A) R_i is a quasilocal ring with stable maximal ideal M_i such that $M_i^2 \subseteq M_{i-1}$, and if $R_i \neq R_{i+1}$, then $M_i = R_{i+1}M$ and the residue field of R_i is R_{i-1}/M_{i-1}; if $R_i = R_{i+1}$, then R_i is a minimal overring of R_{i-1} and the residue field of R_i is either R/M or a quadratic extension field of R/M.

(B) R_i has two or three maximal ideals, each of which is a principal ideal of R_i, $R_i = R_j$ for all $j \geqslant i$, and R_i is a finitely generated R-module.

We make frequent use of the following lemmas.

Lemma 2.1. *Let R be a quasilocal stable domain with stable maximal ideal $M = R_1 m$. The following statements hold for R.*

(i) $\overline{R} = \bigcup_i R_i$.

(ii) *$\overline{R}$ has at most two maximal ideals.*

(iii) *If $\overline{R}$ is not quasilocal, then $\overline{R} = R_i$ for some $i \geqslant 0$ such that $R_{i-1} \neq R_i$, $\overline{R}$ is a finitely generated R-module and the product of maximal ideals of $\overline{R}$, i.e. the Jacobson radical of $\overline{R}$, is $\overline{R}m$.*

(iv) *$\overline{R}$ is a strongly discrete Prüfer domain (i.e. $\overline{R}$ is a Prüfer domain for which every non-zero prime ideal P satisfies $P \neq P^2$).*

(v) *Every R-submodule of $\overline{R}$ containing R is a ring.*

Proof. Statement (i) is given by [18, Theorem 4.8]; statement (ii) by [14, Corollary 2.4]; statement (v) by [20, Proposition 2.1]. Only (iii) and (iv) need justification. Statement (iii) is immediate from the properties of the construction and [18, Corollary 4.3]. Statement (iv) follows from [16, Theorem 4.6] and the fact that every overring of a stable domain is stable [18, Theorem 5.1]. ■

LEMMA 2.2. *Let R be a quasilocal stable domain.*

(i) *If P is a non-maximal non-zero prime ideal of R, then R_P is a strongly discrete valuation domain, $E(P) = R_P$, and for all $r \in R \backslash P$, $P \subseteq Rr$.*

(ii) *The prime ideals of R are linearly ordered by inclusion and satisfy the ascending chain condition.*

(iii) *The set of non-maximal prime ideals of $\overline{R}$ coincides precisely with the set of non-maximal prime of ideals of R.*

PROOF. The claims are proved in [18, Theorem 4.11], with the exception of the last assertion of statement (i). However, this claims follows at once, for since $P = PR_P$, we have $P = Pr \subseteq Rr$. ■

Any quasilocal stable domain of Krull dimension greater than one gives rise to a Cartesian square. The use of this square is fundamental to the approach taken in this paper. It is particularly useful when one selects P to be the largest non-maximal prime ideal of R, since in this case R/P is one-dimensional, and when the maximal ideal of R is finitely generated (as it will be for totally divisorial and Warfield domains), the ring R/P will be Noetherian and stable.

COROLLARY 2.3. *Let R be a quasilocal stable domain of Krull dimension greater than one. Let P be a non-zero non-maximal prime ideal of R. Then $P = PR_P$ and there is a commutative diagram,*

$$\begin{array}{ccc} R & \longrightarrow & R/P \\ \downarrow & & \downarrow \\ R_P & \longrightarrow & R_P/P\,. \end{array}$$

where R/P is a stable domain, R_P is a strongly discrete valuation domain and R_P/P is the quotient field of R/P.

PROOF. That $P = PR_P$, R_P is a valuation domain and R_P/P is the quotient field of R/P are all immediate from Lemma 2.2. To see that R/P is stable, let I be an ideal of R properly containing P. Since R is stable and quasilocal, $I^2 = Ii$ for some $i \in I$. Necessarily, $i \notin P$.

Therefore, $i^{-1}P = P$ and $P \subseteq Ii = I^2$. Thus $I^2/P = (I/P)(Ri + P)/P$ and I/P is stable, proving the claim. ■

A more general statement involving pullbacks of stable domains is proved in [14].

COROLLARY 2.4. *Let R be a quasilocal stable domain. If P is a non-maximal prime ideal of R, then $R_i/P = (R/P)_i$ for all $i > 0$.*

PROOF. Since by Lemma 2.2, P is a prime ideal of R_i for all $i > 0$, it suffices to prove the claim for $i = 1$. If $R = R_1$, the claim is trivial, so suppose $R \neq R_1$. By Corollary 2.3, R_P/P is the quotient field of R/P. Let $q + P \in R_P/P$ such that $(q + P)(M/P) \subseteq M/P$. Then $qM + P \subseteq M$ and $q \in [M : M] = R_1$. Thus $(R/P)_1 \subseteq R_1/P$. The reverse inclusion is clear, so the claim is proved. ■

3. Totally divisorial domains.

As noted in the introduction, the integrally closed and Noetherian cases of totally divsorial domains are decided in [2]. For ease of reference, we recall these characterizations here. Recall that an integral domain R is *h-local* if each non-zero ideal of R is contained in at most finitely many maximal ideals of R and each non-zero prime ideal of R is contained in a unique maximal ideal of R.

THEOREM 3.1 (Bazzoni and Salce). *Let R be an integral domain.*

(i) *R is a Noetherian totally divisorial domain if and only if every ideal of R is 2-generated.*

(ii) *R is an integrally closed totally divisorial domain if and only if R is an h-local strongly discrete Prüfer domain.*

PROOF. This theorem is not explicitly stated by Bazzoni and Salce in [2]. However, it follows easily from their results. Noetherian divisorial domains are one-dimensional and hence every overring of a Noetherian divisorial domain is Noetherian [8]. Noetherian divisorial domains are reflexive [11, Theorem 40] and Noetherian totally divisorial domains are Warfield domains, which implies every ideal is 2-generated [2, Theorem 7.3]. This confirms statement (i). For statement (ii), apply [2, Propositions 5.4 and 7.6]. ■

If R is a divisorial domain, then R is h-local and R_M is a divisorial domain for all $M \in Max(R)$ (see [8] or [11]). The converse is also true [2, Proposition 5.4].

Integrally closed totally divisorial domains admit an interesting description: They are precisely the integral domains for which every ideal can be written as a product of prime ideals and invertible ideals [15].

By [2, Lemma 5.2], totally divisorial domains are stable and hence we have access to the results and constructions of Section 2. Recall that an integral domain R is coherent if and only if the intersection of any two finitely generated ideals of R is finitely generated. Not all stable domains are coherent [14, Proposition 5.2]; totally divisorial domains, however, are coherent, and consequently, the maximal ideals of a totally divisorial domain are finitely generated [14, Lemma 4.1].

