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A Problem of Transversal Anisotropic Ellipticity.

MARCO MUGHETTI (*)

Introduction.

In the celebrated paper of L. Boutet - A. Grigis - B. Helffer [2], the authors consider a class of pseudodifferential operators $P \in \mathrm{OPS}^m(X)$, whose symbol $p(x, \xi) \sim \sum p_{m-j}(x, \xi)$ vanishes of order $k \geq 1$ on a closed conic submanifold Σ of $T^*X \backslash 0$ (precisely, $p_{m-j}(x, \xi)$ vanishes of order $k-2j$ at least on Σ when $j \leq k/2, j \geq 0$). They show how the hypoellipticity (or micro-hypoellipticity) in C^∞ of P, with minimal loss of $k/2$ derivatives, depends on the injectivity in L^2, when ϱ belongs to Σ, of a suitable «test» differential operator P_ϱ defined in an invariant fashion. In [2] it is assumed the transversal ellipticity of $p_m(x, \xi)$ with respet to Σ (i.e., p_m vanishes exactly of order k on Σ). Later on, B. Helffer and J. F. Nourrigat [9] have suggested to remove the condition of transversal ellipticity, by considering the case $\Sigma = \Sigma_1 \cap \Sigma_2$ with transversal intersection. They require that, for some integer $h \geq 1$, $p_m(x, \xi)$ is equivalent to $|\xi|^m (\mathrm{dist}^h_{\Sigma_1}(x, \xi) + \mathrm{dist}_{\Sigma_2}(x, \xi))^k$ (where $\mathrm{dist}_{\Sigma_i}(x, \xi)$ denotes the distance of $(x, \xi/|\xi|)$ to Σ_i, $i = 1, 2$) and $|p_{m-j}(x, \xi)| \lesssim |\xi|^{m-j} (\mathrm{dist}^h_{\Sigma_1}(x, \xi) + \mathrm{dist}_{\Sigma_2}(x, \xi))^{k-2j}$ for $j \leq k/2$. In this situation they again obtain a necessary and sufficient condition for the C^∞ hypoellipticity of P with loss of $k/2$ derivatives in term of the injectivity in L^2 of a «test» differential operator P_ϱ, $\varrho \in \Sigma$, defined in an invariant way.

The classical example of Grushin (in $\mathbb{R}^2$) $D_x^2 + x^{2h} D_y^2$, that, from well known results (see Hörmander Theorem 22.2.1 in [13], Rothschild-Stein [20] and Fefferman-Phong [3]), is hypoelliptic with loss of $2h/(h+1)$ derivatives, is not included (for $h > 1$) in the framework of the papers cited above. M. Mascarello - L. Rodino in [15] have suggested a variation of

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Helffer-Nourrigat's approach which should contain the Grushin model. However, it seems to us that the classes of O.P.D. $\Lambda_h^{m,M}(\Sigma_1, \Sigma_2, \varrho)$ defined in [15] do not have an invariant meaning and therefore the results of [15] seemingly refer to the «flat» case only (i.e., when Σ_1 and Σ_2 are flat cones).

In this paper we give an *invariant* approach in the spirit of Boutet-Grigis-Helffer [2] when Σ is a symplectic cone of codimension 2ν, which contains the Grushin model. The paper is organized as follows.

In Section 1 the flat case is studied (with $h \geq 1$) following a suitable anisotropic version of the calculus developed by Boutet de Monvel in [1].

In Section 2 we show how the previous calculus, based on a flat model, can be used to treat the general case of two involutive cones Σ_1, Σ_2 of $T^*X\backslash 0$ with transversal and symplectic intersection Σ. We consider a class of classical pseudodifferential operators $P = p(x, D)$ with «double characteristics», whose principal and subprincipal symbols satisfy suitable vanishing conditions on Σ_1 and Σ_2. Moreover, we suppose that the principal symbol $p_m(x, \xi)$ of P is transversally elliptic in the following anisotropic sense, i.e.

$$|p_m(x, \xi)| \gtrsim |\xi|^m (\mathrm{dist}^h_{\Sigma_1}(x, \xi) + \mathrm{dist}_{\Sigma_2}(x, \xi))^2.$$

We show that the hypoellipticity of $P(x, D)$ with loss of $2h/(h+1)$ derivatives depends on the injectivity in L^2 of a suitable parameter-dependent differential operator $P_\varrho(x', D_{x'})$, $\varrho \in \Sigma$, in ν variables. In particular, in Section 2.3 we show how it is possible to reduce the above spectral condition on $P_\varrho(x', D_{x'})$ to an explicit algebraic condition when the cone Σ has codimension 2.

1. The flat case.

1.1. *Definition of symbols.*

Let us fix the notations used throughout in this Chapter (1). For any $n \geq 2$, consider $T^*\mathbb{R}^n \cong \mathbb{R}^n_x \times \mathbb{R}^n_\xi$ and fix a decomposition $\mathbb{R}^n_x = \mathbb{R}^\nu_{x'} \times \mathbb{R}^{n-\nu}_{x''}$

(1) Unexplained notations used throughout are standard, and can be found in Hörmander's book [13] vol.I and III.

for some $1 \leqslant \nu < n$ (write, accordingly, $\mathbb{R}^n_\xi = \mathbb{R}^\nu_{\xi'} \times \mathbb{R}^{n-\nu}_{\xi''}$). Define:

$$\Sigma_1 = \{(x, \xi) \in T^\star \mathbb{R}^n \backslash 0 \mid x' = 0\}, \qquad \Sigma_2 = \{(x, \xi) \in T^\star \mathbb{R}^n \backslash 0 \mid \xi' = 0\}$$

$$\Sigma = \Sigma_1 \cap \Sigma_2 = \{(x, \xi) \in T^\star \mathbb{R}^n \backslash 0 \mid x' = 0 = \xi'\}.$$

For some fixed positive integer h, consider in $T^\star \mathbb{R}^n \backslash 0$ the weight function

$$d_\Sigma(x, \xi) = |x'|^h + \frac{|\xi'|}{|\xi|} + \frac{1}{|\xi|^{h/(h+1)}}.$$

As in [1], for two non-negative functions f, g defined on some open conic subset $\Gamma \subseteq T^\star \mathbb{R}^n \backslash 0$, we use the notation

$$f \lesssim g \ (or\ g \gtrsim f)$$

to mean that, for every subcone $\Gamma' \subseteq \Gamma$ with compact base and for any $\varepsilon > 0$, there exists a constant $C = C_{\Gamma', \varepsilon} > 0$ such that $f(x, \xi) \leqslant C g(x, \xi)$, for any $(x, \xi) \in \Gamma'$, $|\xi| \geqslant \varepsilon$ (we simply write $f \approx g$ for $f \lesssim g$ and $g \lesssim f$).

