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# Some Remarks on Global Solutions to Nonlinear Dissipative Mildly Degenerate Kirchhoff Strings.

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ABSTRACT - We investigate the evolution problem

$$u_{tt} + \delta u_t - m\left(\int_\Omega |\nabla u|^2\right)\Delta u + f(u) = 0\,,$$

$$u(0, x) = u_0(x)\,, \qquad u_t(0, x) = u_1(x)\,, \qquad x \in \Omega\,,\ t \geq 0$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, $\delta > 0$, and $m$ is a locally Lipschitz continuous function, with $m(0) = 0$ and $m(r) > 0$ in a neighbourhood of $0$. We prove that, if $\beta = \max\{1, [n/2]\}$, this problem has a unique global solution for positive times, provided that $(u_0, u_1) \in (H_0^\beta \cap H^{\beta+1})(\Omega) \times H_0^\beta(\Omega)$ and $u_0, u_1, f$ satisfy suitable smallness assumptions and the non-degeneracy condition $u_0 \neq 0$ holds. We prove also that $(u(t), u_t(t), u_{tt}(t)) \to (0, 0, 0)$ in $(H_0^\beta \cap H^{\beta+1})(\Omega) \times H^\beta(\Omega) \times H^{\beta-1}(\Omega)$ as $t \to \infty$.

## 1. Introduction.

Let $\Omega \subseteq \mathbb{R}^n$ be an open domain, $H := L^2(\Omega)$, with norm $\|\cdot\|$ and scalar product $\langle \cdot, \cdot \rangle$. Let us set $A := -\Delta$, with domain $D(A) := (H_0^1 \cap \cap H^2)(\Omega)$. We consider the Cauchy problem

$$\text{(1.1)} \qquad \begin{cases} u''(t) + \delta u'(t) + m(\|A^{1/2}u(t)\|^2)Au(t) + f(u(t)) = 0, & t \geq 0, \\ u(0) = u_0, \quad u'(0) = u_1, \end{cases}$$

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where $\delta > 0$, $m : [0, +\infty[ \to [0, +\infty[$ is a locally Lipschitz continuous function.

If $\Omega$ is an interval of the real line, this equation is a model for the damped small transversal vibrations of an elastic string with fixed endpoints.

The case $\delta = 0, f = 0$ (free vibrations) has long been studied: the interested reader can find appropriate references in the surveys of A. Arosio [1] and S. Spagnolo [15].

In the case $\delta = 0$, $f(u) = \pm |u|^\alpha u$ with large $\alpha$ and $m(r) \geqslant \nu > 0$, P. D'Ancona and S. Spagnolo [4] proved that if $u_0, u_1 \in C_0^\infty(\mathbb{R}^n)$ are small, then problem (1.1) has a global solution.

The case $\delta > 0$ and $f = 0$ was considered by E. H. De Brito, Y. Yamada and K. Nishihara [2,14,3,10] if $m(r) \geqslant \nu > 0$ and by K. Nishihara and Y. Yamada [11] and in [6] if $m(r) \geqslant 0$. In [6] it was proved that if $(u_0, u_1) \in D(A) \times D(A^{1/2})$ are small enough and $m(\|A^{1/2} u_0\|^2) \neq 0$, there exists a unique global solution $u(t)$ of (1.1) and that $(u, u', u'') \to (u_\infty, 0, 0)$ in $D(A) \times D(A^{1/2}) \times H$ as $t \to +\infty$; moreover either $u_\infty = 0$ or $m(\|A^{1/2} u_\infty\|^2) = 0$.

Here we are interested in the case in which $f \neq 0$, i.e. we have a nonlinear perturbation effect (for example the presence of an external force).

The case $m(r) \geqslant \nu > 0$, $\delta > 0$, $f(u) = |u|^\alpha u$ has been considered by M. Hosoya and Y. Yamada [7] under the following condition:

$$(1.2) \qquad 0 \leqslant \alpha < \frac{2}{n-4} \quad \text{if} \quad n \geqslant 5, \qquad 0 \leqslant \alpha < +\infty \quad \text{if} \quad n \leqslant 4 .$$

They proved that, if the initial data are small enough, problem (1.1) has a global solution and such a solution decays exponentially as $t \to +\infty$.

Degenerate equations ($m(r) \geqslant 0$) were considered by K. Ono [12] and in [5] when $n \leqslant 3$, $\delta > 0$ and $f(u)u \geqslant 0$. In [12] it was proved that if $m(r) = r^\gamma$, $f(u) \simeq |u|^\alpha u$ and the initial data $(u_0, u_1) \in D(A) \times D(A^{1/2})$ are small enough, $u_0 \neq 0$, and:

$$(1.3) \qquad \alpha > 2\gamma - 1 \quad \text{if} \quad n = 1, 2, \qquad \alpha > 4\gamma - 2 \quad \text{if} \quad n = 3 ,$$

then problem (1.1) has a global solution, that decay as $t \to +\infty$.

In [5] the quoted result was extended to a general function $m(r)$ with $m(0) = 0$, $m(r) > 0$ in $]0, r_0]$ when, for some $\varepsilon > 0$:

either $f(y)y \geqslant 0$ and:

$$\max_{|y| \leqslant s} |f'(y)| \leqslant \begin{cases} Cm(s^{2+\varepsilon}) & \text{if } n = 1, 2, \\ Cm(s^4) & \text{if } n = 3, \end{cases} \tag{1.4}$$

or $f(0) = 0$, $f' \geqslant 0$ and:

$$\max_{|y| \leqslant s} |f'(y)| \leqslant \begin{cases} Cm(s^{2+\varepsilon})\, s^{-1+\varepsilon} & \text{if } n = 1, 2, \\ Cm(s^4)\, s^{-2+\varepsilon} & \text{if } n = 3. \end{cases} \tag{1.5}$$

Moreover K. Ono [13] proved that (1.1) has a global solution, if $f(u) \simeq \pm$ $\pm |u|^{\alpha} u$, $(u_0, u_1) \in D(A) \times D(A^{1/2})$ are small enough and at least one of the following conditions is verified:

1. $m(r) = r^{\gamma}$, $u_0 \neq 0$, $n \leqslant 3$, and

$$\alpha > 2\gamma \quad \text{if} \quad n = 1, 2 \qquad \alpha > 4\gamma - 2 \quad \text{if} \quad n = 3. \tag{1.6}$$

2. $m(r) \geqslant \nu > 0$, and satisfies (1.2) (see also R. Ikehata [8]).