LEMMA 3.2. *Totally divisorial domains are stable and coherent.*

PROOF. As noted above, totally divisorial domains are stable. A stable domain is coherent if and only if the maximal ideals of R are finitely generated [14]. Assume first R is a quasilocal totally divisorial domain with maximal ideal M. Then by [2, page 856], R_1 is a finitely generated R-module. Since R is stable and M is a principal ideal of R_1, it follows that M is finitely generated and R is coherent. If R is an arbitrary totally divisorial domain, then R is locally coherent. Moreover, every non-zero ideal of R is contained in at most finitely many maximal ideals of R, so R is coherent [11, Theorem 26]. ■

LEMMA 3.3 [2, Proposition 7.1]. *A totally divisorial domain R is one-dimensional if and only if R is Noetherian.*

LEMMA 3.4. *The maximal ideal of a quasilocal stable domain R is 2-generated if and only if every proper R-submodule of $\overline{R}$ containing R is equal to R_i for some $i > 0$.*

PROOF. Let R be a quasilocal stable domain with 2-generated maximal ideal M. By Lemma 2.2, there is a largest non-maximal prime ideal P of R. If $P = 0$, then R is a one-dimensional stable Noetherian domain of multiplicity 2. Hence, by [22, Proposition 2.1], every ideal of R is 2-generated. Thus R is totally divisorial and the claim follows from [2, p. 856]. Otherwise, if $P \neq 0$, then by Corollary 2.3, R/P is a stable domain with quotient field R_P/P. Moreover, by Lemma 2.2, $R/P \subsetneq \overline{R}/P \subsetneq R_P/P$ and $\overline{R}/P$ is the integral closure of R/P in R_P/P. Now R/P is necessarily Noetherian, since its maximal ideal is finitely generated, and so another application of [22, Proposition 2.1] shows that since M/P is 2-generated and R/P is stable, every ideal of R/P is 2-generated. Thus, by [2, p. 856], every proper R/P-submodule of $\overline{R}/P$ containing R/P is of the form $(R/P)_i$ for some $i \geq 0$. The claim now follows from Corollary 2.4. To prove the converse, note that since the only R-submodules of R_1 that contain R are R and R_1, it follows that $R_1 = R + Rx$, for every $x \in R_1 \setminus R$. In particular, $M \cong R_1$ is a two-generated ideal of R. ∎

An integral domain R has the *2-generator property* provided every finitely generated ideal of R can be generated by 2 elements.

LEMMA 3.5. *Let R be a stable domain. The following are equivalent for R.*

(1) *The maximal ideals of R are 2-generated.*

(2) *R has the 2-generator property.*

(3) *$\overline{R}/R$ is a distributive R-module.*

(4) *Every finitely generated ideal of R is divisorial.*

PROOF. An R-module is distributive if and only if it is locally a uniserial module. Also, since R is stable, every ideal of R is contained in at most finitely many maximal ideals of R. Thus statements (1) and (2) hold if and only if they hold locally [11, Theorem 26]. For these reasons, we prove the equivalence of statements (1)-(3) under the assumption that R is quasilocal.

(1) $\Rightarrow$ (2) (R quasilocal) By Lemma 2.2, there exists a largest non-maximal prime ideal P contained in M and by Corollary 2.3, R/P is a stable domain with 2-generated maximal ideal; hence R/P is a Noetherian domain and every ideal of R/P is 2-generated [21, Lemma 3.2]. Let $I = Ra_1 + Ra_2 + \ldots + Ra_n$ be a proper finitely generated ideal of R. Then $E(I)$ is integral over R; hence, by Lemma 3.4, $E(I) = R_i$ for some i with $R_{i-1} \subset R_i$. Thus $I \cong R_i \cong M_{i-1}$. Also, there exists k such that $M_{i-1}^k \subseteq M$ (see Section 2). Clearly, $M_{i-1}^k \nsubseteq P$, since $M^k \subseteq M_{i-1}^k$. Hence, there exists x in the quotient field of R such that $Ix \nsubseteq P$ and $Ix \subseteq M$. The-

refore, $R_P xa_1 \cap R_P xa_2 \cap \ldots \cap R_P xa_n \not\subseteq PR_P = P$. It follows that $xa_i \notin P$ for all $i = 1, 2, \ldots, n$, proving the first claim. Since R/P is a 2-generated ring, xI/P is a 2-generated ideal, and by Lemma 2.2(i), $xI \cong I$ is 2-generated.

(2) $\Rightarrow$ (3) (R quasilocal) Assume (2). If $x, y \in R_1$ and neither x nor y is an element of R, then $(R + Rx + Ry)/M$ is an R/M-vector space. Since R has the 2-generator property, $(R + Rx + Ry)/M$ is a 2-dimensional R/M-vector space. Thus $R + Rx = R + Ry$. It follows that $R_1 = R + Rx$ for all $x \in R_1 \setminus R$. Since $M \cong R_1$, M is a 2-generated maximal ideal of R. Statement (3) now follows from Lemma 3.4.

(3) $\Rightarrow$ (1) (R quasilocal) By (3) and Lemma 2.1 (v), the R/M-vector space R_1/R must have dimension one. Hence R_1, and consequently M, are 2-generated.

(1) $\Rightarrow$ (4) Suppose first that (1) $\Rightarrow$ (4) holds for each localization of R at a maximal ideal of R. Let I be a finitely generated ideal of R. Since R is stable and the maximal ideal of R is finitely generated, R is coherent [14], and $[R : I]$ is a finitely generated fractional ideal of R. Hence, if M is a maximal ideal of R, $[R : [R : I]]\, R_M = [R_M : [R_M : IR_M]] = IR_M$ and it follows that I is a divisorial ideal of R. Therefore, it is sufficient to prove (1) $\Rightarrow$ (4) for quasilocal domains. Let R be quasilocal. By Lemma 3.4, if I is a non-zero finitely generated ideal of R, then $E(I) \subseteq \overline{R}$ and $E(I) = R_i$ for some $i > 0$. Thus $E(I)$ and hence the principal $E(I)$-ideal I is a divisorial ideal of R.

(4) $\Rightarrow$ (1) Assume (4). Let M be a maximal ideal of R. If $E(M) = R$, then since M is stable, M is invertible. Since every non-zero ideal of R is contained in at most finitely many maximal ideals of R, M is 2-generated. So we suppose $E(M) \neq R$. Let $x \in E(M) \setminus R$, and define $S = R + Rx$. Since R is stable, S is a ring (Lemma 2.1). Also, since $E(M) \neq R$, M is a divisorial ideal of R. For otherwise, if M is not divisorial, then necessarily $(R : M) = R$ and hence $E(M) = R$. Thus $M = [R : [R : M]] = [R : E(M)] \subseteq [R : S] \subseteq R$. Since $S \neq R$, $M = [R : S]$. By (4), S is divisorial, so $S = [R : M] = E(M)$. Thus $E(M)$ is a 2-generated fractional ideal of R. Hence $MR_M \cong E(M)R_M$ is a 2-generated ideal of R_M and, since every non-zero ideal of R is contained in at most finitely many maximal ideals of R (because R is stable), M is a 2-generated ideal of R (see [11]), proving the claim. ■

Sally and Vasconcelos prove that if R is a one-dimensional Macaulay ring with module-finite integral closure, then R is stable if and only if every ideal of R is 2-generated [22, Proposition 2.1]. Thus, Proposition 3.8 (which ultimately depends on the result of Sally and Vasconcelos) can be considered a generalization of their theorem to arbitrary quasilocal domains. Two lemmas are needed first.

LEMMA 3.6. *Let R be a quasilocal stable domain. If $\overline{R} = R_i$ for some $i > 0$, then R has the 2-generator property.*

PROOF. The hypothesis that $\overline{R} = R_i$ implies the maximal ideal of R is finitely generated [14, Theorem 4.6] and that $\overline{R}$ is a finitely generated R-module [18, Corollary 4.3]. Thus if R is one-dimensional, R is Noetherian and the claim follows from the result of Sally and Vasconcelos. So suppose R is not one-dimensional. By Lemma 2.2, there exists a largest non-maximal prime ideal P of R, and by Lemma 2.2 and Corollary 2.4, R/P is a Noetherian stable domain with finitely generated integral closure $\overline{R}/P$. By the theorem of Sally and Vasconcelos, every ideal of R/P is 2-generated. By Lemma 2.2(i), it follows that the maximal ideal of R is 2-generated. Lemma 3.5 completes the claim. ■

An integral domain R is a *Bass ring* if R has module-finite integral closure and every ideal of R is 2-generated. In particular, Bass rings are totally divisorial.

