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On the Existence of Special Divisors on Singular Curves.

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1. Introduction.

Let X be an integral projective curve with $g := p_a(X) \geq 2$ defined over an algebraically closed field $\boldsymbol{K}$. We want to study the Brill-Noether theory of special line bundles on X. However, if $L \in \mathrm{Pic}(X)$, $h^0(X, L) \geq 2$ but L is not spanned by its global sections at some point of $\mathrm{Sing}(X)$ the subsheaf, L', of L generated by $H^0(X, L)$ may be not locally free but only torsion free. This observation explains why in the theory of special line bundles on singular curves one has to consider also torsion free sheaves. For all positive integers d, r set $W_d^r(X) :=$ {rank 1 torsion free sheaves L on X with $\deg(L) = d$ and $h^0(X, L) \geq r+1$} and $W_d^r(X,^{**}) := W_d^r(X) \cap \mathrm{Pic}(X)$. The definitions of the sets $G_d^r(X)$ and $G_d^r(X,^{**})$ are completely analogous. It is an elementary fact that, under weak hypotheses, if $\varrho(g, r, d) := g - (r+1)(g-d+r) \geq 0$ then $W_d^r(X) \neq \emptyset$ (see Proposition 2.1). On the other hand we will give an explicit example of a singular curve X with $W_d^r(X,^{**}) = \emptyset$ although $\varrho(g, r, d) \geq 0$. The singularities of such a curve are very mild (just ordinary nodes in Example 2.2 or nodes and cusps in the variation described in Remark 2.3). These examples motivate the search for existence theorems for $W_d^r(X,^{**})$ when the curve X has only A_k-singularities under assumptions stronger than just $\varrho(g, r, d) \geq 0$. See Proposition 2.4 for a result in this direction based on the clasification of torsion free modules on the completion of the local ring of an A_k-singularity ([GK]). The main re-

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sults of this paper are existence theorems for Cartier special divisors (i.e. $W_d^r(X,^{**}) \neq \emptyset$) on «general curves with prescribed singularities. In section 3, after fixing the notation and giving a few remarks, we will prove two results for $W_d^r(X,^{**})$ (see Theorems 3.3 and 3.5) and an easier one for $W_d^r(X)$ (Corollary 3.4). Next we will construct rank 1 torsion free sheaves on X with low degree, many sections and such that the formal completion of the sheaf at each point $P \in \mathrm{Sing}(X)$ is a prescribed rank 1 torsion free $O_{X,P^\wedge}$-module. A key tool for our proofs will be the notion and properties of generalized parabolic bundles (or just generalized parabolic line bundles) considered in [Bh1], [Bh2] and [Co].

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2. The basic example.

Using the existence theorem for special divisors on smooth projective curves ([ACGH], Ch, IV) it is very easy to prove the following result which is well-known to the specialists.

PROPOSITION 2.1. *Fix integers g, r, d with $g \geq 2$, $d > 0$, $r > 0$ and $\varrho(g, d, r) \geq 0$. Let X be an integral projective curve with $p_a(X) = g$ and whose only singularities are smoothable singularities. Then $W_d^r(X) \neq \emptyset$ and* $\dim(W_d^r(X)) \geq \varrho(g, r, d)$.

PROOF. Since X has only smoothable singularities, X is a flat limit of an integral family of connected smooth curves of genus g. Hence it is sufficient to use the existence of the relative generalized Jacobian for flat families of integral projective curves ([AKI]), The semicontinuity of cohomology and the semicontinuity of the fiber dimension for proper maps.

Notice that every rank 1 torsion free sheaf on an integral curve X is the flat limit of a family of line bundles if and only if X has only planar singularities (see [Re] or [AKI] and references therein).

Let $k \geq 3$ be an integer and let C be a smooth curve of genus $g = 2k - 2$. Then $\varrho(g, l, k) = 0$ and the gonality of C is at most k. But if C is singular this is not always the case, as we can see by taking $q = z = k$ in the following example.