DEFINITION 1.1. *Let $m, k \in \mathbb{R}$ and $\Gamma \subseteq T^\star \mathbb{R}^n \backslash 0$ be an open conic set. We denote by $S^{m,k}_h(\Gamma)$ (simply, by $S^{m,k}_h$ if $\Gamma = T^\star \mathbb{R}^n \backslash 0$) the set of all smooth functions $a(x, \xi)$ defined in Γ such that for any multi-index $\alpha, \beta \in \mathbb{Z}^\nu_+$, $\gamma, \theta \in \mathbb{Z}^{n-\nu}_+$, we have*

$$(1) \qquad |\partial^\alpha_{x'} \partial^\beta_{\xi'} \partial^\gamma_{x''} \partial^\theta_{\xi''} a(x, \xi)| \lesssim |\xi|^{m - |\beta| - |\theta|} d_\Sigma(x, \xi)^{k - (|\alpha|/h) - |\beta|}.$$

Define $S^{m,\infty}_h(\Gamma) = \bigcap_k S^{m,k}_h(\Gamma)$ and note that, for any k, $\bigcap_m S^{m,k}_h(\Gamma) = S^{-\infty}(\Gamma)$.

In particular, when $\Gamma = T^\star X \backslash 0$ (with X open set of $\mathbb{R}^n$) we write $S^{m,k}_h(X)$, $S^{m,\infty}_h(X)$ instead of $S^{m,k}_h(\Gamma)$, $S^{m,\infty}_h(\Gamma)$.

Since

$$(2) \qquad |\xi|^{-\frac{h}{h+1}} \lesssim d_\Sigma(x, \xi) \lesssim 1,$$

it follows that $S^{m,k}_h \subseteq S^{m + \frac{h}{h+1} k_-}_{\frac{1}{h+1}, \frac{1}{h+1}}$ (with $k_- = \max\{0, -k\}$). Moreover, it is easy to see that if $m \leqslant m'$ and $m - \frac{h}{h+1} k \leqslant m' - \frac{h}{h+1} k'$ then $S^{m,k}_h \subseteq S^{m',k'}_h$. From the Leibniz formula, it follows that $ab \in S^{m+m', k+k'}_h$ if $a \in S^{m,k}_h$, $b \in S^{m',k'}_h$. Finally, if $f \in S^{m,k}_h$ and $|f| \gtrsim |\xi|^m d^k_\Sigma$ then $f^{-1} \in S^{-m,-k}_h$.

PROPOSITION 1.1. *Let $a \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ be a classical symbol with asymptotic expansion $a \sim \sum_{j \geq 0} a_{m-j}$. Let $k \geq 0$. The following statements are equivalent:*

1) $a \in S_h^{m,k}$;

2) $\partial_{x'}^{\alpha} \partial_{\xi'}^{\beta} a_{m-j}|_{\Sigma} \equiv 0$ *for any* $j \in \mathbb{Z}_+$, $\alpha, \beta \in \mathbb{Z}_+^{\nu}$ *with* $\frac{|\alpha|}{h} + |\beta| < \left(k - j\frac{h+1}{h}\right)_+$ *(where $(d)_+ = \max\{0, d\}$ denotes the positive part of d).*

PROOF. By the same arguments seen in Example 1.4 [1].

DEFINITION 1.2. *We define, with the above notations,*

$$\mathcal{H}_h^m(\Gamma) = \bigcap_j S_h^{m-j, -j\frac{h+1}{h}}(\Gamma).$$

Its elements are symbols of degree $-\infty$ out of Σ and are called Hermite symbols (simply, $\mathcal{H}_h^m(X)$ when $\Gamma = T^\star X \backslash 0$, or $\mathcal{H}_h^m$ when $\Gamma = T^\star \mathbb{R}^n \backslash 0$).

Adapting the arguments of Proposition 1.11 [1], we prove

PROPOSITION 1.2. *We have the following results about asymptotic expansions:*

1) *If $a_j \in S_h^{m, k+j/h}$ with $j = 0, 1, 2, \ldots$, then there exists $a \in S_h^{m,k}$, unique modulo $S_h^{m,\infty}$, such that, for all $N \in \mathbb{N}$,*

$$a - \sum_{j<N} a_j \in S_h^{m, k+\frac{N}{h}}.$$

2) *If $a_j \in S_h^{m-j/(h+1), k-j/h}$ with $j = 0, 1, 2, \ldots$, then there exists $a \in S_h^{m,k}$, unique modulo $\mathcal{H}_h^{m-\frac{h}{h+1}k}$, such that, for all $N \in \mathbb{N}$,*

$$a - \sum_{j<N} a_j \in S_h^{m-\frac{N}{h+1}, k-\frac{N}{h}}.$$

In the following section, we show how it is possible to construct a pseudodifferential calculus based on the symbols defined above.