LEMMA 3.7. *Let R be a quasilocal stable domain. If $\overline{R}$ is an h-local domain with more than one maximal ideal, then R is a Bass ring.*

PROOF. If P is a non-zero non-maximal prime ideal of R, then by Lemma 2.2, P is a prime ideal of $\overline{R}$ contained in the Jacobson radical of $\overline{R}$, and hence $\overline{R}$ is not h-local, contrary to assumption. Thus R is one-dimensional and by Lemma 2.2, $\overline{R}$ is a finitely generated R-module. This latter fact, along with the stability of R, imply R is a Bass ring [14, Proposition 4.5]. ■

PROPOSITION 3.8. *Let R be an integral domain. If $\overline{R}$ is a finitely generated R-module, then R is a totally divisorial domain if and only if R is a stable domain and $\overline{R}$ is h-local.*

PROOF. Suppose R is a stable domain and $\overline{R}$ is h-local. We check first that R is h-local. Because R is stable, every non-zero ideal of R is contained in at most finitely many maximal ideals of R ([18, Theorem 3.3]). Suppose there exists a non-zero non-maximal prime ideal P of R.

Then by Lemma 2.2, PR_P is a prime ideal of $\overline{R}_M$ for all maximal ideals M of R that contain P. If more than one maximal ideal, say M and N, of R contains P, define $A = R_{R\setminus(M\cup N)}$. Then AP is a prime ideal of A that is contained in the maximal ideals AM and AN. Every overring of an h-local Prüfer domain is h-local [2, Lemma 7.4], so $\overline{A}$ is an h-local domain. Overrings of stable domains are stable [18, Theorem 5.1], so A is a stable domain. By Lemma 2.2, $A_{AM}P$ is a prime ideal of $\overline{A}_{AM}$ and $A_{AN}P$ is a prime ideal of $\overline{A}_{AN}$. Thus AP is a prime ideal of $\overline{A}$ that is contained in at least two maximal ideals of $\overline{A}$, a contradiction. Hence R is h-local. In fact, our argument shows that since every overring of $\overline{R}$ is h-local and every overring of R is stable, we may conclude every overring of R is h-local. So to show R is totally divisorial, it suffices to prove the claim locally. In particular, we assume R is a quasilocal stable domain and $\overline{R}$ is h-local, and we show that R is totally divisorial.

If R is one-dimensional, then, since $\overline{R}$ is a finitely generated R-module, R is a Bass ring [14, Proposition 4.5]. Hence R is totally divisorial. Suppose R is not one-dimensional. By Lemmas 3.5 and 3.6, the maximal ideal of R is 2-generated. We verify first that R is a divisorial domain. Since $\overline{R}$ is a finitely generated R-module and $\overline{R} = \bigcup_{i>0} R_i$, it follows that $\overline{R} = R_i$ for some i. By their construction, the R_i's are divisorial fractional ideals of R. If L is a non-maximal prime ideal of R, then, by Lemma 2.2, $[R : L] = R_L$ and so R_L is a divisorial ideal of R. Since $\overline{R}$ is h-local, it must be the case that $\overline{R}$ is a quasilocal domain. For the non-zero non-maximal prime ideals of R (by assumption, there is at least one) coincide with the non-maximal prime ideals of $\overline{R}$. This shows that every overring of R of the form R_i for some $i \geqslant 0$ or of the form R_L for some non-maximal prime ideal L of R, is a divisorial fractional ideal of R. Observe, however, that every overring S of R properly contained in the quotient field of R conforms to one of these two possiblities. For, if $S \neq R_i$ for some i, then by Lemma 3.4, $S \not\subseteq \overline{R}$; hence $\overline{R}S = R_L$ for some non-maximal prime ideal L of R since $\overline{R}$ is a valuation domain and, as already noted, the non-maximal prime ideals of $\overline{R}$ coincide with the non-maximal prime ideals of R. By the properties of the construction (Section 2), there exists $k > 0$ such that $\overline{R}M^k \subseteq \subseteq M_i^k \subseteq R$. Thus $S \subseteq R_L = R_L M_i^k \subseteq \overline{R}SM_i^k \subseteq S$ and it follows that $S = R_L$. This shows that every fractional overring of R is a divisorial fractional ideal of R. Since R is stable, every fractional ideal of R is isomorphic to a fractional overring of R, so R is a divisorial domain.

To see next that R_i is a divisorial domain for all $i > 0$, fix $i > 0$ and let I be an ideal of R_i. Assume without loss of generality that $R_i \neq R_{i-1}$. Then since M_{i-1} is a principal ideal of R_i, I is an ideal of R_i and $R_i = [R_i : M_{i-1}^i]$. Thus

$$\begin{aligned}[R_i:[R_i: I]] &= [R : M_{i-1}^i[R : IM_{i-1}^i]] \\ &= [R :[R: R_i I]] \\ &= [R :[R : I]] \\ &= I\,.\end{aligned}$$

Thus R_i is a divisorial domain for all $i \geqslant 0$. In particular, $\overline{R}$ is a divisorial domain and since $\overline{R}$ is quasilocal, $\overline{R}$ must be a strongly discrete valuation domain. Since each R_i is a divisorial domain and $\overline{R}$ is totally divisorial domain, every overring is a divisorial domain. This completes the proof that R is a totally divisorial domain. The converse is clear. ∎

LEMMA 3.9. *Let R be a quasilocal stable divisorial domain. If $\overline{R} \neq R_i$ for all $i \geqslant 0$, then R is a local one-dimensional domain for which every ideal is 2-generated.*

PROOF. Assume $R_i \neq R_{i+1}$ for all $i > 0$. Then by Lemma 2.1, $\overline{R}$ is quasilocal. Suppose there exists a non-zero non-maximal prime ideal P of R. By Lemma 2.2, P is a prime ideal of $\overline{R}$, and we may assume without loss of generality that P is the largest non-maximal prime ideal of R. Thus $P \subseteq [R : \overline{R}]$. Suppose $P \subset [R : \overline{R}]$. Observe $\overline{R}/P$ is a DVR with maximal ideal $\overline{R}m$. Furthermore, $P = PR_P \subsetneq \overline{R}m^k$ for all $k > 0$. Thus $[R : \overline{R}] =$ $= \overline{R}m^k$ for some $k > 0$ and consequently, $[R : \overline{R}m^k] = \overline{R}$, which implies $\overline{R}m^k \subseteq R$. Hence $\overline{R}m^k \subseteq M \subsetneq R_1 m$, and so $\overline{R}m^{k-1} \subseteq R_1$. If $k = 1$, we have a contradiction, so $k > 1$ and, since $\overline{R} \neq R_1$, we have $\overline{R}m^{k-1} \subsetneq R_2 m$ and $\overline{R}m^{k-2} \subsetneq R_2$. Continuing in this manner, we eventually arrive at the contradiction, $\overline{R} = R_k$. Thus $[R : R_P] = P = [R : \overline{R}]$, and since R is a divisorial domain, $R_P = \overline{R}$, a contradiction. Thus R is one-dimensional and, since R is divisorial, R has the 2-generator property by Lemma 3.5. By Cohen's Theorem, R must be Noetherian since it has finitely generated prime ideals. Thus every ideal of R is 2-generated. ∎

Even in the one-dimensional case, the assumption on the finite generation of $\overline{R}$ in Proposition 3.8 cannot be dropped [21, Example 5.4]. Moreover, if one trades the finite generation of $\overline{R}$ for the requirement that the maximal ideal of R be 2-generated, the modified claim is false. For there exists a local domain R for which every ideal is 2-generated and $\overline{R}$ is not a finitely generated R-module [21, p. 328]. So, if we define $S = R + XK[X]_{(X)}$, where K is quotient field of R, then by [14, Theorem 2.6], S is a two-dimensional stable domain with 2-generated maximal ideal. Also, $S_i \neq S_{i+1}$ for all i, and hence, by Lemma 3.9, S cannot be totally divisorial.