He use the modified potential well method and the general theory on the energy decay in Nakao [9]. Unfortunately this method does not seem to be extendible to the case of more general $m$.

Our purpose is to consider problem (1.1) where $m$ is any non-negative locally Lipschitz continuous function, and $m(0) = 0$, $m(r) > 0$ in a neighbourhood of 0 and $f(u)u$ is not necessary positive.

Let us denote

$$\beta = \max\left\{1, \left[\frac{n}{2}\right]\right\} \quad \text{and} \quad B := A^{1/2}$$

where $[x]$ is the integer part of $x$.

We prove that there exists a unique global solution provided that $(u_0, u_1) \in D(B^{\beta+1}) \times D(B^{\beta})$ and $u_0, u_1, f$ satisfy suitable smallness assumptions (cf. Theorem 2.2) and the non-degeneracy condition $u_0 \neq 0$ holds. Moreover we prove that $u(t) \to 0$ as $t \to \infty$. (cf. Theorem 2.3).

The differences with respect to the case considered in [5] are of two different types: first the term $f$ is «not positive» and this compels us to modify the estimates (for example there exist no positive conserved energies...); here we must estimate with care some terms that in the case in [5] are negligible. Second we consider the case of all space dimensions, then we need more accurate estimates, in particular to reduce the requests at the minimum on the perturbation term.

*Notations*

In this paper, we denote by $a_{i,n}$ some constants such that

$$\begin{array}{ll} \|u\| \leqslant a_{1,n}\|Bu\| & u \in D(B) \\ \|u\|_\infty \leqslant a_{2,n}\|Bu\|^\lambda \|B^{\beta+1}u\|^{1-\lambda} & u \in D(B^{\beta+1}) \\ \|u\|_{p_0} \leqslant a_{3,n}\|Bu\| & u \in D(B) \\ \|Bu\| \leqslant a_{4,n}\|B^{\beta+1}u\| & u \in D(B^{\beta+1}) \end{array}$$

where $p_0 = \dfrac{2n}{n-2}$ if $n \geqslant 3$ and

$$\lambda = \begin{cases} 1 & \text{if } n = 1\,, \\ 1-\varepsilon & \text{if } n = 2\,, \\ 1 - \dfrac{n-2}{2\beta} & \text{if } n \geqslant 3\,. \end{cases}$$

## 2. Statement of the results.

In this section we state the main results of this paper. For sake of completeness, we recall the following local existence result.

THEOREM 2.1. *(Local existence) Let* $\delta > 0$, *let* $m$ *be a locally Lipschitz continuous function,* $f \in C^\beta(\mathbb{R})$, *and let* $(u_0, u_1) \in D(B^{\beta+1}) \times \times D(B^\beta)$ *with* $m(\|Bu_0\|^2) > 0$.

*Then there exists* $T > 0$ *such that problem* (1.1) *has a unique solution*

$$u \in C^2([0, T]; D(B^{\beta-1})) \cap C^1([0, T]; D(B^\beta)) \cap C^0([0, T]; D(B^{\beta+1})).$$

*Moreover,* $u$ *can be uniquely continued to a maximal solution defined in an interval* $[0, T_*[$, *and at least one of the following statements holds:*

(i) $T_* = \infty$;

(ii) $\limsup_{t \to T_*^-} (\|B^\beta u'(t)\|^2 + \|B^{\beta+1}u(t)\|^2) = +\infty$;

(iii) $\liminf_{t \to T_*^-} m(\|Bu(t)\|^2) = 0$.

The proof is standard and we can obtain it by following the outline of the one in [5] with the obvious changes in the notations. We can state the global existence's result.

THEOREM 2.2. *(Global existence) Let $\delta > 0$, and let $m$ be a locally Lipschitz continuous function with $m(0) = 0$ and $m(r) > 0$ in $]0, r_0]$ for some $r_0 > 0$. Let us assume that $f \in C^{\beta}(\mathbb{R})$ verifies in a neighbourhood of 0 the following conditions:*

(i) *if $n = 1, 2$: $f(0) = 0$ and for some $\varepsilon_1, \varepsilon_2 > 0$:*

$$\max_{|y| \leq s} |f'(y)| \leqslant Cm(s^{2+\varepsilon_1})\, s^{\varepsilon_2} \tag{2.1}$$

(ii) *if $n \geqslant 3$: $f(0) = f'(0) = \ldots = f^{(\beta-1)}(0) = 0$ and there exists some $\eta \geqslant 0$ such that:*

$$\max_{|y| \leq s} \frac{|f^{\beta}(y)|}{|y|^{\eta}} \leqslant Cm(s^{2/\lambda})\, s^{-(\eta - \tilde{\eta} + \beta - \beta_0) + (\varepsilon - \tilde{\eta})/\lambda} \tag{2.2}$$

*for some $\varepsilon > 0$, where $\tilde{\eta} = \min\left\{\eta, \dfrac{2}{n-2}\right\}$ and $\beta_0 := \max\{1, [\beta/2]\}$. Moreover let us assume that the initial data $(u_0, u_1) \in D(B^{\beta+1}) \times \times D(B^{\beta})$ are small enough and satisfy the non-degeneracy condition $u_0 \neq 0$.*

*Then problem* (1.1) *admits a unique global solution*

$$u \in C^2([0, +\infty[; D(B^{\beta-1})) \cap C^1([0, \infty[; D(B^{\beta})) \cap C^0([0, \infty[; D(B^{\beta+1})).$$

If $n \leqslant 3$, $m(r) = r^{\gamma}$ and $|f'(u)| \leqslant k|u|^{\alpha}$, by Theorem 2.2 we obtain the result in [12] (cf. (1.6)).

Finally we have the following result.

THEOREM 2.3. *(Asymptotic behaviour) Let us assume that all the hypotheses of Theorem 2.2 are satisfied.*

*Then we have that:*

(i) *$m(\|Bu(t)\|^2) > 0$ for all $t \geqslant 0$;*

(ii) *$(u(t), u'(t), u''(t)) \to (0,0,0)$ in the space $D(B^{\beta+1}) \times D(B^{\beta}) \times \times D(B^{\beta-1})$ as $t \to \infty$.*

The proof of Theorem 2.3 relies on a result about the asymptotic behaviour of solutions of the linearization of (1.1) (see Lemma 3.1 for the precise statement).