EXAMPLE 2.2. Fix integers g, z and q with $q \geq z \geq 2$ and $z + 2q \leq 2g + 1$. Let X be a smooth hyperelliptic curve of genus q, $\sigma : X \to X$ the hyperelliptic involution, $R \in \mathrm{Pic}(X)$ the hyperelliptic line bundle and $f: X \to \boldsymbol{P}^1$ the associated degree 2 covering. Fix $2g-2q$ distinct points P_i, Q_i, $1 \leqslant i \leqslant g-q$, of X with $f(P_i) \neq f(Q_i)$ for every i. Let Y be the nodal curve with $p_a(Y) = g$ obtained from X by gluing together the point P_i and the point Q_i, $1 \leqslant i \leqslant g-q$. Call $\pi : X \to Y$ the associated birational morphism. Here we will show that there is no $L \in \mathrm{Pic}^t(Y)$ with $t \leqslant z$ and $h^0(Y, L) \geqslant 2$.

Assume the existence of such an $L \in \mathrm{Pic}^t(Y)$ and set $M := \pi^*(L)$. Taking t minimal we may assume $h^0(Y, L) = 2$ and that either L is base-point-free or the scheme-theoretic base locus of L does not contain any Cartier divisor of Y. Let F be the subsheaf of L spanned by $H^0(Y, L)$. By the minimality of t we have $\mathrm{Supp}(L/F) \subseteq \mathrm{Sing}(Y)$ and the sheaf F is not locally free at any point of $\mathrm{Supp}(L/F)$. Set $N := \pi^*(F)/\mathrm{Tors}(\pi^*(F))$.

Hence $N \in \mathrm{Pic}(X)$. Call M' (resp. N') the base point free part of M (resp N). It is easy to check working in a Zariski dense open subset $f^{-1}(Y_{\mathrm{reg}})$ that the induced map $H^0(Y, F) \to H^0(X, M)$ is injective. Hence $h^0(X, M) \geqslant 2$, and the base point free part, M', of M induces a non-constant morphism $u : X \to \boldsymbol{P}(H^0(X, M')) \cong \boldsymbol{P}(H^0(X, M))$.

Since $\deg(M') \leqslant \deg(M) = t \leqslant q = p_a(X)$ and X is hyperelliptic, u factors through the double covering f. Hence the rational map α from Y into $\boldsymbol{P}^1$ induced by $H^0(Y, L)$ factors through the rational degree 2 map induced by f. since $\sigma(P_i) \neq Q_i$ for every i, we see that α cannot be defined at any point $\pi(P_i)$, $1 \leqslant i \leqslant g-q$. Thus $\mathrm{Sing}(Y) \subseteq \mathrm{Supp}(L/F)$ and hence $\mathrm{Supp}(L/F) = \mathrm{Sing}(Y)$. In particular, $\deg(F) \leqslant \deg(L) - \mathrm{card}(\mathrm{Sing}(Y)) = t - g + q$. Furthermore, F is not locally free at any point of $\mathrm{Sing}(Y)$.

By [EHKS], Lemma 1 of the appendix with J. Harris, we have $\deg(N) \leqslant \deg(F) - \mathrm{card}(\mathrm{Sing}(Y)) \leqslant t - 2g + 2q \leqslant z + 2q - 2g \leqslant 1$. Since $h^0(Y, N) \geqslant 2$ and X is not rational, this is impossible.

Remark 2.3. Take q, X, P_i $(1 \leqslant i \leqslant g-q)$ and Q_i $(1 \leqslant i \leqslant q)$ as in Example 2.2. Fix an integer h with $1 \leqslant h \leqslant g-q$ and assume that $\sigma(P_i) \neq P_i$ for every $i \leqslant h$, i.e. assume that no P_i, $1 \leqslant i \leqslant h$, is a ramification of the degree 2 pencil $f: X \to \boldsymbol{P}^1$. Let Y be curve with $p_a(Y) = g$ obtained from X by gluing together the point P_i and the point Q_i, $h+1 \leqslant i \leqslant g-q$, and creating an ordinary cusp at the point P_j, $1 \leqslant j \leqslant h$. Thus Y has h ordinary cusps and $g-q-h$ ordinary nodes. The proof of Example 2.2 shows that there is no $L \in \mathrm{Pic}^t(Y)$ with $t \leqslant z$ and $h^0(Y, L) \geqslant 2$.