LEMMA 3.10. *Let R be a quasilocal stable domain with quotient field K. Suppose the Krull dimension of R is greater than one. Then R is a totally divisorial domain if and only if $\overline{R}$ is a finitely generated R-module and the set of overrings of R is*

$$\{R_1, R_2, \ldots, R_i\} \cup \{R_P \colon P \in Spec(R)\} \cup \{K\},$$

for some $i \geq 0$. Moreover, if R is totally divisorial, then each overring of R properly contained in K is a fractional ideal of R and $\overline{R}$ is a valuation domain.

PROOF. Suppose R is totally divisorial. If R is an integrally closed domain, then R is strongly discrete and the claim follows at once from [7, Proposition 5.3.1], so we assume $R \neq \overline{R}$. There exists a largest non-zero non-maximal prime ideal P of R. By Lemma 3.9, $\overline{R} = R_i$ for some $i > 0$, and hence by Lemma 2.1, $\overline{R}$ is a finitely generated R-module. Also, by Lemmas 3.4 and 3.6, the integral overrings of R are of the form R_j for some $j \geq 0$. Let S be an overring of R such that $S \neq R_j$ for all $j \geq 0$. For each j such that $R_j \neq R_{j+1}$, R_{j+1} is the unique minimal overring of R_j [2, p. 856]. Thus $\overline{R} \subseteq S$ and hence, since by Lemma 3.7, $\overline{R}$ is a valuation domain, Lemma 2.2 implies $S = \overline{R}_L = R_L$ for some prime ideal L of R. That each overring of R properly contained in K is a fractional ideal follows from Lemma 2.2. The converse follows from Proposition 3.8. ■

LEMMA 3.11. *Let R be a integral domain such that R_M is a totally divisorial domain for each $M \in Max(R)$. Then R is h-local if and only if $\overline{R}$ is h-local.*

PROOF. Suppose first that R is h-local. Then R, and hence $\overline{R}$, is stable, so, since every non-zero ideal of $\overline{R}$ is contained in at most finitely many maximal ideals, it is enough to check that each non-zero prime ideal of $\overline{R}$ is contained in a unique prime ideal of $\overline{R}$. Suppose P is a non-zero prime ideal of $\overline{R}$ contained in two maximal ideals N_1 and N_2 of $\overline{R}$. Note that since R is h-local and $P \cap R \subseteq (N_1 \cap R) \cap (N_2 \cap R)$ it must be that $N_1 \cap R = N_2 \cap R$. Let $M = N_1 \cap R$. Since R_M is a totally divisorial domain, $\overline{R}_M$ is h-local and by Lemma 3.7, the ring $\overline{R}_M$ is quasilocal or a Bass ring. Thus $N_1 = N_2$ and $\overline{R}$ is h-local.

Conversely, suppose $\overline{R}$ is h-local and that P is a non-zero non-maximal prime ideal of R contained in two distinct maximal ideals M_1 and M_2 of R. Define $A = R_S$, where $S = R \backslash (M_1 \cup M_2)$. Then A is a locally stable domain and since A is semi-quasilocal, A is stable. Therefore, $\overline{A}$ is a Prüfer domain. Also, A has maximal ideals AM_1 and AM_2, both containing the prime ideal AP. Let N_1 and N_2 be maximal ideals of $\overline{A}$ lying over AM_1 and AM_2, respectively. Then since overrings of h-local Prüfer domains are h-local [2, Lemma 7.4], $\overline{A}$ is h-local. By Lemma 2.2, $PA_{AM_1} = PA_P = = PA_{AM_2}$ and $PA_P = AP$ is a prime ideal of both $\overline{A}_{AM_1}$ and $\overline{A}_{AM_2}$. Thus AP is a prime ideal of $\overline{A}$ that is contained in at least two maximal ideals of $\overline{A}$, a contradiction. ■

THEOREM 3.12 (Ascent from R). *An integral domain R is totally divisorial if and only if R is a stable divisorial domain.*

PROOF. Suppose first R is a stable divisorial domain. Let M be a maximal ideal of R. If $\overline{R}_M$ is not quasilocal, then by Lemma 2.1, $\overline{R}_M = (R_M)_i$ for some $i \geq 0$. As in the proof of Proposition 3.8, the fact that R_M is divisorial implies that $(R_M)_i$ is a divisorial domain and hence h-local; so $\overline{R}_M$ is h-local and by Lemma 3.7, R_M is one-dimensional and Noetherian. By Lemma 3.5, R_M is totally divisorial. So suppose $\overline{R}_M$ is quasilocal and not one-dimensional. By Lemma 3.9, $\overline{R}_M = (R_M)_i$ for some i. Thus $\overline{R}_M$ is divisorial and h-local. By property (A) of the construction (Section 2), $\overline{R}_M$ is a finitely generated R_M-module. Hence, by Proposition 3.8, R_M is a totally divisorial domain. This holds for every maximal ideal of R, so by Lemma 3.11, $\overline{R}$ is h-local. Every overring of an h-local Prüfer domain is h-local [2, Lemma 7.4]. Thus, by Lemma 3.11, every overring of R is h-local. Thus every overring of R is a divisorial domain, proving the claim. The converse is clear. ■

Heinzer proves in [8] that a divisorial domain R is h-local and for each maximal ideal M of R, R_M is a divisorial domain. The converse is established in [2, Proposition 5.4]. These assertions remain true if «divisorial» is replaced by «totally divisorial:»

COROLLARY 3.13. *An integral domain R is totally divisorial if and only if R is h-local and R_M is totally divisorial for all maximal ideals M of R.*

PROOF. Since h-local locally stable domains are stable and h-local locally divisorial domains are divisorial, the claim follows at once from Theorem 3.12. ■

THEOREM 3.14 (Descent from $\overline{R}$). *An integral domain R is a totally divisorial domain if and only if*

(i) *$\overline{R}$ is an h-local strongly discrete Prüfer domain,*

(ii) *$\overline{R}/R$ is a distributive R-module,*

(iii) *$\overline{R}_M/R_M$ is a finitely generated Artinian R_M-module for all maximal ideals M of height greater than one, and*

(iv) *there are at most two maximal ideals of $\overline{R}$ lying over each maximal ideal of R.*

PROOF. In [14, Corollary 2.5 and Theorem 4.9], it is shown that (a) an integral domain R is coherent and stable if and only if $\overline{R}$ is stable, every R-submodule of $\overline{R}$ containing R is a ring, $\overline{R}_M/R_M$ is an Artinian R_M-module for all maximal ideals M of R and there are at most 2 maximal ideals of $\overline{R}$ lying over each maximal ideal of R; and (b) an integral domain R is one-dimensional and stable if and only if $\overline{R}$ is a Dedekind domain, every R-submodule of $\overline{R}$ containing R is a ring and there are at most 2 maximal ideals of $\overline{R}$ lying over each maximal ideal of R.