1.2. *Estimates of some oscillatory integrals.*

The study of the stability of PDO's associated with the previously defined symbols requires having some estimates on oscillatory integrals that are obtained in this section. It is useful to work in $T^\star \mathbb{R}^{2n}$ instead of

$T^\star \mathbb{R}^n$ with the standard identification $T^\star \mathbb{R}^{2n} \cong \mathbb{R}^{2n}_{(x,\,z)} \times \mathbb{R}^{2n}_{(\xi,\,\zeta)}$; as in the previous section, we fix a decomposition $\mathbb{R}^n_z = \mathbb{R}^\nu_{z'} \times \mathbb{R}^{n-\nu}_{z''}$ and, accordingly, $\mathbb{R}^n_\zeta = \mathbb{R}^\nu_{\zeta'} \times \mathbb{R}^{n-\nu}_{\zeta''}$. In this way, Σ can be identifed with $\tilde{\Sigma} = \{(x, z, \xi, \zeta) \in \mathbb{R}^{2n} \times (\mathbb{R}^{2n} \setminus \{0\}) \,|\, x' = \xi' = 0,\ z = \zeta = 0\}$, a subcone of the cone $\Delta = \{(x, z, \xi, \zeta) \in \mathbb{R}^{2n} \times (\mathbb{R}^{2n} \setminus \{0\}) \,|\, z = 0,\ \zeta = 0\}$; so it is convenient to introduce the following weight functions on $T^\star \mathbb{R}^{2n} \setminus 0$:

$$d_\Delta(x, z, \xi, \zeta) = \left(|z|^{2h} + \left(\frac{|\zeta|}{r} \right)^2 + \frac{1}{r^{2h/(h+1)}} \right)^{1/2}$$

$$d_{\tilde{\Sigma}}(x, z, \xi, \zeta) = \left(|x'|^{2h} + |z|^{2h} + \left(\frac{|\xi'|}{r} \right)^2 + \left(\frac{|\zeta|}{r} \right)^2 + \frac{1}{r^{2h/(h+1)}} \right)^{1/2}$$

where $r = r(x, z, \xi, \zeta) = |(\xi, \zeta)|$.

Definition 1.3. *Given any m,k,l, we denote by $QS_h^{m,k,l}$ the set of all C^∞ functions $a(x, z, \xi, \zeta)$ on $T^\star \mathbb{R}^{2n} \setminus 0$ such that for all $\alpha_1, \beta_1 \in \mathbb{Z}^n_+$, $\alpha_2, \beta_2 \in \mathbb{Z}^\nu_+$ and $\gamma, \theta \in \mathbb{Z}^{n-\nu}_+$*

$$|\partial_z^{\alpha_1} \partial_{x'}^{\alpha_2} \partial_\zeta^{\beta_1} \partial_{\xi'}^{\beta_2} \partial_{x''}^{\gamma} \partial_{\xi''}^{\theta} a(x, z, \xi, \zeta)| \lesssim$$

$$\lesssim r^{m + \frac{|\alpha_1|}{h+1} - \frac{|\beta_1|}{h+1} - |\beta_2| - |\theta|} d_{\tilde{\Sigma}}^{k - \frac{|\alpha_2|}{h} - |\beta_2|} \left(r^{\frac{h}{h+1}} d_\Delta \right)^{l + \frac{|\alpha_2|}{h} + |\beta_2|}.$$

Let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, we denote by xy their standard scalar product in $\mathbb{R}^n$. We are now in a position to state the main result of this section.

Proposition 1.3. *Let $a \in QS_h^{m,k,l}$ and suppose that there exists a constant $c > 0$ such that $a(x, z, \xi, \zeta)$ vanishes for $|\zeta|/|\xi| + |z| \geq c$. Then*

$$I(a)(x, \xi) := \int e^{iz\zeta} a(x, z, \xi, \zeta)\, dz\, d\zeta \in S_h^{m,k}.$$

Proof. The proof is similar to the one in Proposition 2.7 [1] with some modifications required because in the weight functions d_Δ, $d_{\tilde{\Sigma}}$, d_Σ we have chosen $r^{-h/(h+1)}$ instead of $r^{-1/2}$. First of all, we note that

$$\partial_{x'}^\alpha \partial_{\xi'}^\beta \partial_{x''}^\gamma \partial_{\xi''}^\theta I(a) = I(\partial_{x'}^\alpha \partial_{\xi'}^\beta \partial_{x''}^\gamma \partial_{\xi''}^\theta a),$$

where $\partial_{x'}^\alpha \partial_{\xi'}^\beta \partial_{x''}^\gamma \partial_{\xi''}^\theta a \in QS_h^{m - |\beta| - |\theta|,\, k - \frac{|\alpha|}{h} - |\beta|,\, l + \frac{|\alpha|}{h} + |\beta|}$.

Since $a(x, z, \xi, \zeta)$ vanishes for $|\zeta|/|\xi| + |z| \geq c$, we can suppose that $|\zeta|/|\xi| \leqslant c$, so that $r \approx |\xi|$; hence, from now on we assume that $r = |\xi|$. Therefore, it suffices to prove that $|I(a)| \lesssim r^m d_\Sigma^k$. To this purpose, we define the operator

$$P = (r^{\frac{h}{h+1}} d_\Delta)^{-2}(1 - (r^{\frac{2}{h+1}} \Delta_\zeta)^h - r^{-\frac{2}{h+1}} \Delta_z)$$

and its transpose

$$^tP = (1 - (r^{\frac{2}{h+1}} \Delta_\zeta)^h - r^{-\frac{2}{h+1}} \Delta_z)((r^{\frac{h}{h+1}} d_\Delta)^{-2} \bullet)$$

in such a way that, for every integer $N > 0$ we get $I(a) = I((^tP)^N a)$ and $(^tP)^N a \in QS_h^{m,k,l-2N}$. From this point the proof is similar to the one in Proposition 2.7 [1] replacing the classical $QS^{m,k,l}$ by $QS_h^{m,k,l}$. ∎

1.3. *Pseudodifferential operators associated with* $S_h^{m,k}(X)$.

Let X be an open set in $\mathbb{R}^n$ and $a = a(x, \xi)$ be a C^∞ function on $X \times \mathbb{R}^n$ such that $a \in S_h^{m,k}(X)$. Then we define the pseudodifferential operator $a(x, D)$ as

$$a(x, D) f = (2\pi)^{-n} \int e^{ix\xi} a(x, \xi) \hat{f}(\xi) \, d\xi$$

where $f \in C_0^\infty(X)$ and $\hat{f}(\xi)$ denotes the usual Fourier transform of $f(x)$.