We conclude this section with the promised existence result.

PROPOSITION 2.4. *Assume* $\mathrm{char}(\boldsymbol{K}) = 0$. *Fix integers* d, g, r *with* $r > 0$ $g \geqslant 2$ *and* $2 \leqslant d \leqslant 2g-2$. *Let* C *be an integral projective curve with* $p_a(C) = g$ *and only with singularities of type* A_k. *Let* Y *the the normalization of* C. *Set* $\delta := p_a(C) - p_a(Y)$.

(*a*) *If* $\varrho(g, r, d-\delta) \geqslant 0$, *then* $W_d^r(C, ^{**}) \neq \emptyset$.

(*b*) *If* C *has only ordinary nodes and ordinary cusps,* Y *has general moduli and* $\varrho(g, r, d-\delta+1) \geqslant 0$, *then* $W_d^r(C, ^{**}) \neq \emptyset$.

PROOF. By 2.1 $W_{d-\delta}^r(C) \neq \emptyset$ and $\dim(W_{d-\delta}^r(C)) \geqslant \varrho(g, r, d-\delta)$. Fix $F \in W_{d-\delta}^r(C)$ and look at its stalks at the singular points of C. Fix $P \in \mathrm{Sing}(C)$, say a singularity of type A_m. If F is not locally free at P, then its completion is isomorphic as a module over the completion of the local ring $\boldsymbol{O}_{C,P}$ to the local ring of a singularity of type A_k for some integer k with $0 \leqslant k < m$ by the classification of torsion free modules on complete local rings corresponding to A_m singularities (see [GK] or [Co], p. 24). It follows that we may enlarge the stalks of F at each point of $\mathrm{Sing}(C)$ and obtain $L \in \mathrm{Pic}(C)$ with $\deg(L) \leqslant \deg(F) + \delta = d$ and $F \subseteq L$ (and hence $h^0(C, L) \geqslant h^0(C, F) \geqslant r+1$), proving part ($a$). Now make the assumptions of part (b) and call $f\colon Y \to C$ the normalization map. We may obviously assume $\delta > 0$. The proof of part (a) works unless every $F \in W_{d-\delta+1}^r(C)$ is non-locally free at each point of $\mathrm{Sing}(C)$. Assume that this is the case and fix $F \in W_{d-\delta+1}^r(C)$. By the classification of rank 1 torsion free sheaves on nodal or cuspidal singularities (a very particular case of [GK] or [Co], p. 24) there is $L \in \mathrm{Pic}(Y)$ with $\deg(L) = \deg(F) - \delta = d - 2\delta + 1$ and $F \cong f_*(L)$. Hence $h^0(Y, L) = h^0(C, F)$. Since Y has general moduli we have $\dim(W_{d-2\delta+1}^r(Y)) = \varrho(g-\delta, r, d-2\delta+1)$ by the clasical Brill-Noether theory for curve with general moduli ([ACGH], Ch, IV). Since this is true for every F we obtain $\dim(W_{d-\delta+1}^r(C) \leqslant \dim(\mathrm{W}_{\mathrm{d}-2\delta+1}^{\mathrm{r}}(\mathrm{Y})) = \varrho(\mathrm{g}-\delta, \mathrm{r}, \mathrm{d}-2\delta+1) = \varrho(\mathrm{g}, \mathrm{r}, \mathrm{d}-\delta+1) - \delta < \varrho(\mathrm{g}, \mathrm{r}, \mathrm{d}-\delta+1)$, contradicting 2.1.