Suppose R is a totally divisorial domain. Then (i) is clear, and (ii) follows from Lemma 3.5 and the fact that a module is distributive if and only if it is locally a uniserial module. Conditions (iii) and (iv) are consequences of (a).

Conversely, assume (i)-(iv) hold. Consider first the case that R is quasilocal. If R is one-dimensional, then by (b), R is stable. If R is not one-dimensional, then the maximal ideal of R has height greater than one, so by (a), R is stable. In either case, R is a stable domain. Let M be the maximal ideal of R. By Lemma 3.5, the maximal ideal of R is 2-generated and R has the 2-generator property. Thus, if R is a one-dimensional domain, R is Noetherian and R is totally divisorial. Otherwise, if R is not one-dimensional, then by (i), (iii) and Proposition 3.8, R is a totally divisorial domain. To complete the proof, assume R is a not necessarily quasilocal domain satisfying (i)-(iv). Then R_M is totally divisorial for all maximal ideals M of R. By (i), Lemma 3.11 and Corollary 3.13, R itself is totally divisorial. ■

COROLLARY 3.15. *Let B be a quasilocal integral domain. Define $k = B/M$, where M is the maximal ideal of B and let $\nu : B \to k$ denote the canonical projection of B onto the residue field of B. Let A be a quasilocal one-dimensional subring of k and consider the pullback:*

$$\begin{array}{ccc} R = \nu^{-1}(A) & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \xrightarrow{\nu} & k \end{array}$$

Then R is a quasilocal totally divisorial domain if and only if k is the quotient field of A, A is a local Bass domain, $\overline{A}$ is local and B is a strongly discrete valuation domain.

PROOF. Let R, A, B and k be as in the statement of the claim and suppose A is a local Bass domain, $\overline{A}$ is local and B is a strongly discrete valuation domain. In this case, the stability of A is sufficient to guarantee the stability of R [14, Theorem 2.6], so R is a quasilocal stable domain. Also, by Lemma 3.5, the R-module $\overline{R}/R$ is uniserial. Now by Corollaries 2.3 and 2.4, $\overline{R} = \nu^{-1}(\overline{A})$, and by [7, Proposition 5.3.3], $\overline{R}$ is a strongly discrete valuation domain. Since $\overline{R}$ is a finitely generated R-module, Theorem 3.14 applies and R is a totally divisorial domain. Conversely, given that R is a quasilocal totally divisorial domain, the claim follows from Corollary 2.3 and Theorem 3.14. For let P be the largest non-maximal prime ideal of R. Then $A \cong R/P$, $B \cong R_P$ and $k \cong R_P/P$. Also, $\overline{R}/P$ is the integral closure of R/P in its quotient field R_P/P. By Theorem 3.14, $\overline{R}/P$ is a finitely generated R/P-module and hence R/P is a Bass domain. ■

COROLLARY 3.16 (Classification of totally divisorial domains). *An integral domain R is totally divisorial if and only if R is h-local and for each $M \in Max(R)$, R_M satisfies one of the following conditions:*

(i) *every ideal of R_M is 2-generated,*

(ii) *R_M is a strongly discrete valuation domain, or*

(iii) *R_M is a pullback, in the sense of Corollary 3.15, of quasilocal rings A and B such that $\overline{A}$ is a finitely generated A-module, $\overline{A}$ is local, A satisfies* (i) *and B satisfies* (ii).

PROOF. This is an immediate consequence of Corollaries 3.13 and 3.15. ∎

4. Warfield domains.

Since Warfield domains are totally divisorial, this section makes heavy use of the characterizations given in Section 3. We recall the descriptions of Noetherian and integrally closed Warfield domains given in [2, Theorem 7.3]:

THEOREM 4.1 (Bazzoni and Salce). *Let R be an integral domain.*

(i) *R is a Noetherian Warfield domain if and only if every ideal of R is 2-generated.*

(ii) *R is an integrally closed Warfield domain if and only if R is an almost maximal strongly discrete Prüfer domain.*

Recall that an integral domain R is *almost maximal* if and only if every proper homomorphic image of R is linearly compact. A Prüfer domain R with quotient field K is almost maximal if and only if R is h-local and K/R is an injective R-module [4]; equivalently, a Prüfer domain R is almost maximal if and only if each (prime) ideal of R has injective dimension one as an R-module [13, Theorem 3.5 and Corollary 3.6]. An integral domain R is *complete* if it is complete in the R-topology; equivalently, R is not a field and $Ext_R(K, R) = 0$, where K is the quotient field of R (see [11]). A complete almost maximal valuation domain is a *maximal* valuation domain.

Most of the arguments in this section are homological in nature, and we frequently rely on flatness properties to enable us to «switch rings;» i.e. if $R \to S$ is a homomorphism of commutative rings, A is an R-module and B is an S-module, then if $Tor_n^R(A, S) = 0$ for all $n > 0$, there is a natural isomorphism

$$Ext_R(A, B) \cong Ext_S(A \otimes_R S, B).$$

By a *switch of rings* from R to S, we mean an appeal to this property.

Like totally divisorial domains, the study of Warfield domains reduces to the quasilocal case. Surprisingly, however, the local properties of locally Warfield domains are so strong that one does not need the full strength of the h-local property to globalize Warfield duality from the quasilocal case:

LEMMA 4.2. *An integral domain R is a Warfield domain if and only if every non-zero ideal of R is contained in at most finitely many maximal ideals of R and R_M is a Warfield domain for all $M \in$ $\in Max(R)$.*

PROOF. The integrally closed case of the claim is proved (in different but equivalent terminogy) by Facchini in [6, Theorem 4.3]. Now if R is locally a Warfield domain, R is locally a stable domain. By assumption, each non-zero ideal of R is contained in at most finitely many maximal ideals of R, so R is stable [18, Theorem 3.3], and hence, since overrings of stable domains are stable [18, Theorem 5.1], $\overline{R}$ is also stable. Thus $\overline{R}$ is a Warfield domain; in particular, $\overline{R}$ is h-local. Since every overring of an h-local Prüfer domain is h-local [2, Lemma 7.4], it follows from Lemma 3.11 that every overring of R is h-local and hence R is a Warfield domain [2, Proposition 4.9]. The converse is clear from [2]. ■

THEOREM 4.3 (Ascent from R). *An integral domain R with quotient field K is a Warfield domain if and only if R is a stable domain and K/R is an injective R-module.*

PROOF. Assume first R is a stable domain and K/R is injective. We check that R is a reflexive ring. To do this, it suffices to prove K/R is a universal injective R-module [11, Theorem 29]. Thus we must show K/R contains a copy of every simple R-module. Let M be a maximal ideal of R. If $[R : M] = R$, then $E(M) = R$, and since M is stable, M is invertible, a contradiction to assumption that $[R : M] =$ $= R$. Thus $[R : M]/R$ is a non-zero R-module, and the R/M-vector space $[R : M]/R$ embeds into K/R, proving K/R contains a copy of R/M. Thus R is reflexive. Also, by Theorem 3.12, R is totally divisorial, and in particular, every overring of R is h-local. A module over an h-local domain is injective if and only if it is locally an injective module (with respect to maximal ideals) [11, Theorem 24]. So, to show every overring of R is reflexive, it is enough to prove the claim locally; i. e. we show that if M is a maximal ideal of R, then for every overring S of R_M, K/S is an injective S-module. For, as discussed in the Introduction, totally reflexive domains are Warfield domains. Also, an integral domain S is reflexive if and only if it is a divisorial domain and K/S is injective [11, Theorem 29]. Thus, having justified the reduction, we suppose R is a quasilocal domain.