Definition 1.4. *We denote by* $\mathrm{OPS}_h^{m,k}(X)$ *(simply,* $\mathrm{OPS}_h^{m,k}$ *when* $X = \mathbb{R}^n$*) the set of all operators of the form* $a(x, D) + R$*, where* $a(x, \xi)$ *is as above and* R *is an operator with* C^∞ *kernel.*

Accordingly, we define:

$$\mathrm{OP}\mathcal{H}_h^m(X) = \bigcap_{j \in \mathbb{N}} \mathrm{OPS}_h^{m-j,\, -j(h+1)/h}(X)$$

$$\mathrm{OPS}_h^{m,\infty}(X) = \bigcap_{j \in \mathbb{N}} \mathrm{OPS}_h^{m,j}(X)$$

As usual, we write $\mathrm{OP}\mathcal{H}_h^m$ *and* $\mathrm{OP}S_h^{m,\infty}$ *when* $X = \mathbb{R}^n$.

Now we are ready to give the following crucial result.

Proposition 1.4. *Let* $a \in S_h^{m,k}$, $b \in S_h^{m',k'}$ *and suppose that* $b(x, \xi) = 0$ *for* $x \notin K$*, where* K *is some compact set in* $\mathbb{R}^n$*. Then*

$$a(x, D) \circ b(x, D) \in \mathrm{OP}S_h^{m+m',\, k+k'}.$$

Moreover, if $c(x, D) = a(x, D) \circ b(x, D)$, *then it follows that, for any integer* $N > 0$,

$$c(x, \xi) - \sum_{|\alpha| < N} \frac{i^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha a \partial_x^\alpha b \in S_h^{m+m'-N,\, k+k'-\frac{h+1}{h}N}.$$

PROOF. By definition of symbol, we have

$$c(x, \xi) = e^{-ix\xi} a(x, D)\, b(x, D)(e^{ix\xi}) = e^{-ix\xi} a(x, D)(e^{ix\xi} b(x, \xi))$$

$$= (2\pi)^{-n} \int\int e^{iz\zeta} a(x, \xi + \zeta)\, b(x - z, \xi)\, dz\, d\zeta = I(c_1),$$

where $c_1(x, z, \xi, \zeta) = (2\pi)^{-n} a(x, \xi + \zeta)\, b(x - z, \xi)$. By using Proposition 1.3, the conclusion follows. ■

The next result is an immediate consequence of the previous proposition.

COROLLARY 1.5. *If* $A \in \mathrm{OP}\mathcal{H}_h^m$ *and* $B \in \mathrm{OP}S_h^{m', k'}$ *then* AB, $BA \in \mathrm{OP}\mathcal{H}_h^{m+m'-\frac{h}{h+1}k}$ *provided* A *or* B *is properly supported. Moreover, if* $B \in \mathrm{OP}S_h^{m', \infty}$ *then* AB, $BA \in \mathrm{OPS}^{-\infty}$.

Let A be a pseudodifferential operator with symbol $a(x, \xi)$; we denote by A^* the «formal» adjoint operator, defined by

$$\langle Au, v \rangle = \langle u, A^* v \rangle, \qquad u, v \in C_0^\infty(\mathbb{R}^n)$$

where $\langle u, v \rangle = \int u(x)\, \overline{v(x)}\, dx$. The next proposition shows that our classes of PDO are closed under the operation of taking formal adjoints; it can be easily proved by using the arguments seen in the proof of Proposition 1.4.

PROPOSITION 1.6. *If* $A = a(x, D)$ *is in* $\mathrm{OPS}_h^{m, k}$, *then* A^* *is also in* $\mathrm{OP}S_h^{m, k}$ *and its symbol* $a^*(x, \xi)$ *has the following asymptotic expansion*

$$a^*(x, \xi) \sim \sum_{\alpha \geq 0} \frac{i^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha \partial_x^\alpha \overline{a(x, \xi)}$$

in the sense that, for any positive integer N

$$a^*(x, \xi) - \sum_{|\alpha| < N} \frac{i^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha \partial_x^\alpha \overline{a(x, \xi)} \in S^{m-N,\, k-N\frac{h+1}{h}}.$$

We are now going to discuss the continuity of the operators $\mathrm{OP}S_h^{m,k}$ in Sobolev spaces. In order to do that, we shall introduce the distribution spaces $H_h^{s,k}$ which are related to operators in $\mathrm{OP}S_h^{m,k}$ in the same way usual Sobolev spaces are related to the usual pseudodifferential operators.

DEFINITION 1.5. *Let X be an open set of $\mathbb{R}^n$ and $s \in \mathbb{R}$, $k \in \mathbb{Z}$. We denote by $H_h^{s,k}(X)$ (simply, $H_h^{s,k}$ when $X = \mathbb{R}^n$) the space of all distributions f on X such that, for any properly supported operator $A \in \mathrm{OP}S_h^{m,k}(X)$, we have $Af \in L^2_{\mathrm{loc}}(X)$. $H_h^{s,k}(X)$ is equipped with the weakest locally convex topology for which the maps $f \rightarrow Af$ as above are continuous for any $A \in \mathrm{OP}S_h^{m,k}(X)$.*

We observe that

$$H_h^{s,0}(X) = H^s_{\mathrm{loc}}(X),$$

$H_h^{m,k}(X) \hookrightarrow H_h^{m',k'}(X)$ continuously, if

$$m \geqslant m' \text{ and } m - \frac{h}{h+1} k \geqslant m' - \frac{h}{h+1} k'.$$

Hence, by Proposition 1 we have

LEMMA 1.7. *Fix $m \in \mathbb{R}_+$, $k \in \mathbb{Z}_+$, and let A be a properly supported operator of $\mathrm{OP}S_h^{-m,-k}(X)$. Then, given any $s \in \mathbb{R}$, A maps $H^s_{\mathrm{loc}}(X) \rightarrow H_{\mathrm{loc}}^{s+m-\frac{h}{h+1}k}(X)$ ($H^s_{\mathrm{comp}}(X) \rightarrow H_{\mathrm{comp}}^{s+m-\frac{h}{h+1}k}(X)$) continuously.*