3. The main result.

Let X be an integral projective curve and $\pi : Y \to X$ its normalization. For every $P \in \mathrm{Sing}(X)$, let $\boldsymbol{P}'_P := \Pi_{Q \in \pi^{-1}(P)} \boldsymbol{O}_{Y,Q}$ be the integral closure of $\boldsymbol{O}_{X,P}$ in its total ring of fractions and C_P the conductor of $\boldsymbol{O}_{X,P}$ in $\boldsymbol{O}'_P$. We will add a superscript ^ to denote the corresponding objects for the formal completion of $\boldsymbol{O}_{X,P}$. Set $\boldsymbol{I}_P := C_P$ if X is either Gorenstein or unibranch at P and $\boldsymbol{I}_P := C_{P^2}$ otherwise; this choice is motivated by [Co], Remark 2 at p. 21, and (in the unibrach case) by the key result [GP], 1.4; the sheaf $\boldsymbol{I}_P$ defines the ideal sheaf of an effective divisor, D_P, of Y supported on $\pi^{-1}(P)$ and with $\deg(D_P) = \dim_{\boldsymbol{K}}(\boldsymbol{O}_{X,P}/\boldsymbol{I}_P) = \dim_{\boldsymbol{K}}(\boldsymbol{O}^{\wedge}_{X,P}/\boldsymbol{I}^{\wedge}_P)$; let $\delta''(Q)$ be the degree of the connected component of D_P supported by Q; hence $\delta''(Q) > 0$ for every $Q \in \pi^{-1}(P)$ and $\Sigma_{Q \in \pi^{-1}(P)} \delta''(Q) = \deg(D_P)$. Set

$$\delta(P) := \dim_{\boldsymbol{K}}(\boldsymbol{O}'_P/\boldsymbol{O}_{X,P}), \qquad \delta'(P) := \dim_{\boldsymbol{K}}(\boldsymbol{O}'_P/C_P),$$

$$\delta''(P) := \dim_{\boldsymbol{K}}(\boldsymbol{O}_{X,P}/\boldsymbol{I}_P) = \Sigma_{Q \in \pi^{-1}(P)} \delta''(Q), \qquad \delta(X) := \Sigma_{P \in \mathrm{Sing}(X)} \delta(P),$$

$$\delta'(X) := \Sigma_{P \in \mathrm{Sing}(X)} \delta'(P) \quad \text{and} \quad \delta''(X) := \Sigma_{P \in \mathrm{Sing}(X)} \delta''(P).$$

Hence if X is Gorenstein we have $\delta''(X) = \delta'(X) = 2\delta(X)$. Set $g'' := g - \delta(X) = p_a(Y)$.

DEFINITION 3.1. Fix a finite number of complete one-dimensional local $\boldsymbol{K}$-algebras, say $\{R_{P^\wedge}\}_{P \in S}$, S a finite set; we will call $\{R_{P^\wedge}\}_{P \in S}$, S a finite set; we will call $\{R_{P^\wedge}\}_{P \in S}$ the formal rank 1 datum of a curve; call $\{\delta(P)\}_{P \in S}$, $\{\delta'(P)\}_{P \in S}$ and $\{\delta''(P)\}_{P \in S}$ the corresponding invariants. Fix an integer $g'' \geq 0$ and set $g := g'' \Sigma_{P \in S} \delta(P)$. Let X be an integral projective curve with $p_a(X) = g$ and a fixed bijection between $\mathrm{Sing}(X)$ and S such that $\boldsymbol{O}_{X,P^\wedge} \cong R_{P^\wedge}$ for every $P \in \mathrm{Sing}(X)$ (with an obvious abuse of notation). Let $\pi : Y \to X$ be the normalization of X. We will say that X is general for the fixed formal rank 1 datum $\{R_{P^\wedge}\}_{P \in S}$ if Y is a general smooth curve of genus g'' and $\pi^{-1}(\mathrm{Sing}(X))$ are general points of Y. We will say that X is general for the fixed normalization Y and the fixed formal rank 1 datum $\{R_{P^\wedge}\}_{P \in S}$ if $\pi^{-1}(\mathrm{Sing}(X))$ are general points of Y.