If R is a one-dimensional domain, then since R is totally divisorial, R is Noetherian (Lemma 3.3) and every ideal of R is 2-generated; hence R is a Warfield domain. So suppose R is not one-dimensional. We consider each overring of R, and we claim first that for each non-maximal prime ideal L of R, R_L has injective dimension one as an R_L-module. Since $L \cong R_L$, we prove the claim for L. Observe that since by Lemma 2.2, $L = [R : R_L]$, there is an exact sequence

$$0 \to L \to K \to Hom_R(R_L, K/R) \to Ext_R(R_L, R).$$

Since R has injective dimension one as an R-module, it follows that $K/L \cong Hom_R(R_L, K/R)$. Let A be an R_L-module. Since K/R is an injective module, we have that

$$Ext_{R_L}(A, Hom_R(R_L, K/R)) \cong Hom_R(Tor_1^{R_L}(A, R_L), K/R) = 0,$$

since R_L is a flat R_L-module [11, p. 8]. Thus K/L is an injective R_L-module, proving $L \cong R_L$ has injective dimension one as an R_L-module.

If R is integrally closed, then, since this implies R is a valuation domain, every overring of R is of the form R_L for some prime ideal L of R, and the claim follows from the preceding considerations. Thus we suppose $R \neq R_1$. Now we claim that for all $i > 0$, R_i has injective dimension one as an R_i-module. Of course, it suffices to show R_i is a reflexive ring. The proof is by induction on i. Suppose R_j is a reflexive ring for all $j < i$. Let G be a torsionless R_i-module. Then $R_i = [R_{i-1}: M_{i-1}]$, and R_i can be identified with $Hom_{R_{i-1}}(M_{i-1}, R_{i-1})$. Thus the fact that M_{i-1} is a principal ideal of R_i and an easy application of the Adjoint Isomorphism Theorem yields,

$$Hom_{R_i}(Hom_{R_i}(G, R_i), R_i) \cong Hom_{R_{i-1}}(Hom_{R_{i-1}}(G, R_{i-1}), R_{i-1}).$$

This isomorphism is natural, so by the induction hypothesis, the canonical mapping $G \to Hom_{R_i}(Hom_{R_i}(G, R_i), R_i)$ is an isomorphism and it follows that R_i is reflexive, and in particular, K/R_i is injective module. By Lemma 3.10, we have shown every overring S of R has injective dimension one as an S-module. This completes the proof, since the converse is clear. ∎

THEOREM 4.4 (Descent from $\overline{R}$). *An integral domain R is a Warfield domain if and only if*

(i) $\overline{R}$ *is an almost maximal strongly discrete Prüfer domain,*

(ii) $\overline{R}/R$ *is a distributive* R*-module,*

(iii) $\overline{R}_M/R_M$ *is a finitely generated Artinian* R_M*-module for all maximal ideals* M *of height greater than one, and*

(iv) *there are at most two maximal ideals of* $\overline{R}$ *lying over each maximal ideal of* R.

PROOF. Suppose R satisfies (i)-(iv). Then by Theorem 3.14, R is totally divisorial and so, since each overring of R is h-local, we may assume without loss of generality that R is quasilocal (use Lemma 4.2). If R is one-dimensional, then R is Noetherian and R is a Warfield domain, so we suppose R has a non-zero largest non-maximal prime ideal P.

Since R is stable, it is enough by Theorem 4.3 to show R has injective dimension one as an R-module. To do this, it suffices to prove $Ext_R(I, R) = 0$ for all ideals I of R. By Lemma 3.10 and the fact that R is stable, it is enough to prove $Ext_R(S, R) = 0$ for each overring S of R properly contained in the quotient field of R. We consider overrings of the form R_i first. The proof that $Ext_R(R_i, R) = 0$ is by induction on i. Let $i > 0$ and suppose $Ext_R(R_n, R) = 0$ for all $n < i$. If $R_i = R_{i-1}$, there is nothing to prove, so suppose $R_i \neq R_{i-1}$. There is an exact sequence,

$$0 \to Hom_R(R_i, R) \to Hom_R(R_{i-1}, R) \to Ext_R(R_i/R_{i-1}, R) \to Ext_R(R_i, R) \to 0 .$$

Now $R_i/R_{i-1} \cong R/M$ and hence $Ext_R(R_i/R_{i-1}, R) \cong Ext_R(R/M, R)$. Consider the exact sequence,

$$0 \to Hom_R(R, R) \to Hom_R(M, R) \to Ext_R(R/M, R) \to 0 .$$

The R-module $Hom_R(M, R)$ can be identified with $[R : M] = R_1$. Thus $R/M \cong R_1/R \cong Ext_R(R/M, R)$. Therefore, $Ext_R(R_i/R_{i-1}, R)$ is a simple R-module and since the mapping $Hom(R_i, R) \to Hom(R_{i-1}, R)$ is nontrivial and not surjective, we have

$$Hom(R_{i-1}, R) \to Ext(R_i/R_{i-1}, R) \to 0 .$$

It follows that $Ext(R_i, R) = 0$, as claimed, and the induction is complete.

If $\overline{R} \neq R_i$ for all $i > 0$, then by Lemma 3.9, R is one-dimensional, contrary to assumption. So $\overline{R} = R_i$ for some $i > 0$ and $Ext_R(\overline{R}, R) = 0$. It remains to show that $Ext_R(R_Q, R) = 0$ for all non-maximal prime ideals Q of R. Consider the exact sequence,

$$Hom_R(R_Q, \overline{R}/R) \to Ext_R(R_Q, R) \to Ext_R(R_Q, \overline{R}).$$

Since R_Q is a flat R-module, we may switch rings from R to $\overline{R}$; consequently, $Ext_R(R_Q, \overline{R}) \cong Ext_{\overline{R}}(R_Q, \overline{R}) = 0$, since $R_Q \cong Q$ and $\overline{R}$ has injective dimension one as an $\overline{R}$-module. We verify now that $Hom_R(R_Q, \overline{R}/R) = 0$. Let $f \in Hom_R(R_Q, \overline{R}/R)$. Then by Lemma 3.5, $\overline{R}/R$ is a uniserial R-module. By Lemma 2.1, every R-submodule of $\overline{R}$ containing R is a ring. But by Lemma 3.10, these rings are all of the form R_i for some j. Thus $R_Q/Ker\, f \cong R_j/R$ for some $j \geq 0$. Now R_j is annhilated by some element $m \in R$ not contained in Q. This implies $f = 0$ and we conclude $Ext_R(R_Q, R) = 0$. This completes the proof that R has injective dimension one as an R-module. The converse is clear from Theorem 3.14. ■

LEMMA 4.5. *Let R be an integral domain of Krull dimension greater than one. Let P be a non-zero non-maximal prime ideal of R. If $P = PR_P$, then R is an almost maximal valuation domain if and only if R/P is a maximal valuation domain and R_P is an almost maximal valuation domain.*

PROOF. Assume R/P and R_P are as in the statement of the Lemma. Then R is a valuation domain [7, Corollary 1.1.9]. To show that R is almost maximal, it suffices to show $Ext(I, R) = 0$ for all ideals I of R. Let I be an ideal of R. There is an exact sequence,

$$Ext_R(I, P) \to Ext_R(I, R) \to Ext_R(I, R/P).$$

Since R is valuation domain, I is a flat R-module. Thus $Ext_R(I, P) \cong$ $\cong Ext_{R_P}(IR_P, P) = 0$, since P has injective dimension one as an R_P-module. Also, $Ext_R(I, R/P) = Ext_{R/P}(I/IP, R/P) = 0$, since R/P is a maximal valuation domain [12, Theorem 9]. Thus $Ext_R(I, R) = 0$ and R has injective dimension one as an R-module. The converse is well-known. ■