1.4. *The localized polynomial.*

Let $p(x, \xi) \in S^m \cap S_h^{m,k}$ be a classical symbol with asymptotic expansion $p \sim \sum_{j \geqslant 0} p_{m-j}$. We recall that the Weyl-symbol $p^W(x, \xi)$ associated with $p(x, \xi)$ is defined as

$$p^W(x, \xi) = e^{\langle D_x, D_\xi \rangle / 2i} p(x, \xi) \sim \sum_{j \geqslant 0} p^W_{m-j}(x, \xi),$$

where

$$p^W_{m-j}(x,\xi)=\sum_{l+r=j}\frac{1}{r!}\left(\frac{1}{2i}\langle D_x, D_\xi\rangle\right)^r p_{m-l}(x,\xi).$$

For any fixed $\varrho=(0,x'',0,\xi'')$, we define the localized polynomial $p_\varrho(x',\xi')$ in ϱ associated with $p(x,\xi)$ as

$$(3)\qquad p_\varrho(x',\xi')=\sum_{\substack{j\in\mathbb{Z}_+\\ k-j\frac{h+1}{h}\geq 0}}\ \sum_{\substack{\alpha,\beta\in Z^\nu_+\\ \frac{|\alpha|}{h}+|\beta|=k-j\frac{h+1}{h}}}\frac{1}{\alpha!\beta!}(\partial^\alpha_{x'}\partial^\beta_{\xi'}p^W_{m-j})(\varrho)(x')^\alpha(\xi')^\beta$$

and denote by $P_\varrho(x',D_{x'})$ its Weyl-quantization.

Later on, we shall show that the problem of hypoellipticity of $p(x,D)$ can be reduced to the study of some spectral conditions on the operator $P_\varrho(x',D_{x'})$.

We define

$$(4)\qquad p_\Sigma(x',x'',\xi',\xi'')=$$

$$=\sum_{\substack{j\in\mathbb{Z}_+\\ k-j\frac{h+1}{h}\geq 0}}\ \sum_{\substack{\alpha,\beta\in Z^\nu_+\\ \frac{|\alpha|}{h}+|\beta|=k-j\frac{h+1}{h}}}\frac{1}{\alpha!\beta!}(\partial^\alpha_{x'}\partial^\beta_{\xi'}p_{m-j})(0,x'',0,\xi'')(x')^\alpha(\xi')^\beta.$$

From the vanishing properties of the terms p_{m-j} of the asymptotic expansion of p (as shown in Proposition 1.1), we obtain after a few computations

$$e^{\langle D_{x'},D_{\xi'}\rangle/2i}p_\Sigma(x',x'',\xi',\xi'')=p_\varrho(x',\xi')$$

so that

$$(5)\qquad P_\varrho(x',D_{x'})=\mathrm{Op}^W(p_\varrho(x',\xi'))=p_\Sigma(x',x'',D_{x'},\xi'')=$$

$$=\sum_{\substack{j\in\mathbb{Z}_+\\ k-j\frac{h+1}{h}\geq 0}}\ \sum_{\substack{\alpha,\beta\in Z^\nu_+\\ \frac{|\alpha|}{h}+|\beta|=k-j\frac{h+1}{h}}}\frac{1}{\alpha!\beta!}(\partial^\alpha_{x'}\partial^\beta_{\xi'}p_{m-j})(0,x'',0,\xi'')(x')^\alpha D^\beta_{x'}.$$

Actually, in the flat case considered there is no real need to use the Weyl-symbol and the related quantization as we have done above. Nevertheless, the reasons of our approach will be made clear in the Section 2, when we treat our hypoellipticity problem in a «no-flat» context.

Finally, assume that $k \in \mathbb{N}$; with a little work we can show that

$$p(x, \xi) - p_\Sigma(x, \xi) \in S_h^{m, k + \frac{1}{h}}. \tag{6}$$

1.5. *A class of parameter-dependent pseudodifferential operators.*

The crucial idea of this paper is that the hypoellipticity with loss of $hk/(h+1)$ derivatives of the pseudodifferential operator $P = p(x, D)$ is strictly related to the existence of a left inverse of $p_\Sigma(x', x'', D_{x'}, \xi'')$ in a suitable class. In this section we develop the «machinery» by means of which we can construct such a left inverse starting from the injectivity of $p_\Sigma(x', x'', D_{x'}, \xi'')$ in $\mathcal{S}(\mathbb{R}^\nu)$. In order to do that, we introduce a pseudodifferential calculus based on symbols, whose model is represented by $p_\Sigma(x', x'', \xi', \xi'')$ regarded as a smooth function in $(x', \xi') \in \mathbb{R}^\nu \times \mathbb{R}^\nu$ depending on a parameter $(x'', \xi'') \in \mathbb{R}^{n-\nu} \times (\mathbb{R}^{n-\nu} \setminus \{0\})$. To make the exposition clearer and more readable, we begin by considering symbols without parameter (for more details we refer to Chapter 7 of Mascarello-Rodino [16]); afterwards we shall put in evidence the changes which are required when introducing the dependence on $(x'', \xi'') \in \mathbb{R}^{n-\nu} \times \times (\mathbb{R}^{n-\nu} \setminus \{0\})$.

DEFINITION 1.6. *Given $k \in \mathbb{R}$, we denote by S_h^k the space of all smooth functions $a(x', \xi')$ defined in $\mathbb{R}^\nu_{x'} \times \mathbb{R}^\nu_{\xi'}$ such that, for any multi-index $\alpha, \beta \in \mathbb{Z}^\nu_+$, there exists a constant $C = C(\alpha, \beta) > 0$ for which*

$$|\partial^\alpha_{x'} \partial^\beta_{\xi'} a(x', \xi')| \leq C(1 + |x'|^h + |\xi'|)^{k - \frac{|\alpha|}{h} - |\beta|} \tag{7}$$

whenever $(x', \xi') \in \mathbb{R}^\nu \times \mathbb{R}^\nu$.

Finally, we define

$$\mathrm{S}_h^{-\infty} = \bigcap_k \mathrm{S}_h^k.$$

We observe that S_h^k (with $k \in \mathbb{R} \cup \{-\infty\}$) is a Fréchet space when equipped with the semi-norms defined by the best possible constants in inequality (7).