REMARK 3.2. Assume X Gorenstein and $G := p_a(X) \geq 2$. The line bundle ω_X is spanned, i.e. the canonical map $u_\omega \colon X \to \boldsymbol{P}^{g-1}$ is a morphism ([Ca], Th. D, or [R]). The canonical map u_ω is not birational if and only if it is a two-to-one morphism and in this case X is «hyperelliptic» ([Ca], Prop. 3.10, or [R]). If X is not hyperelliptic, then u_ω is an embedding ([R], Th. 15, or [Ha], Th. 1.6). For a discussion of this topic, even in the non-Gorenstein case, see [S]. Fix positive integers, d, r with $\varrho(g, r, d) \geq 0$ and assume X not hyperelliptic. The standard proof of the inequality $\dim(G^r_d(X, {**})) \geq \varrho(g, r, d)$ given in [GH], p. 260, just uses $u_\omega(X)$ and the Grassmannian $G(g - d + r, g)$ of all projective subspaces of dimension $d - r - 1$ of $\boldsymbol{P}^{g-1}$: since u_ω is an embedding, this proof shows that for every $L \in W_d^r(X,**)$ every irreducible component of $W_d^r(X,**)$ containing L has dimension at least $\varrho(g, r, d)$.

THEOREM 3.3. *Fix integers g, d, r, x, and $\delta(i)$, $1 \leqslant i \leqslant x$, with $x \geqslant 0$, $\delta(i) > 0$ for every $i > 0$, $\delta(i) \geqslant \delta(j)$ if $i \geqslant j$, $g \geqslant \Sigma_{1 \leqslant i \leqslant x} \delta(i) \geqslant 0$, $g \geqslant 2$. Assume* $\mathrm{char}(\boldsymbol{K}) = 0$ *or* $\mathrm{char}(\boldsymbol{K}) > g$. *Set $S := \{1, \ldots, x\}$. Let $\{R_{i^\wedge}\}_{i \in S}$ be a formal* rank 1 *datum for Gorenstein curves with numerical data g and with $\delta(i) = \delta(R_{i^\wedge})$. Set $\delta := \Sigma_{1 \leqslant i \leqslant x} \delta(i)$ and assume $\varrho(g - d, r + \delta, d) \geqslant 0$. Let Y be a smooth curve with genus $g - \delta$. Let X be a general curve with Y as normalization and with $\{R_{P^\wedge}\}_{P \in S}$ as formal* rank 1 *datum. Then $W_d^r(X,**) \neq \emptyset$ and there is an irreducible component T of $W_d^r(X,**)$ with* $\dim(T) \geqslant \varrho(g, r, d)$.

PROOF. We will use induction on $\mathrm{card}(S)$, the case $S = \emptyset$ (i.e. $X = Y$) being true by the classical Brill-Noether theorem for smooth curves ([ACGH], Ch. IV). Assume $x := \mathrm{card}(S) > 0$ and fix $x \in S$. Fix a general curve X with Y as normalization and with formal rank 1 datum $\{R_{i^\wedge}\}_{i \in S}$. Identify S with $\mathrm{Sing}(X)$ and call A the point of $\mathrm{Sing}(X)$ corresponding to $X \in S = \{1, \ldots, x\}$; set $r'' := r + \delta(A)$: Let $f: Z \to X$ be the partial normalization of X at A. Hence Z and X are Gorenstein curves and Z is a curve of arithmetic genus $g - \delta(A)$ general forthe fixed normalization Y and the formal rank 1 datum $\{R_{P^\wedge}\}_{P \in S \setminus \{x\}}$. We have $\varrho(g - \delta(A) - (\delta - \delta(A)), r'' + (\delta - \delta(A)), d) = \varrho(g - \delta, r + \delta, d) \geqslant 0$ by assumption. Hence by the inductive assumption the scheme $W_d^{r''}(Z,**)$ is not empty. Fix $L \in W_d^{r''}(Z,**)$. By the generality of X we may assume that $\pi^{-1}(A)$ is general in Z. Set $\{Q_1, \ldots, Q_t\} := \pi^{-1}(A)$. Let D be the effective divisor on Y of degree $\delta''(A) = 2\delta(A)$ with $D_{\mathrm{red}} = \pi^{-1}(A)$ and such that the connected component of D supported by each point Q_i of $\pi^{-1}(A)$ has degree $\delta''(Q_i)$.

Claim: For general $\pi^{-1}(A)$ the restriction map $t: H^0(Z, L) \to H^0(D, L|D)$ is surjective.