LEMMA 4.6. *If R is an integral domain such that $\overline{R}$ is a fractional ideal of R that is complete in the $\overline{R}$ topology, then R is complete.*

PROOF. Let K denote the quotient field of R and consider the exact sequence,

$$Hom_R(K, \overline{R}/R) \to Ext_R(K, R) \to Ext_R(K, \overline{R}).$$

Since $\overline{R}/R$ is a bounded torsion R-module, $Hom_R(K, \overline{R}/R) = 0$. By a switch of rings from R to $\overline{R}$ (use the fact that K is a flat R-module), we have $Ext_R(K, R) \cong Ext_{\overline{R}}(K, \overline{R}) = 0$. It follows that $Ext_R(K, R) = 0$, proving the claim. ∎

LEMMA 4.7. *If R is a Warfield domain and L is a non-zero non-maximal prime ideal of R, then R/L is complete in the R/L-topology.*

PROOF. Since R is h-local, $R/L \cong R_M/L_M$, where M is the unique maximal ideal of R containing L. Thus we assume without loss of generality that R is a quasilocal Warfield domain. By Lemma 2.2, R_L/L is the quotient field of R/L, and to show that R/L is complete, it suffices to show that $Ext_{R/L}(R_L/L, R/L) = 0$ [11, Theorem 10]. Let P be the largest non-maximal prime ideal of R, and suppose first that $L = P$. It follows from Lemma 2.1 that $\overline{R}/P$ is the integral closure of R/P in R_P/P. By Theorem 4.1, $\overline{R}$ is an almost maximal Prüfer domain and hence $\overline{R}/P$ is a complete valuation domain. Since R is not one-dimensional, $\overline{R}$ is a finitely generated R-module (Lemma 3.10). Hence, by Lemma 4.6, R/P is a complete domain in the R/P-topology. This settles the case $P = L$; we now suppose $L \subset P$.

There is an exact sequence,

$$0 \to Hom_R(R_L/L, R_P/R) \to Ext_R(R_L/L, R/L) \to Ext_R(R_L/L, R_P/L).$$

Since R_P is an almost maximal valuation domain, R_P/L is a complete ring. By Lemma 2.2, both L and R_L are flat R-modules. Thus we may switch rings from R to R_P/L and obtain $Ext_R(R_L/L, R_P/L) = 0$. To complete the claim, we check that $Hom_R(R_L/L, R_P/R) = 0$. Observe that by Lemma 3.10 the R-module R_P/R is a uniserial R-module, and if $M = R_1 m$, then the submodules of R_P/R are of the form

$$\{R_1/R, R_2/R, \ldots, R_i/R\} \cup \{R_i m^{-k}/R : k > 0\}.$$

Thus, if $f \in Hom_R(R_L/L, R_P/R)$ and f is not surjective, then $Im\, f$ is annhilated as an R-module by m^k for some k. But $Im\, f \cong R_L/A$ for some R-submodule A of R_L. Since $R_L m^k = R_L$, this forces $A = R_L$ and $f = 0$. Otherwise, if f is surjective, then $R_L/B \cong R_P/R$ for some R-submodule B of R_L. Comparing annhilators, this forces $B = R_L$ and $f = 0$. We conclude $Hom_R(R_L/L, R_P/R) = 0$ and $Ext_{R/L}(R_L/L, R/L) = 0$, and so R/L is complete. ∎

LEMMA 4.8. *Let B be a quasilocal integral domain. Define $k = B/M$, where M is the maximal ideal of B and let $\nu : B \to k$ denote the canonical projection of B onto the residue field of B. Let A be a quasilocal one-dimensional subring of k and consider the pullback:*

$$\begin{array}{ccc} R = \nu^{-1}(A) & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \xrightarrow{\nu} & k \end{array}$$

Then R is a quasilocal Warfield domain of Krull dimension greater than one if and only if k is the quotient field of A, A is a complete local Bass domain and B is an almost maximal strongly discrete valuation domain.

PROOF. Suppose R is a Warfield domain. Let P be the largest non-zero non-maximal prime ideal of R. By Corollary 3.15, $A \cong R/P$ is a local Bass domain, $\overline{A}$ is quasilocal and $B \cong R_P$ is a strongly discrete valuation domain. By Lemma 4.7, A is complete. Since B is a Warfield valuation domain, B is almost maximal.

Conversely, suppose A is a complete local Bass domain. Since A is complete and Noetherian, $\overline{A}$ is local and B is a strongly discrete almost maximal valuation domain. Then there exists a non-maximal prime ideal P of R such that $R/P \cong A$ and $R_P \cong B$. By Corollary 3.15, R is totally divisorial. Now $\overline{R}$, as a pullback of the complete DVR $\overline{A}$ and the almost maximal valuation domain B, is an almost maximal valuation domain (Lemma 4.5). Thus, by Theorem 4.4, R is a Warfield domain. ■

Lemma 4.8 substantiates our classification of Warfield domains:

THEOREM 4.9 (Classification of Warfield domains). *An integral domain R is a Warfield domain if and only if each non-zero ideal of R is contained in at most finitely many maximal ideals of R and for all $M \in Max(R)$, R_M satisfies one of the following statements:*

(i) *every ideal of R_M is 2-generated,*

(ii) *R_M is an almost maximal strongly discrete valuation domain, or*

(iii) *R_M is a pullback, in the sense of Theorem 4.8, of quasilocal rings A and B such that A is complete and satisfies* (i) *and B satisfies* (ii).

COROLLARY 4.10. *Let R be a complete Bass domain, and let K be the quotient field of R. Then $R + XK[X]_{(X)}$ is a non-Noetherian, non-integrally closed Warfield domain.*

PROOF. This is immediate from Lemma 4.8 or Theorem 4.9. ■

5. A canonical decompositions of torsion-free modules.

In this section, we characterize Warfield domains by the decomposition of their *torsionless modules,* those modules which are a submodule of some finitely generated free R-module. To show that Warfield domains indeed possess the decomposition discussed in the Introduction, we extend an argument of Rush (which is in turn an extension of a result in [5]) about the trace ideal of a torsionless module [20, Lemma 4.2]. Rush proves our Lemma 5.2 for one-dimensional Macaulay rings with maximal ideals that are not minimal. We replace these hypotheses with the assumption that the ring be an h-local integral domain, thus generalizing the domain case of Rush's result.

LEMMA 5.1. *Let I be a non-zero ideal of an h-local integral domain. Then $I = I_1 \ldots I_n$ for pairwise comaximal ideals $I_1, \ldots, I_n$ of R such that each I_i is contained in at most one maximal ideal of R.*

PROOF. Since R is h-local, there are finitely many maximal ideals $M_1, \ldots, M_n$ of R that contain I. For each $i \leq n$, define $I_i = IR_{M_i} \cap R$. If M is a maximal ideal of R, then since R is h-local, $R_M \otimes_R \cap_{N \neq M} R_N \cong K$, where K is the quotient field of R and the intersection ranges over all maximal ideals of R not equal to M [11]. Thus $I_i R_{M_i} = IR_{M_i}$ while $I_i R_N = R_N$ for all maximal ideals N of R distinct from M_i. It follows that M_i is the unique maximal ideal of R containing I_i and local verification shows that $I = I_1 \ldots I_n$. ■

The *trace* of an R-module G is the ideal $\tau(G)$ of R, where $\tau(G) = \sum_{f \in Hom_R(G, R)} f(G)$.