As usual, we define the pseudodifferential operator $a(x', D_{x'})$ associated with the symbol $a(x', \xi') \in \mathrm{S}_h^k$ (with $k \in \mathbb{R} \cup \{-\infty\}$) as

$$a(x', D_{x'}) f = (2\pi)^{-\nu} \int e^{ix'\xi'} a(x', \xi')\, \hat{f}(\xi')\, d\xi' \tag{8}$$

where $f \in \mathcal{S}(\mathbb{R}^\nu)$. This class of pseudodifferential operators is denoted by $\mathrm{OP}\,\mathbb{S}_h^k$; notice that every operator in $\mathrm{OP}\,\mathbb{S}_h^k$ (resp., in $\mathrm{OP}\,\mathbb{S}_h^{-\infty}$) maps $\mathcal{S}(\mathbb{R}^\nu) \to \mathcal{S}(\mathbb{R}^\nu)$ (resp. $\mathcal{S}'(\mathbb{R}^\nu) \to \mathcal{S}(\mathbb{R}^\nu)$) continuously.

From now on, we denote by U the cone $T^*\mathbb{R}^{n-\nu}\backslash 0 \simeq \mathbb{R}^{n-\nu}_{x''} \times \times (\mathbb{R}^{n-\nu}_{\xi''}\backslash\{0\})$. Let Γ be an open subcone of U and $a((x'', \xi''), x', \xi') \in \in C^\infty(\Gamma, \mathbb{S}_h^k)$, i.e. $a((x'', \xi''), x', \xi')$ is a smooth function in $\Gamma \times \mathbb{R}^\nu_{x'} \times \mathbb{R}^\nu_{\xi'}$ such that, for any compact set H of Γ, for any multi-index γ, $\theta \in \mathbb{Z}_+^{n-\nu}$, $\alpha, \beta \in \mathbb{Z}_+^\nu$, there exists a constant $C = C(H, \gamma, \theta, \alpha, \beta) > 0$ for which

$$|\partial_{x''}^\gamma \partial_{\xi''}^\theta \partial_{x'}^\alpha \partial_{\xi'}^\beta a((x'', \xi''), x', \xi')| \leq C(1 + |x'|^h + |\xi'|)^{k - \frac{|\alpha|}{h} - |\beta|}$$

whenever $(x', \xi') \in \mathbb{R}^\nu \times \mathbb{R}^\nu$, $(x'', \xi'') \in H$.

We denote by $a((x'', \xi''), x', D_{x'})$ the pseudodifferential operator of type (8) depending on the parameter $(x'', \xi'') \in \Gamma$. We say that $a((x'', \xi''), x', \xi')$ is semi-homogeneous of degree μ if, for any $\lambda > 0$, one has

$$a((x'', \lambda\xi''), \lambda^{-\frac{1}{h+1}} x', \lambda^{\frac{1}{h+1}} \xi') = \lambda^\mu a((x'', \xi''), x', \xi').$$

The above definitions are motivated by the fact that $(x'', \xi'') \to \to p_\Sigma(x', x'', \xi', \xi'') \in C^\infty(U, \mathbb{S}_h^k)$ and is semi-homogeneous of degree $m - hk/(h+1)$. Given any integer $j \geq 0$ such that $k - j\frac{h+1}{h} \geq 0$, define:

$$p_{\Sigma, j} = \sum_{\substack{\alpha, \beta \in \mathbb{Z}_+^\nu \\ \frac{|\alpha|}{h} + |\beta| = k - j\frac{h+1}{h}}} \frac{1}{\alpha!\beta!} (\partial_{x'}^\alpha \partial_{\xi'}^\beta p_{m-j})(0, x'', 0, \xi'')(x')^\alpha (\xi')^\beta$$

hence,

$$p_\Sigma(x, \xi) = \sum_{\substack{j \in \mathbb{Z}_+ \\ k - j\frac{h+1}{h} \geq 0}} p_{\Sigma, j}(x, \xi).$$

Notice that, as in the classical case, the formal adjoint P_Σ^* of $P_\Sigma = = p_\Sigma(x', x'', D_{x'}, \xi'')$ has symbol

$$p_\Sigma^*(x, \xi) = \sum_{\alpha \in \mathbb{Z}_+^\nu} \frac{i^{-|\alpha|}}{\alpha!} \partial_{\xi'}^\alpha \partial_{x'}^\alpha \overline{p_\Sigma(x', x'', \xi', \xi'')} = \sum_{\substack{j \in \mathbb{Z}_+ \\ k - j\frac{h+1}{h} \geq 0}} p_{\Sigma, j}^*(x, \xi)$$

with

$$p^*_{\Sigma, j} = \sum_{\substack{s \in \mathbb{Z}_+;\, \alpha \in \mathbb{Z}^\nu_+ \\ s + |\alpha| = j}} \frac{i^{-|\alpha|}}{\alpha!} \partial^\alpha_{\xi'} \partial^\alpha_{x'} \overline{p_{\Sigma, s}} \in C^\infty(U, \mathbb{S}_h^{k - j\frac{h+1}{h}}).$$

Since p_Σ is a polynomial in (x', ξ'), the above sums are finite and can be deduced by a direct computation without using asymptotic expansions.

The next lemma puts in evidence the relation between $C^\infty(U, \mathbb{S}^k_h)$ (resp., $C^\infty(U, \mathbb{S}_h^{-\infty})$) and $S_h^{m,k}$ (resp., $\mathcal{H}^m_h$).

LEMMA 1.8. *Every function $a((x'', \xi''), x', \xi')$ in $C^\infty(U, \mathbb{S}^k_h)$ (resp., $C^\infty(U, \mathbb{S}_h^{-\infty})$), semi-homogeneous of degree $m - \frac{hk}{h+1}$ (resp., m), thought as map $(x', x'', \xi', \xi'') \mapsto a((x'', \xi''), x', \xi')$, belongs to $S_h^{m,k}$ (resp., $\mathcal{H}^m_h$). We simply denote by $a(x, D)$ the standard pseudodifferential operator in $\mathrm{OP}S_h^{m,k}$ obtained by quantizing the map $(x', x'', \xi', \xi'') \mapsto a((x'', \xi''), x', \xi')$.*

PROOF. As we shall make clear later on, for our purpose we can suppose that $|\xi''| \geqslant C|\xi'|$ for some positive constant C, so that $|\xi''| \approx |\xi|$. An easy computation completes the proof. ∎

The following proposition contains results which represent an anisotropic parameter-dependent version of well-known facts (see Chapter 7 of Mascarello-Rodino [16], Helffer [11] or Shubin [21]). They are the basic tools necessary to develop an elliptic theory based on the pseudodifferential operators above described.