## 3. Proofs.

3.1. *Proof of Theorem* 2.2.

Case $n=1,2$

We use the following notations:

$$\phi_\varepsilon(n) := \begin{cases} (a_{1,1}^{\varepsilon} a_{2,1})^{-2/\varepsilon} & n=1 \\ (a_{2,2})^{-2/\varepsilon} & n=2 . \end{cases}$$

$$\mu_f(s) := \max_{|y| \leqslant s} |f'(y)|, \ \sqrt{c} := C .$$

With these notations we can rewrite, without loss of generality, (2.1) as follows:

$$\mu_f(s^{1-\varepsilon_1}) \leqslant \sqrt{c}\, m(s^2)\, s^{\varepsilon_2} \qquad s \in [0, \sqrt{r_0}] \tag{3.1}$$

for some constants $0 < \varepsilon_1, \varepsilon_2 < 1$.

Let us set:

$$\sigma := \min\{\phi_{\varepsilon_1}, r_0 a_{1,n}^{-2}, (2\sqrt{c}\, a_{1,n}^{2+\varepsilon_2})^{-2/\varepsilon_2}\},$$

$$M := \max_{|r| \leqslant r_0} m(r), \qquad L := \sup_{|r| \leqslant r_0} |m'(r)| .$$

Let us assume that, for a suitable $0 < \sigma_1 \leqslant \sigma$:

$$F_0 := F(0) + H_0 \frac{a_{1,n}^{2\varepsilon_2} c}{\delta} \sigma_1^{\varepsilon_2} < \sigma_1, \ LG_0 \sqrt{F_0} < \frac{\delta}{4}$$

where

$$H_0 := \frac{4}{\delta} a_{1,n}^2 M \sigma_1 + 2\langle u_0, u_1 \rangle + \delta \|u_0\|^2$$

$$F(0) := \frac{\|Bu_1\|^2}{m(\|Bu_0\|^2)} + \|B^2 u_0\|^2$$

$$G_0 := \max\left\{ \frac{\|u_1\|}{m(\|Bu_0\|^2)}, \ \frac{2}{\delta}(\sqrt{c r_0^{\varepsilon_2}}\, a_{1,n}^2 + 1)\sqrt{F_0} \right\}.$$

We prove that under these smallness assumptions the solution $u$ of (1.1) is a global solution.

In the following let us denote

$$c(t) = m(\|Bu(t)\|^2).$$

Let us assume that $m \in C^1([0, +\infty[; \mathbb{R})$, and let $[0, T_*[$ be the maximal interval where the solution exists.

*Step* 1. Let us define:

$$F(t) := \frac{\|Bu'(t)\|^2}{c(t)} + \|B^2 u(t)\|^2 + \frac{\delta}{2}\int_0^t \frac{\|Bu'(s)\|^2}{c(s)} ds,$$

$$T := \sup\left\{\tau \in [0, T_*[: c(t) > 0, \left|\frac{c'(t)}{c(t)}\right| \leq \frac{\delta}{2}, F(t) \leq \sigma_1 \ \forall t \in [0, \tau]\right\}.$$

We show that $T = T_*$. Let us assume by contradiction that $T < T_*$. Since $|c'(t)| \leq \frac{\delta}{2} c(t)$ in $[0, T[$ we have that

$$0 < c(0)\, e^{-\delta T/2} \leq c(t) \leq c(0)\, e^{\delta T/2} \qquad t \in [0, T[. \tag{3.2}$$

Moreover, by $\|B^2 u(t)\|^2 \leq \sigma$ we obtain:

$$\|Bu(t)\|^2 \leq a_{1,n}^2 \|B^2 u(t)\|^2 \leq r_0 \qquad t \in [0, T[.$$

Since $c(\cdot)$, $c'(\cdot)$, and $F(t)$ are continuous functions, by the maximality of $T$ we have that necessarily

$$\left|\frac{c'(T)}{c(T)}\right| = \frac{\delta}{2}; \tag{3.3}$$

or

$$F(T) = \sigma_1. \tag{3.4}$$

*Step* 2. Firstly, let us remark that, since:

$$\|u\|_\infty \leq \phi_{\varepsilon_1}^{-\varepsilon_1/2} \|Bu\|^{1-\varepsilon_1} \|B^2 u\|^{\varepsilon_1} \leq \|Bu\|^{1-\varepsilon_1} \tag{3.5}$$

then, using $f(0) = 0$:

$$\int_\Omega |f(u)\, u|\, dx = \int_\Omega |f'(\xi_u)\, u^2|\, dx \leq a_{1,n}^2 \mu_f(\|Bu\|^{1-\varepsilon_1}) \|Bu\|^2. \tag{3.6}$$

Furthermore, by taking the scalar product of the equation (1.1) with $u$, and integrating on $[0, t]$ we obtain:

$$\int_0^t (c(s)\|Bu(s)\|^2 + \langle f(u(s)), u(s)\rangle)\, ds =$$

$$= \int_0^t \|u'(s)\|^2 ds + \langle u_0, u_1\rangle - \langle u(t), u'(t)\rangle + \frac{\delta}{2}\|u_0\|^2 - \frac{\delta}{2}\|u(t)\|^2$$

$$\leq a_{1,n}^2 M\left(\int_0^t \frac{\|Bu'(s)\|^2}{c(s)} ds + \frac{\|Bu'(t)\|^2}{2\delta c(t)}\right) + \langle u_0, u_1\rangle + \frac{\delta}{2}\|u_0\|^2$$

$$\leq \frac{2}{\delta} a_{1,n}^2 M\sigma_1 + \langle u_0, u_1\rangle + \frac{\delta}{2}\|u_0\|^2.$$

Hence, for $t \in [0, T[$, by (3.6):

$$\int_0^t c(s)\|Bu(s)\|^2 ds - \sqrt{c}\, a_{1,n}^2 \int_0^t c(s)\|Bu(s)\|^{2+\varepsilon_2} \leq \frac{1}{2} H_0. \tag{3.7}$$