Proof of the Claim: Since $\mathrm{char}(\boldsymbol{K}) = 0$ or $\mathrm{char}(\boldsymbol{K}) > g$, for a general $Q_1 \in Z$ the order sequence of L at Q_1 is $\{0, 1, 2, 3, \ldots, r'' - 1\}$ ([L], Th. 15); more precisely, calling u: $Y \to Z$ the normalization map, this is the order sequence at a general point of Y of $u^*(L)$ with respect to the non-complete linear system $u^*(H^0(Z, L))$. In particular for a general $Q_1 \in Z$ we have $h^0(Y, L(-\delta''(Q_1)\, Q)) = h^0(Y, L) - \delta''(Q_1) = r'' + 1 - \delta''(Q_1)$. then we continue taking $L(-\delta''(Q_1)\, Q_1)$ instead of L and a general $Q_2 \in Z$ instead of Q_1. After t steps we obtain $h^0(Z, L(-D)) = h^0(Z, L) - \delta''(A) = h^0(Z, L) - \deg(D)$ and hence the Claim.

Set $\delta(Q_i) := \delta''(Q_i)/2$. Since D is a zero dimensional scheme, the restriction map $\alpha : L|D \to L|(\bigcup_{1 \leqslant i \leqslant t} \delta(Q_i)\, Q_i)$ is surjective. Set $B := \mathrm{Ker}(\alpha)$. By the Claim the counterimage of B into $H^0(Z, L)$ has dimension at least $h^0(Z, L) - \delta''(A)/2 = r + 1$. Call R the sheaf on X (a priori not a sheaf of $\boldsymbol{O}_X$-modules) defined by the kernel of the surjective map $\pi_*(L) \to \pi_*((L|D)/B)$. By [Co], pp. 48-50, R is locally free and $\deg(R) = \deg(f_*(L)) - \deg(D) + \deg(B) = \deg(L) + \delta(A) + \delta(A) = d$; here we use that X is Gorenstein at A. Since $h^0(X, R) \geqslant r + 1$, we have $R \in W_d^r(X, **)$. the last assertion follows from Remark 3.2.

As in the proof of Proposition 2.1 just using the semicontinuity theorem for the dimensions of the fibers of proper morphisms from Theorem 3.3 we obtain the following result.

COROLLARY 3.4. *Fix integers g, d, r, x, and $\delta(i)$, $1 \leqslant i \leqslant x$, with $x \geqslant 0$, $\delta(i) > 0$ for every $i > 0$, $g \geqslant \delta := \Sigma_{1 \leqslant i \leqslant x} \delta(i) \geqslant 0$, $g \geqslant 2$. Assume* $\mathrm{char}(\boldsymbol{K}) = 0$ *or* $\mathrm{char}(\boldsymbol{K}) > g$. *Set $S := \{1, \ldots, x\}$. Let $\{R_{i^\wedge}\}_{i \in S}$ be a formal* rank 1 *datum for Gorenstein curves with numerical data g and with $\delta(i) = \delta(R_{i^\wedge})$. Assume $\varrho(g - \delta, r + \delta, d) \geqslant 0$. Let Y be a smooth curve with genus $g - \delta$. Let X be any curve with Y as normalization and with $\{R_{P^\wedge}\}_{P \in S}$ as formal* rank 1 *datum. Then $W_d^r(X) \neq \emptyset$ and there is an irreducible component T of $W_d^r(X)$ with dimension at least $\varrho(g, r, d)$.*

When the normalization of X has low gonality the following result may be useful.

THEOREM 3.5. *Let X be an integral projective curve such that every singularity of X is either Gorenstein or unibranch. Let $\pi : Y \to X$ be the normalization. Set $\delta(X) := p_a(X) - p_a(Y)$. Fix positive integers r and d. Assume that Y has a g_d^r. Then $W_{d + 2\delta(x)}^r(X, **) \neq \emptyset$.*