PROPOSITION 1.9. *With the above notations, one has:*

1) *if $a \in C^\infty(\Gamma, \mathbb{S}^k_h)$ and for any compact set H of Γ there exists a constant $C = C(H) > 0$ such that $|a((x'', \xi''), x', \xi')| \geqslant C(1 + |x'|^h + |\xi'|)^k$ whenever $(x'', \xi'') \in H$, $(x', \xi') \in \mathbb{R}^\nu \times \mathbb{R}^\nu$, then $a^{-1} \in C^\infty(\Gamma, \mathbb{S}_h^{-k})$;*

2) *if $a_j \in C^\infty(\Gamma, \mathbb{S}_h^{k - \frac{h+1}{h}j})$ with $j = 0, 1, 2, \dots$, then there exists $a \in C^\infty(\Gamma, \mathbb{S}^k_h)$, unique modulo $C^\infty(\Gamma, \mathbb{S}_h^{-\infty})$, such that, for all $N \in \mathbb{N}$,*

$$a - \sum_{j<N} a_j \in C^\infty(\Gamma, \mathbb{S}_h^{k - \frac{h+1}{h}N}); \tag{9}$$

3) *if $a \in C^\infty(\Gamma, \mathbb{S}^k_h)$, $b \in C^\infty(\Gamma, \mathbb{S}^{k'}_h)$, then $a((x'', \xi''), x', D_{x'})\#$*

$\# b((x'', \xi''), x', D_{x'}) \in \mathrm{OP}\,\mathbb{S}_h^{k+k'}$ for any fixed $(x'', \xi'') \in \Gamma$, where $\#$ represents the composition in $\mathcal{S}(\mathbb{R}^\nu)$. Its symbol $c((x'', \xi''), x', \xi') \in$ $\in C^\infty(\Gamma, \mathbb{S}_h^{k+k'})$ and has the following asymptotic expansion

$$c((x'',\xi''),x',\xi') - \sum_{|\alpha|<N} \frac{i^{-|\alpha|}}{\alpha!} \partial_{\xi'}^{\alpha} a((x'',\xi''),x',\xi')\, \partial_{x'}^{\alpha} b((x'',\xi''),x',\xi') \in$$

$$\in C^\infty(\Gamma, \mathbb{S}_h^{k+k'-\frac{h+1}{h}N})$$

for any integer $N > 0$. Moreover, if $a((x'', \xi''), x', \xi')$ and $b((x'', \xi''), x', \xi')$ are semi-homogeneous of degree $m - \dfrac{hk}{h+1}$ and $m' - \dfrac{hk'}{h+1}$, respectively, then $c((x'', \xi''), x', \xi')$ is semi-homogeneous of degree $m + m' - \dfrac{h}{h+1}(k + k')$.

The next proposition is the main result of this section.

PROPOSITION 1.10. *Fixed an integer $k > 0$, for any integer $j \geqslant 0$ such that $k - \dfrac{h+1}{h} j \geqslant 0$, let*

$$a_j((x'', \xi''), x', \xi') = \sum_{\substack{\alpha, \beta \in \mathbb{Z}_+^\nu \\ \frac{|\alpha|}{h} + |\beta| = k - j\frac{h+1}{h}}} a_{j,\alpha,\beta}(x'', \xi'')(x')^\alpha (\xi')^\beta$$

where $a_{j,\alpha,\beta}(x'', \xi'')$ is a smooth homogeneous function of degree $m - j - |\beta|$ in U $\left(\text{with } m \geqslant \dfrac{hk}{h+1}\right)$. *Define*

$$a((x'', \xi''), x', \xi') = \sum_{j \in \mathbb{Z}_+,\, k - \frac{h+1}{h} j \geqslant 0} a_j((x'', \xi''), x', \xi').$$

and assume that, for every fixed $(x'', \xi'') \in U$:

1) *the cone of the values of the principal symbol $a_0((x'', \xi''), x', \xi')$ is not the whole complex plane;*

2) *$a_0((x'', \xi''), x', \xi') \neq 0$ whenever $0 \neq (x', \xi') \in \mathbb{R}^\nu \times \mathbb{R}^\nu$;*

3) *the operator $a((x'', \xi''), x', D_{x'}) : \mathcal{S}(\mathbb{R}^\nu) \to \mathcal{S}(\mathbb{R}^\nu)$ is injective.*

Then, there exists $q((x'', \xi''), x', \xi') \in C^\infty(U, \mathbb{S}_h^{-k})$, semi-homogeneous of degree $-m + hk/(h+1)$, such that $q((x'', \xi''), x', D_{x'}) : \mathcal{S}(\mathbb{R}^\nu) \to$ $\to \mathcal{S}(\mathbb{R}^\nu)$ is a left inverse of $A(x'', \xi'') = a((x'', \xi''), x', D_{x'}) : \mathcal{S}(\mathbb{R}^\nu) \to \mathcal{S}(\mathbb{R}^\nu)$ for any fixed $(x'', \xi'') \in U$.