Furthermore, since $\sqrt{c}\, a_{1,n}^{2+\varepsilon_2} \sigma_1^{\varepsilon_2/2} \leq 1/2$, then:

$$\int_0^t c(s)\|Bu(s)\|^2 ds \leq H_0 \tag{3.8}$$

*Step* 3. We prove that (3.4) is false. A standard calculation show that on $[0, T[$ we have:

$$F'(t) \leq -\left(\frac{3}{2}\delta + \frac{c'(t)}{c(t)}\right)\frac{\|Bu'(t)\|^2}{c(t)} + \frac{2}{c(t)}\|Bu'(t)\|\|f'(u(t))\, Bu(t)\|$$

$$\leq \frac{1}{\delta c(t)}\|f'(u(t))\, Bu(t)\|^2.$$

Using (3.1), and (3.5) we obtain:

$$\|f'(u(t))\, Bu(t)\|^2 \leq \mu_f(\|Bu(t)\|^{1-\varepsilon_1})^2 \|Bu(t)\|^2 \tag{3.9}$$

$$\leq cm(\|Bu(t)\|^2)^2 \|Bu(t)\|^{2+2\varepsilon_2};$$

hence, by (3.8), for all $t \in [0, T]$:

$$F(t) \leq F(0) + \frac{c}{\delta} a_{1,n}^{2\varepsilon_2} \sigma_1^{\varepsilon_2} \int_0^t c(s) \|Bu(s)\|^2 ds \leq F_0 < \sigma_1. \tag{3.10}$$

This contradicts (3.4).

*Step* 4. We prove that (3.3) is false. Let us define $G(t) := \frac{\|u'(t)\|}{c(t)}$. By a simple computation, on $[0, T[$ we obtain:

$$(G^2(t))' \leq -\delta G^2(t) + 2G(t)\|Bu(t)\| + 2G(t) \frac{\|f(u(t))\|}{c(t)}.$$

Moreover, since $f(0) = 0$, by (3.1) and (3.5) we have:

$$\int_\Omega f(u(t, x))^2 \leq a_{1,n}^2 \mu_f(\|Bu(t)\|^{1-\varepsilon_1})^2 \|Bu(t)\|^2 \tag{3.11}$$

$$\leq c r_0^{\varepsilon_2} a_{1,n}^4 m(\|Bu(t)\|^2)^2 \|B^2 u(t)\|^2.$$

By this fact:

$$(G^2(t))' \leq -G(t)(\delta G(t) - 2(1 + \sqrt{c r_0^{\varepsilon_2}} a_{1,n}^2) \sqrt{F_0}).$$

Hence, by a standard ODE's inequality we have:

$$G(T) \leq \max\left\{G(0), \frac{2(1 + \sqrt{c r_0^{\varepsilon_2}} a_{1,n}^2)}{\delta} \sqrt{F_0}\right\} = G_0. \tag{3.12}$$

By (3.10) - (3.12), we have then

$$\left| \frac{c'(T)}{c(T)} \right| = \left| \frac{2m'(|Bu(T)|^2)\langle u'(T), B^2 u(T)\rangle}{c(T)} \right|$$

$$\leq 2L \frac{|u'(T)|}{c(T)} |B^2 u(T)|$$

$$\leq 2L G_0 \sqrt{F_0} < \frac{\delta}{2}.$$

This contradicts (3.3).

*Step* 5. Let us assume by contradiction that $T_* < +\infty$. By (3.2) and (3.8) it follows that

$$\liminf_{t \to T_*^-} m(\|Bu(t)\|^2) \geq m(\|Bu_0\|^2)\, e^{-\delta T_*/2} > 0\,,$$

$$\limsup_{t \to T_*^-}(\|Bu'(t)\|^2 + \|B^2 u(t)\|^2) \leq \max\{1,\, c(0)\, e^{\delta T_*/2}\}\, F_0 < +\infty\,.$$

By the last statement of Theorem 2.1 this is a contradiction. This completes the proof if $m'$ is continuous. If $m$ is only locally Lipschitz continuous, thesis follows from a standard approximation argument. ■

Case $n \geq 3$

In the following we denote by $\overline{a}, \overline{b}, \overline{c} \ldots$ some constants independent from the initial data, which we use in the proof. Moreover let us define:

$$\mu_f(s) := \max_{|y| \leq s} \left| \frac{f^\beta(y)}{y^\eta} \right|$$

With these notations we can rewrite, without loss of generality, (2.2) as follows:

$$\mu_f(s^\lambda) \leq Cm(s^2)\, s^{\varepsilon - \tilde{\eta} - \lambda(\eta - \tilde{\eta} + \beta - \beta_0)} \qquad s \in [0, \sqrt{r_0}] \tag{3.13}$$

for some constant $0 < \varepsilon < 1$. We can also assume $r_0 \leq 1$.

Let us set:

$$\sigma := \min\{1,\, a_{2,n}^{2/(\lambda - 1)},\, r_0\, a_{4,n}^{-2}, (4 a_{1,n}^2 \overline{a})^{-1/\varepsilon}\}\,,$$

$$M := \max_{|r| \leq r_0} m(r), \qquad L := \sup_{|r| \leq r_0} |m'(r)|\,.$$

Let us assume that, for a suitable $0 < \sigma_1 \leq \sigma$:

$$F_0 := F(0) + \frac{\overline{a}}{\delta} H_0 \sigma_1^\varepsilon < \sigma_1,\ LG_0 \sqrt{F_0} < \frac{\delta}{4}$$

where

$$H_0 := \frac{4}{\delta} M\sigma_1 + 2\langle B^\beta u_0, B^\beta u_1 \rangle + \delta \|B^\beta u_0\|^2$$

$$F(0) := \frac{\|B^\beta u_1\|^2}{m(\|Bu_0\|^2)} + \|B^{\beta+1} u_0\|^2$$

$$G_0 := \max\left\{ \frac{\|u_1\|}{m(\|Bu_0\|^2)}, \frac{\bar{b}}{\delta}(F_0^{(\lambda\beta_0+\varepsilon)/2} + \sqrt{F_0}) \right\}.$$

We prove that under these smallness assumptions the solution $u$ of (1.1) is a global solution.

In the following let us denote

$$c(t) = m(\|Bu(t)\|^2).$$

Let us assume that $m \in C^1([0, +\infty[; \mathbb{R})$, and let $[0, T_*[$ be the maximal interval where the solution exists.