LEMMA 5.2. *Let R be an h-local integral domain and suppose G is a torsionless R-module of rank greater than 1. If $\tau(G) = R$, then $G \cong R \oplus H$ for some torsionless module H.*

PROOF. Our proof closely follows Rush's argument in [20, Lemma 4.2], with the exception that we must appeal to the more general property of h-locality rather than a Noetherian one-dimensional assumption. Let K denote the quotient field of R and set $F = R \oplus R$. Then there is a homomorphism $\phi = (f, g): G \to F$ such that $\phi \otimes 1_K$ maps $G \otimes_R K$ onto $F \otimes_R K$. Define $C = coker(\phi)$ and $I = Ann_R(C)$. For all maximal ideals M of R that do not contain I, $\phi(G)_M = F_M$ and hence f_M and g_M are onto.

We now construct a homomorphism $h \in Hom_R(G, R)$ such that $h \otimes_R 1_{R/I}$ is surjective. Using Lemma 5.1, write $I = I_1 \dots I_n$, where $\{I_1, \dots, I_n\}$ is a set of pairwise comaximal ideals of R, and let M_i denote the unique maximal ideal of R containing I_i. Since $\tau(G) = R$, there exists a surjective map $f_i \in Hom_{R_{M_i}}(G_{M_i}, R_{M_i}) \cong Hom_R(G, R) \otimes_R R_{M_i}$. (Since R is h-local, Hom localizes for torsionless modules [17]). Thus we may write $f_i = h_i/s_i$ with $h_i \in Hom_R(G, R)$ and $s_i \in R \backslash M_i$. For each $i \leq n$, use the Chinese Remainder Theorem to choose $u_i \in R$ such that $u_i \equiv \delta_{ij} \, mod(I_j)$ for all $j \leq n$, where δ_{ij} is Kroenecker delta. Define $h = u_1 h_1 + \dots + u_n h_n$. Then $h \in Hom_R(G, R)$ and $h \otimes 1_{R_{M_i}/IR_{M_i}}$ is surjective for each $i \leq n$. Since torsion modules over h-local domains can be decomposed into their primary components (see [11]), $R/I \cong \bigoplus_{i \leq n} R_{M_i}/IR_{M_i}$, and it follows that the mapping $h^*: G \to R/I$ defined by $h^*(g) = h(g) + I$, for all $g \in G$, is surjective. Thus $h(G) + I = R$, and for all $i \leq n$, $h(G_{M_i}) = R_{M_i}$; consequently, h_M is onto for all maximal ideals M that contain I.

Define $\phi' = (f, h): G \to F$. Then, by the construction, if M is a maximal ideal of R, then f_M or h_M is onto. For if $I \subseteq M$, then h_M is onto, and if $I \nsubseteq M$, then f_M is onto. Define $C' = coker(\phi')$. Then, since R is h-local, $C' \cong \bigoplus_{M \in Max(R)} C'_M$ and hence, since either f_M or h_M is onto, $C' \cong R_{N_1}/J_1 R_{N_1} \oplus \dots \oplus R_{N_l}/J_l R_{N_l}$ for some ideals $J_1, \dots, J_l$ of R and maximal ideals $N_1, \dots, N_l$. For each $i \leq l$, define $A_i = J_{N_i} \cap \cap \left(\bigcap_{N \neq N_i} R_N \right)$, where the intersection ranges over all maximal ideals N of R such that $N \neq N_i$. As in the proof of Lemma 5.1, the h-locality of R implies that the only maximal ideal of R containing A_i is N_i. Set $A = A_1 \cap \dots \cap A_n$. Then local verification and the h-local hypothesis show $C' \cong R/A$ and C' is a cyclic module over R/A. Now R/A has only finitely many maximal ideals, so 1 is in the stable range of R/A (for example, see [23]). If e_1, e_2 is a basis for F, then there exists $r \in R$ such that $e_1 + re_2$ generates C'. Thus $e = e_2 - b(e_1 + re_2) \in \in Im \phi'$ for some $b \in R$ and $\{e, e_1 + re_2\}$ is a basis for F. Thus $Im \phi' =$

$=Re\oplus A$ for some some submodule A. It follows that G maps onto R. ∎

An interesting consequence of Lemma 5.2 is that finitely generated projective modules over h-local integral domains decompose like finitely generated projectives over Dedekind domains. Namely, if R is an h-local domain and P is a finitely generated projective module, then P is isomorphic to a direct sum of a free R-module and an invertible ideal.

If K is the quotient field of an integral domain R, then the *coefficient ring* S of a torsionless R-module G is defined to be the overring $S=\{q\in K : qG\subseteq G\}$.

THEOREM 5.3. *Let R be an integral domain. The following statements are equivalent for R.*

(1) *R is a Warfield domain.*

(2) *Every torsionless R-module G can be decomposed as*

$$G\cong S_1\oplus S_2\oplus\ldots\oplus S_{n-1}\oplus I,$$

where $S_1\subseteq S_2\subseteq\ldots\subseteq S_n$ are overrings of R and I is an invertible ideal of S_n.

(3) *Every torsionless R-module is isomorphic to a direct sum of stable ideals.*

PROOF. (1) $\Rightarrow$ (2) Let G be a torsionless R-module, and denote the coefficient ring of G by R_1. Then, by assumption, G is a reflexive R_1-module and hence $\tau(G)$, the trace of G with respect to R_1, is R_1 [20, Lemma 4.1]. By Lemma 5.2, $G\cong R_1\oplus A$, for some torsionless R-module A. If A does not have rank one, then, viewing A as a torsionless module over its coefficient ring R_2, we have A is a reflexive R_2-module and another application of Lemma 5.2 yields a module B such that $A\cong R_2\oplus B$. Continuing in this manner, we arrive at the decomposition of (2) and since Warfield domains are stable, the ideal I is stable.

(2) $\Rightarrow$ (3) This is clear.

(3) $\Rightarrow$ (1) Given (3), every ideal of R is necessarily stable. Thus every ideal of R is contained in at most finitely many maximal ideals of R, so it suffices to show R_N is a Warfield domain for all $N\in\max(R)$ (Lemma 4.1). For this reason, we reduce to the case that R is a quasilocal domain. Bass's theorem implies every finitely generated ideal of R is 2-generated (see [1, Proposition 7.5] or [11, Theorem 56]). If R is one-dimensional,

then by Lemma 3.5, R is a Noetherian ring for which every ideal is 2-generated and hence R is a Warfield domain. If, on the other hand, R has a non-zero non-maximal prime ideal P, then we may suppose P is the largest non-maximal prime ideal of R. Moreover, by Lemma 3.5, $\overline{R} \subseteq E(P)$, so $\overline{R}$, as a fractional overring of R, inherits the decomposition property (3). By Lemmas 3.7 and 3.9, $\overline{R}$ is quasilocal and a finitely generated R-module. Thus $\overline{R}$ is an almost maximal valuation domain [12, Theorem 4], and since $\overline{R}$ is strongly discrete, it follows that $\overline{R}$ is a Warfield domain. To justify an application of Theorem 4.4, note that (i) is immediately satisfied. Also, (ii) and (iii) hold, for, since $\overline{R}/R \cong (\overline{R}/P)/(R/P)$ and R-submodules of $\overline{R}/R$ are R/P-modules, $\overline{R}/R$ is a uniserial, finitely generated Artinian R-module. Since R is stable, (iv) is satisfied (Lemma 2.1(ii)). Thus R is a Warfield domain.

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