PROOF. Fix $M \in W_d^r(Y)$. Let D be only effective divisor on Y with $\deg(D) = 2\delta(X)$, $\mathrm{Supp}(D) = \pi^{-1}(\mathrm{Sing}(X))$ and such that for every $P \in \mathrm{Sing}(X)$ and every $Q \in \pi^{-1}(P)$ the connected component of D supported by Q has degree $\delta''(Q)$. Set $L := M(D)$. Hence $\deg(L) = d + 2\delta$ and $h^0(Y, L) \geqslant h^0(X, M)$. take a suitable effective divisor $B \subset D$ with $\deg(B) = \delta(X)$ as in the proof of Theorem 3.3 and repeat exactly the same reasoning, only using [GP], 1.4, instead of [Co], pp. 48-50, at each unibranch point of X. We obtain a rank 1 subsheaf R of $\pi_*(L)$ with $h^0(X, R) \geqslant h^0(X, M) \geqslant r + 1$. By [Co], pp. 48-50, R is locally free and

$\deg(R) = \deg(R) = \deg(\pi_*(L)) - \deg(D) + \deg(B) = \deg(L) + \delta(X) - 2\delta(X) + \delta(X) = d + 2\delta(X)$.

Now we want to construct rank 1 torsion free sheaves on the curve X with many sections and whose formal completion at each singular point $P \in \mathrm{Sing}(X)$ is a prescribed rank 1 torsion free module over $\boldsymbol{O}_{X,P^\wedge}$.

DEFINITION 3.6. Fix a formal rank 1 datum $\{R_{P^\wedge}\}_{P \in S}$. A formal datum of rank 1 modules for $\{R_{P^\wedge}\}_{P \in S}$ is a set $\{M_{P^\wedge}\}_{P \in S}$, where $M_{P^\wedge}$ is a torsion free rank 1-module. Set $l(\{M_{P^\wedge}\}_{P \in S}) := \Sigma_{P \in \mathrm{Sing}(X)} l(M_{P^\wedge}, P)$. If L is a rank 1 torsion free sheaf on a projective curve, call $L_{P^\wedge}$ the completion of its stalk at $P \in \mathrm{Sing}(X)$; $\{L_{P^\wedge}\}_{P \in \mathrm{Sing}(X)}$ will be called the formal datum of L along $\mathrm{Sing}(X)$.

THEOREM 3.7. *Let X be an integral projective curve such that every singularity of X is either Gorenstein or unibranch. Let $\pi: Y \to X$ be the normalization. Set $\delta(X) := p_a(X) - p_a(Y)$. Fix a formal* rank 1 *datum $\{M_{P^\wedge}\}_{P \in \mathrm{Sing}(X)}$ for the formal datum $\{\boldsymbol{O}_{X,P^\wedge}\}_{P \in \mathrm{Sing}(X)}$ and set $\boldsymbol{m} := l(\{M_{P^\wedge}, P\}_{P \in \mathrm{Sing}(X)})$. Fix positive integers r and d. Assume that Y has a g_d^r. Then there exists $L \in W^r_{d+2\delta(X)+\boldsymbol{m}}(X)$ such that $L_{P^\wedge} \cong M_{P^\wedge}$ for every $P \in \mathrm{Sing}(X)$.*

PROOF. The recipe in [Co], Ch. III, allows the construction of such a sheaf, just modifying the proof of Theorem 3.5 with the quotation of [Co], Prop. 3.4.8. The computation of $\deg(L)$ is given in [Co], Prop. 3.4.1, part 2). We give an alternative proof which shows why this type of results with the additional term «$+\boldsymbol{m}$» is easy. By Theorem 3.5 there exists $W \in W^r_{d+2\delta(X)}(X, **)$. there exists a rank 1 torsion free sheaf U on X such that $W \subseteq U$, $\mathrm{Supp}(U/W) \subseteq \mathrm{Sing}(X)$, $U_{P^\wedge} \cong M_{P^\wedge}$ for every $p \in \mathrm{Sing}(X)$ and such that the connected component of U/W supported by P has length $l(M_{P^\wedge}, P)$ (see e.g. [B], Remark 1.14). Thus $\deg(U) = \deg(W) + \boldsymbol{m}$. Since $h^0(X, U) \geqslant h^0(X, W) \geqslant r+1$, we conclude.

We leave to the interested reader the extension of Theorem 3.3 to the non-locally free case.

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