*Step* 1. Let us define

$$F(t) := \frac{\|B^\beta u'(t)\|^2}{c(t)} + \|B^{\beta+1} u(t)\|^2 + \frac{\delta}{2}\int_0^t \frac{\|B^\beta u'(s)\|^2}{c(s)} ds.$$

Let us set

$$T := \sup\left\{ \tau \in [0, T_*[ : c(t) > 0, \left| \frac{c'(t)}{c(t)} \right| \leq \frac{\delta}{2}, F(t) \leq \sigma_1 \forall t \in [0, \tau] \right\}.$$

We show that $T = T_*$. Let us assume by contradiction that $T < T_*$. Since $|c'(t)| \leq \frac{\delta}{2} c(t)$ in $[0, T[$ we have that

$$0 < c(0)e^{-\delta T/2} \leq c(t) \leq c(0)e^{\delta T/2} \qquad t \in [0, T[. \tag{3.14}$$

Moreover, by $\|B^{\beta+1} u(t)\|^2 \leq \sigma$ we obtain:

$$\|Bu(t)\|^2 \leq a_{4,n}^2 \|B^{\beta+1} u(t)\|^2 \leq r_0 \qquad t \in [0, T[,$$

and

$$\|u(t)\|_\infty \leq \|Bu(t)\|^\lambda \qquad t \in [0, T[.$$

Since $c(\cdot)$, $c'(\cdot)$, and $F(t)$ are continuous functions, by the maximality of $T$

we have that necessarily

$$\left| \frac{c'(T)}{c(T)} \right| = \frac{\delta}{2}; \tag{3.15}$$

or

$$F(T) = \sigma_1. \tag{3.16}$$

*Step* 2. In this step we denote various constants depending only from $n$ by $c$. Let us set

$$q = \frac{n}{n-2}, \, p = \begin{cases} +\infty & \text{if } \tilde{\eta} = 0 \\ \dfrac{n}{(n-2)\,\tilde{\eta}} & \text{if } \tilde{\eta} > 0 . \end{cases}$$

Let $\beta_1, \ldots, \beta_\nu > 0$ be integers such that $\beta_1 + \ldots + \beta_\nu = \beta$.

Let us suppose that $k$ of the $\beta_j$ are equal to 1. We can assume that they are the first $k$. Let us define for $j = k+1, \ldots, \nu$:

$$\frac{1}{p_j} = \frac{2}{n-2}\left(\frac{n}{2} - \beta - 1 + \beta_j\right).$$

Now let us assume that $k \geqslant 1$ and let us set, for $\nu \geqslant 2$:

$$\frac{1}{p_{0,\nu}} = \frac{1}{k}\left(1 - \frac{2}{n-2}\left(\frac{n}{2}(\nu - k) + \beta(1 - \nu + k) - \nu\right)\right).$$

Using the Sobolev inequalities we have then:

$$\|B^\beta u\|_{2q} \leqslant c\|B^{\beta+1}\|, \qquad \|B^{\beta_j} u\|_{2qp_j} \leqslant c\|B^{\beta+1}\| \quad j = k+1, \ldots, \nu .$$

Furthermore, since:

$$0 < \theta = \left(\frac{1}{2} - \frac{1}{2qp_{0,\nu}}\right)\frac{n}{\beta} = \frac{1-\nu}{k} + 1 + (\nu - 1)\frac{n-2}{2\beta k} < 1$$

we have:

$$\|Bu\|_{2qp_{0,\nu}} \leqslant c\|Bu\|^{1-\theta}\|B^{\beta+1}u\|^{\theta}.$$

By $k/p_{0,\nu} + 1/p_{k+1} + \ldots + 1/p_\nu = 1$ we can then also deduce:

$$\|B^{\beta_1} u \ldots B^{\beta_\nu} u\|_{2q}^2 \leqslant \begin{cases} c\|B^{1+\beta} u\|^{2\nu} & \text{if } k = 0 \\ c\|Bu\|^{2(\nu-1)\lambda} \|B^{\beta+1} u\|^{2+2(\nu-1)(1-\lambda)} & \text{if } k \geqslant 1 . \end{cases}$$

We are now able to estimate $\|B^\beta f(u)\|$.

Since, for all $b = (b_1, \ldots, b_n)$ we have:

$$\partial^b f(u) = \sum_{\nu=1}^{|b|} \sum_{\substack{B_1 + \ldots + B_\nu = b \\ |B_i| > 0}} c_{b, B_1, \ldots, B_\nu} f^{(\nu)}(u)\, \partial^{B_1} u \ldots \partial^{B_\nu} u$$

then:

$$\|B^\beta f(u)\|^2 \leqslant c \sum_{\nu=1}^{\beta} \sum_{\substack{\beta_1 + \ldots + \beta_\nu = \beta, \\ \beta_i > 0}} \|f^{(\nu)}(u)\, B^{\beta_1} u \ldots B^{\beta_\nu} u\|^2$$

$$\leqslant c\mu_f(\|Bu\|^\lambda)^2 \sum_{\nu=1}^{\beta} \sum_{\substack{\beta_1 + \ldots + \beta_\nu = \beta \\ \beta_i > 0}} \|u^{\eta+\beta-\nu} B^{\beta_1} u \ldots B^{\beta_\nu} u\|^2$$

$$\leqslant c\mu_f(\|Bu\|^\lambda)^2 \|u\|_{2n/(n-2)}^{2\tilde{\eta}} \times$$

$$\times \sum_{\nu=1}^{\beta} \sum_{\substack{\beta_1 + \ldots + \beta_\nu = \beta \\ \beta_i > 0}} \|u\|_\infty^{2(\eta - \tilde{\eta} + \beta - \nu)} \|B^{\beta_1} u \ldots B^{\beta_\nu} u\|_{2q}^2$$

Now let us remark that if $\beta \geqslant 2$ and $\nu \leqslant \beta_0$ then there exists $\beta_1 + \ldots + \beta_\nu = \beta$ with $\beta_i \geqslant 2$ for all $i = 1, \ldots, \nu$. Moreover if $\nu > \beta_0$ and $\beta_1 + \ldots + \beta_\nu = \beta$ then at least one of the $\beta_i$ is equal to 1. Furthermore:

$$2\lambda(\beta_0 - \nu) + 2(\nu - 1)\lambda = 2\lambda(\beta_0 - 1) \geqslant 0 .$$

Hence, using $\|Bu\| \leqslant a_{4,n} \|B^{1+\beta} u\| \leqslant 1$ we obtain:

$$\|B^\beta f(u)\|^2 \leqslant cc(t)^2 \|Bu\|^{2\varepsilon} \|B^{\beta+1} u\|^2 \sum_{\nu=1}^{\beta} \|B^{1+\beta} u\|^{2\lambda(\beta_0 - \nu) + 2(\nu-1)} \tag{3.17}$$

$$\leqslant \bar{a} c^2(t) \|Bu\|^{2\varepsilon} \|B^{\beta+1} u\|^2$$

$$\leqslant \bar{a} c^2(t)\, \sigma_1^\varepsilon \|B^{\beta+1} u\|^2 \leqslant \frac{1}{4 a_{1,n}^2} c^2(t) \|B^{\beta+1} u\|^2 .$$

*Step* 3. By applying to the equation (1.1) the operator $B^{\beta-1}$, taking the scalar product of the obtained equation with $B^{\beta+1}$, and integrating on $[0, T]$, we obtain:

$$\int_0^T c(t)\|B^{\beta+1}u(t)\|^2\,dt + \int_0^T \langle B^\beta f(u(t)), B^\beta u(t)\rangle\,dt =$$

$$= \int_0^T \|B^\beta u'(t)\|^2\,dt - \langle B^\beta u'(T), B^\beta u(T)\rangle - \frac{\delta}{2}\|B^\beta u(T)\|^2 +$$

$$+\langle B^\beta u_1, B^\beta u_0\rangle + \frac{\delta}{2}\|B^\beta u_0\|^2$$

$$\leq M\left(\int_0^T \frac{\|B^\beta u'(t)\|^2}{c(t)}\,dt + \frac{1}{2\delta}\,\frac{\|B^\beta u'(T)\|^2}{c(T)}\right) + \frac{H_0}{2} - \frac{2}{\delta}M\sigma_1 \leq \frac{H_0}{2}.$$

Hence, using (3.17):

$$\int_0^T c(t)\|B^{\beta+1}u(t)\|^2\,dt \leq H_0. \tag{3.18}$$

*Step* 4. We prove that (3.16) is false. By a simple calculation using (3.17) in $[0, T[$ we have:

$$F'(t) \leq \frac{1}{\delta}\,\frac{\|B^\beta f(u(t))\|^2}{c(t)} \leq \frac{\bar{a}}{\delta}\sigma_1^\varepsilon c(t)\|B^{\beta+1}u(t)\|^2$$

hence, by (3.18):

$$F(t) \leq F(0) + \frac{\bar{a}}{\delta}\sigma_1^\varepsilon H_0 = F_0 < \sigma_1. \tag{3.19}$$

*Step* 5. We prove that (3.15) is false. Let us firstly remark that, since $f(u) = \frac{f^{(\beta)}(\xi_u)}{\beta!}u^\beta$, hence:

$$\|f(u(t))\|^2 \leq \begin{cases} c_0\,\dfrac{\mu_f(\|Bu(t)\|^\lambda)^2}{\beta!}\|u\|_\infty^{2\beta} & \text{if } \eta = 0 \\[2ex] \dfrac{\mu_f(\|Bu(t)\|^\lambda)^2}{\beta!}\|u\|_\infty^{2(\beta+\eta-\tilde{\eta})}\|u\|_{2\tilde{\eta}}^{2\tilde{\eta}} & \text{otherwise}. \end{cases}$$

Therefore by (3.13):

$$\|f(u(t))\|^2 \leq \bar{c}^2 c^2(t)\|B^{\beta+1}u(t)\|^{2(\varepsilon+\lambda\beta_0)}. \tag{3.20}$$

We can now easy estimate $G(t) := \dfrac{\|u'(t)\|}{c(t)}$ as follows:

$$(G^2(t))' \leqslant - \delta G^2(t) + 2G\left(\|B^2 u(t)\| + \frac{\|f(u(t))\|}{c(t))}\right)$$

$$\leqslant - G(t)(\delta G(t) - \bar{b}(\sqrt{F_0} + F_0^{(\varepsilon + \lambda\beta_0)/2})),$$

hence, by a standard ODE's inequality we obtain $G(t) \leqslant G_0$. Then as in proof of case $n = 1, 2$, step 4:

$$\left| \frac{c'(T)}{c(T)} \right| \leqslant 2LG_0\sqrt{F_0} < \frac{\delta}{2}.$$

*Step* 5. We can conclude as in step 5 of proof of case $n = 1, 2$. ■

3.2. *Asymptotic behaviour.*

In order to study the asymptotic behaviour of the solutions of (1.1), we consider the linearized problem

$$\text{(3.21)} \qquad \begin{cases} v''(t) + \delta v'(t) + c(t)\, B^2 v(t) + f(t, x) = 0\,, & t \geqslant 0\,, \\ v(0) = v_0, \quad v'(0) = v_1. \end{cases}$$

In the following lemma we examine the asymptotic behaviour of the solutions of (3.21).

LEMMA 3.1. *Let* $\delta > 0$. *Let* $c : [0, +\infty[ \to ]0, +\infty[$ *be a Lipschitz continuous bounded function such that*

$$\left| \frac{c'(t)}{c(t)} \right| \leqslant \frac{\delta}{2} \quad \textit{for a.e. } t \geqslant 0\,.$$

*Let* $f : [0, +\infty[ \times \Omega \to \mathbb{R}$ *be a continuous function such that* $f(t, \cdot) \in D(B^\beta)$ *for all* $t \geqslant 0$ *and*

$$\int_0^{+\infty} \frac{1}{c(s)} \|B^\beta f(s)\|^2 ds < +\infty\,, \qquad \sup_{t \geqslant 0} \frac{\|f(t)\|}{c(t)} < +\infty\,.$$

*Let* $v$ *be the unique global solution of* (3.21) *with* $(v_0, v_1) \in D(B^{\beta+1}) \times D(B^\beta)$.

*Then there exists* $v_\infty \in D(B^{\beta+1})$ *such that*

(3.22) $$v(t) \to v_\infty \quad \text{in } D(B^{\beta+1}),$$

(3.23) $$v'(t) \to 0 \quad \text{in } D(B^{\beta}),$$

*as* $t \to \infty$. *Furthermore, if* $v_\infty \neq 0$, *then necessarily* $c(t) \to 0$ *as* $t \to \infty$.

PROOF OF LEMMA 3.1. We only give a sketch of the proof, we refer to [5] for the details.

*Step* 1. Let us consider the function

$$H(t) := \frac{\|B^\beta v'(t)\|^2}{c(t)} + \|B^{\beta+1} v(t)\|^2 - \frac{1}{\delta}\int_0^t \frac{1}{c(s)}\|B^\beta f(s)\|^2 ds .$$

A simple computation shows that

(3.24) $$H'(t) \leqslant -\frac{\delta}{2}\frac{\|B^\beta v'(t)\|^2}{c(t)}.$$

By this fact we obtain:

1. for all $t \geqslant 0$:

$$\frac{\|B^\beta v'(t)\|^2}{c(t)} + \|B^{\beta+1} v(t)\|^2 + \frac{\delta}{2}\int_0^t \frac{\|B^\beta v'(s)\|^2}{c(s)} ds \leqslant$$

$$\leqslant \frac{\|B^\beta v_1\|^2}{c(0)} + \|B^{\beta+1} v_0\|^2 + \int_0^{+\infty} \frac{1}{\delta c(s)}\|B^\beta f(s, \cdot)\|^2 ds =: \gamma_0.$$

2. Since the function $c(\cdot)$ is bounded then:

(3.25) $$\int_0^{+\infty} \|B^\beta v'(t)\|^2 dt < +\infty$$

3. The function $H$ is non-increasing, hence there exists:

$$F_\infty := \lim_{t\to\infty} \frac{\|B^\beta v'(t)\|^2}{c(t)} + \|B^{\beta+1} v(t)\|^2.$$

If $F_\infty = 0$, then (3.22) holds true with $v_\infty = 0$. Since the function $c$ is bounded, then also (3.23) follows from $F_\infty = 0$.

Therefore from now on we assume that $F_\infty > 0$.

*Step* 2. We show that

$$\int_0^\infty c(t)\|B^{\beta+1}v(t)\|^2\,dt < +\infty. \tag{3.26}$$

Indeed, applying the operator $B^{\beta-1}$ to the equation (3.21) and taking its scalar product with $B^{\beta+1}v$ and integrating on $[0, T]$, it follows that

$$\int_0^T c(t)\|B^{\beta+1}v(t)\|^2\,dt \leq \langle B^\beta v_1, B^\beta v_0\rangle + \left(\frac{2}{\delta}\|c\|_\infty + \frac{\delta a_{1,n}^2}{2}\right)\gamma_0 + $$

$$+\frac{1}{2}\int_0^T c(t)\|B^{\beta+1}u(t)\|^2\,dt + \frac{\delta}{2}\|B^\beta v_0\|^2.$$

Hence

$$\int_0^T c(t)\|B^{\beta+1}v(t)\|^2\,dt \leq 2\langle B^\beta v_1, B^\beta v_0\rangle + \delta\|B^\beta v_0\|^2 + 2\left(\frac{2}{\delta}\|c\|_\infty + \frac{\delta a_{1,n}^2}{2}\right)\gamma_0.$$

Passing to the limit as $T \to \infty$, we obtain (3.26).

*Step* 3. From (3.25) and (3.26) it follows that

$$\int_0^\infty c(t)\left(\frac{\|B^\beta v'(t)\|^2}{c(t)} + \|B^{\beta+1}v(t)\|^2\right)dt < +\infty.$$

Since, for $t \geq \overline{T}$:

$$\frac{\|B^\beta v'(t)\|^2}{c(t)} + \|B^{\beta+1}v(t)\|^2 \geq \frac{F_\infty}{2} > 0,$$

then also

$$\int_0^\infty c(t)\,dt < +\infty. \tag{3.27}$$

Since $c(\cdot)$ is Lipschitz continuous, it follows that $c(t) \to 0$ as $t \to \infty$. Since $\|B^\beta v'(t)\|^2 \leq c(t)\gamma_0$, then (3.23) is proved.

*Step* 4. We show that (3.22) holds true with the additional assumptions that $(v_0, v_1) \in D(B^{\beta+3}) \times D(B^{\beta+2})$, $f(t, \cdot) \in D(B^{\beta+2})$ for every $t$ and

$$\int_0^{+\infty} \frac{\|B^{\beta+2} f(t)\|}{c(t)} dt < +\infty, \qquad \sup_{t \geq 0} \frac{\|B^{\beta+1} f(t)\|}{c(t)} < +\infty. \tag{3.28}$$

To this end, let us introduce the function

$$\widehat{H}(t) := \frac{\|B^{\beta+2} v'(t)\|^2}{c(t)} + \|B^{\beta+3} v(t)\|^2 - \frac{1}{\delta} \int_0^t \frac{1}{c(s)} \|B^{\beta+2} f(s)\|^2 ds.$$

As in Step 1, it is possible to prove that $\widehat{H}$ is non-increasing, and that for every $t \geq 0$:

$$\|B^{\beta+3} v(t)\|^2 \leq \widehat{\gamma}_0.$$

Now let us consider the function $\widehat{G}(t) := \frac{\|B^{\beta+1} v'(t)\|}{c(t)}$. By a standard ODE's inequality, it follows that

$$\widehat{G}(t) \leq \max\left\{\widehat{G}(0), \frac{2}{\delta}\left(\sqrt{\widehat{\gamma}_0} + \sup_{t \geq 0} \frac{\|B^{\beta+1} f(t)\|}{c(t)}\right)\right\}.$$

By (3.27), this implies that

$$\int_0^{\infty} \|B^{\beta+1} v'(t)\| dt < +\infty$$

and therefore $B^{\beta+1} v(t)$ has a limit as $t \to \infty$.

*Step* 5. We show that (3.22) hold true for every initial data $(v_0, v_1) \in D(B^{\beta+1}) \times D(B^{\beta})$.

To this end, let us consider a sequence $\{(v_{0n}, v_{1n})\} \subseteq D(B^{\beta+3}) \times D(B^{\beta+2})$ converging to $(v_0, v_1)$ in $D(B^{\beta+1}) \times D(B^{\beta})$ and $f_n$ as in step 4, with:

$$\int_0^{+\infty} \frac{1}{c(t)} \|B^{\beta}(f(t) - f_n(t))\|^2 dt \to 0 \quad \text{as} \quad n \to +\infty.$$

Let $\{v_n\}$ be the corresponding solutions of (3.21), and let us set $w_n :=$

$:= v - v_n$. Since $w_n$ is a solution of (3.21), with $f - f_n$ in place of $f$, we have that

$$\frac{\|B^\beta w_n'(t)\|^2}{c(t)} + \|B^{\beta+1} w_n(t)\|^2 \leq \frac{\|B^\beta (v_{1,n} - v_1)\|^2}{c(0)} + \|B^{\beta+1}(v_{0,n} - v_0)\|^2 + $$

$$+ \frac{1}{\delta} \int_0^{+\infty} \frac{1}{c(t)} \|B^\beta (f(t) - f_n(t))\|^2 \, dt .$$

This proves that $\{B^{\beta+1} v_n\} \to B^{\beta+1} v$ uniformly in $[0, +\infty[$. Since $B^{\beta+1} v_n(t)$ has a limit as $t \to \infty$ for every $n \in \mathbb{N}$ (see Step 4), then necessarily $B^{\beta+1} v(t)$ has a limit as $t \to \infty$.

This completes the proof of (3.22). ■

PROOF OF THEOREM 2.3. We use the same notations as in the proof of Theorem 2.2 case $n = 1, 2$ (resp. case $n \geq 3$). Let us firstly remark that $u$ is the solution of (3.21) with

$$c(t) = m(\|Bu(t)\|^2), \qquad (v_0, v_1) = (u_0, u_1), \qquad f(t, x) = f(u(t, x)).$$

In Step 1 of the proof of Theorem 2.2 case $n = 1, 2$ (resp. Step 1 of case $n \geq 3$ ), we showed that $c(t) > 0$ for every $t \geq 0$ (this proves statement (i)), and

$$\left| \frac{c'(t)}{c(t)} \right| \leq \frac{\delta}{2} \qquad \forall t \geq 0 .$$

Moreover in this step we proved also that $\|Bu\| \leq r_0$, hence $c(\cdot)$ is bounded. Since $m$ is locally Lipschitz continuous, and $\|B^\beta u'(t)\|^2 \leq$ $\leq F(t) c(t) \leq F_0 c(t)$ (see (3.10) (resp. (3.19))), then it turns out that $c(\cdot)$ is globally Lipschitz continuous. Finally, by (3.8) (3.9), (3.11) (resp. (3.17), (3.18), (3.20)):

$$\int_0^{+\infty} \frac{\|B^\beta f(u(t))\|^2}{c(t)} \, dt \leq \begin{cases} \tilde{c} \displaystyle\int_0^{+\infty} c(t) \|Bu(t)\|^2 < +\infty & \text{if } n = 1, 2 \\ \tilde{c} \displaystyle\int_0^{+\infty} c(t) \|B^{\beta+1} u(t)\|^2 < +\infty & \text{if } n \geq 3 \end{cases}$$

$$\frac{\|f(u(t))\|^2}{c^2(t)} \leqslant \tilde{c}\|B^{\beta+1}u(t)\|^2 < c_0$$

for some $c_0$ independent from $t$.

By Lemma 3.1, there exists $u_\infty \in D(B^{\beta+1})$ such that $u \to u_\infty$ in $D(B^{\beta+1})$ and $u' \to 0$ in $D(B^\beta)$. Let us assume that $u_\infty \neq 0$, then by the last statement of Lemma 3.1 we have that $c(t) \to 0$ as $t \to \infty$, hence

$$0 = \lim_{t \to \infty} m(\|Bu(t)\|^2) = m(\|Bu_\infty\|^2).$$

Since $\|Bu_\infty\|^2 \leqslant r_0$, hence must be $u_\infty = 0$. Furthermore, by applying $B^{\beta-1}$ to the equation (1.1), $u'' \to 0$ in $D(B^{\beta-1})$. ■

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## REFERENCES

[1] A. Arosio, *Averaged evolution equations. The Kirchhoff string and its treatment in scales of Banach spaces*, proceedings of the «$2^{nd}$ workshop on functional-analytic methods in complex analysis» (Trieste, 1993), World Scientific, Singapore.

[2] E. H. de Brito, *The damped elastic stretched string equation generalized: existence, uniqueness, regularity and stability*, Applicable Analysis, **13** (1982), pp. 219-233.

[3] E. H. de Brito, *Decay estimates for the generalized damped extensible string and beam equation*, Nonlinear Analysis, 8 (1984), pp. 1489-1496.

[4] P. D'Ancona - S. Spagnolo, *Nonlinear perturbations of the Kirchhoff equation*, Comm. Pure Appl. Math., **47** (1994), pp. 1005-1029.

[5] M. Ghisi, *Global solutions to some nonlinear dissipative mildly degenerate Kirchhoff equations*, Preprint Dip. Mat. Univ. Pisa N° 2.293.1099 (1998).

[6] M. Ghisi - M. Gobbino, *Global Existence for a Mildly Degenerate Dissipative Hyperbolic Equation of Kirchhoff Type*, Preprint Dip. Mat. Univ. Pisa (1997).

[7] M. Hosoya - Y. Yamada, *On some nonlinear wave equations II: global existence and energy decay of solutions*, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math., **38** (1991), pp. 239-250.

[8] R. Ikehata, *A note on the global solvability of solutions to some nonlinear wave equations with dissipative terms*, Differential Integral Equations, 8 (1995), pp. 607-616.

[9] M. Nakao, *A Difference Inequality and its Application to Nonlinear Evolution Equations*, J. Math. Soc. Japan, **30** (1978), pp. 747-762.

[10] K. Nishihara, *Global Existence and Asymptotic Behaviour of the Solution of Some Quasilinear Hyperbolic Equation with Linear Damping*, Funkcial. Ekvac., **32** (1989), pp. 343-355.

[11] K. NISHIHARA - Y. YAMADA, *On Global Solutions of some Degenerate Quasilinear Hyperbolic Equations with Dissipative Terms*, Funkcial. Ekvac., **33** (1990), pp. 151-159.

[12] K. ONO, *Global Existence and Decay Properties of Solutions of Some Mildly Degenerate Nonlinear Dissipative Kirchhoff Strings*, Funkcial. Ekvac., **40** (1997), pp. 255-270.

[13] K. ONO, *Global Existence, Decay and Blowup of Solutions for Some Mildly Degenerate Nonlinear Kirchhoff Strings*, J. Diff. Eq., **137** (1997), pp. 273-301.

[14] Y. YAMADA, *On some quasilinear wave equations with dissipative terms*, Nagoya Math. J., **87** (1982), pp. 17-39.

[15] S.SPAGNOLO, *The Cauchy problem for the Kirchhoff equation*, Rend. Sem. Fis. Matem. di Milano, **62** (1992), pp. 17-51.

[16] R. TEMAM, *Infinite Dimensional Dynamical Systems in Mechanics and Physics«*, Applied Mathematical Sciences, **68**, Springer-Verlag, New York, 1988.

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