PROOF. First of all, we show how to construct a parametrix of $a((x'', \xi''), x', D_{x'})$ in a way similar to the classical case. It is easy to see that $a((x'', \xi''), x', \xi') \in C^\infty(U, \mathrm{S}_h^k)$ and is semi-homogeneous of degree $m - hk/(h+1)$. Using Hypothesis 2, we prove that for any compact set H of U there exists a positive constant $C = C(H)$ for which

$$(10) \qquad |a((x'', \xi''), x', \xi')| \geqslant C(1 + |x'|^h + |\xi'|)^k$$

whenever $(x'', \xi'') \in H$ and $|x'|^h + |\xi'| \geqslant R$. Now, consider a sequence of compact sets $\{H_j\}_{j \in \mathrm{N}}$ of U such that

$$\bigcup_{j \in \mathrm{N}} H_j = U \quad \text{and} \quad H_j \subset \operatorname{interior}(H_{j+1}), \text{ for any } j.$$

Let R_j be the constant such that (10) holds when $(x'', \xi'') \in H_j$. It is no restriction to suppose that $\{R_j\}_{j \in \mathrm{N}}$ be an increasing sequence. Consider now the following sets:

$$F_1 = \bigcup_{j \in \mathrm{N}} (H_{j+1} \times \{(x', \xi') \in \mathbb{R}^\nu \times \mathbb{R}^\nu \mid |x'|^h + |\xi'| \geqslant R_j\})$$

$$F_2 = \bigcup_{j \in \mathrm{N}} (H_j \times \{(x', \xi') \in \mathbb{R}^\nu \times \mathbb{R}^\nu \mid |x'|^h + |\xi'| \geqslant 2R_j\}).$$

Then F_2 is contained in the interior of F_1, and one can find a smooth function χ defined in $U \times \mathbb{R}^\nu \times \mathbb{R}^\nu$, with $0 \leqslant \chi \leqslant 1$, $\chi|_{F_2} \equiv 1$ and $\operatorname{supp}\chi \subseteq F_1$. Put $b_0((x'', \xi''), x', \xi') = \chi(x'', \xi'', x', \xi')/a((x'', \xi''), x', \xi')$ and denote by $B_0(x'', \xi'')$ the operator $b_0((x'', \xi''), x', D_{x'})$. By Proposition 1.9 we get

$$A(x'', \xi'') \# B_0(x'', \xi'') = I + R_0(x'', \xi'')$$

where $R_0(x'', \xi'') \in \mathrm{OP}\,\mathrm{S}_h^{-(h+1)/h}$ for any fixed $(x'', \xi'') \in U$, with symbol in $C^\infty(U, \mathrm{S}_h^{-(h+1)/h})$. Using the asymptotic expansion $\sum(-1)^j R_0(x'', \xi'')^j$ (see 2) of Proposition 1.9) we deduce the existence of $B_1(x'', \xi'') \in \mathrm{OP}\,\mathrm{S}_h^{-k}$ with symbol in $C^\infty(U, \mathrm{S}_h^{-k})$ such that

$$A(x'', \xi'') \# B_1(x'', \xi'') = I + R_1(x'', \xi'')$$

where $R_1(x'', \xi'')$ is a smoothing operator for any $(x'', \xi'') \in U$, whose symbol belongs to $C^\infty(U, \mathrm{S}_h^{-\infty})$. In the same way, we can construct a right parametrix; therefore $B_1(x'', \xi'')$ is a right and left parametrix.

Now, if we consider $A(x'', \xi'')$ as an unbounded linear operator in $L^2(\mathbb{R}^\nu)$ with domain $\{u \in S'(\mathbb{R}^\nu) \mid (x')^\alpha D_{x'}^\beta u(x') \in L^2(\mathbb{R}^\nu)$ for $|\alpha|/h +$

$+ |\beta| \leq k\}$ (see Proposition 7.1.10 in [16]) for any fixed $(x'', \xi'') \in U$, then it turns out that it has compact resolvent and, hence, its spectrum consists entirely of isolated eigenvalues with finite multiplicity. To show that, we can develop a pseudodifferential calculus depending on a complex parameter in the same way as done by Helffer in Section 1.11 of [11] (see also Shubin [21]). It then suffices to replace the weight used in [11] by an anisotropic weight of the type $(1 + |x'|^h + |\xi'|)$ and observe that there exists a closed angle in the complex plane with vertex in the origin, which does not intersect the cone of values of the principal symbol of $A(x'', \xi'')$ (i.e., $a_0((x'', \xi''), x', \xi')$ described above). In fact, the set of values of $a_0((x'', \xi''), .)$ is a closed cone properly contained in the complex plane.

From the existence of the parametrix $B_1(x'', \xi'')$ and the injectivity of $A(x'', \xi'')$ in $\mathcal{S}(\mathbb{R}^\nu)$, it follows that $A(x'', \xi'')$ is injective in $L^2(\mathbb{R}^\nu)$; since its spectrum is discrete, $A(x'', \xi'')$ is invertible and its inverse $Q(x'', \xi'')$ is a continuous linear operator from $\mathcal{S}(\mathbb{R}^\nu) \mapsto \mathcal{S}(\mathbb{R}^\nu)$ depending on a parameter $(x'', \xi'') \in U$. Moreover, we have

$$Q(x'', \xi'') = B_1(x'', \xi'') - Q(x'', \xi'') \# R_1(x'', \xi'') \tag{11}$$

where $Q(x'', \xi'')\#R_1(x'', \xi'')$ is a continuous map from $\mathcal{S}'(\mathbb{R}^\nu)$ to $\mathcal{S}(\mathbb{R}^\nu)$ for any fixed $(x'', \xi'') \in U$; thus $Q(x'', \xi'') \in \mathrm{OP}\,\mathbb{S}_h^{-k}$ for any given $(x'', \xi'') \in U$. Denote by $q((x'', \xi''), x', \xi')$ its symbol. One can prove that $Q(x'', \xi''): \mathcal{S}(\mathbb{R}^\nu) \to \mathcal{S}(\mathbb{R}^\nu)$ is smoothly dependent on $(x'', \xi'') \in U$; whence, the symbol of $Q(x'', \xi'')\#R_1(x'', \xi'')$ belongs to $C^\infty(U, \mathbb{S}_h^{-\infty})$. Therefore, $q((x'', \xi''), x', \xi')$ belongs to $C^\infty(U, \mathbb{S}_h^{-k})$ by identity (11) and is semi-homogeneous of degree $-m + hk/(h+1)$, since $A(x'', \xi'')$ is semi-homogeneous of degree $m - hk/(h+1)$ (see [19]). ■

We conclude this section with the following important result.

PROPOSITION 1.11. *Let $a \in C^\infty(U, \mathbb{S}_h^k)$ be semi-homogeneous of degree $m - \dfrac{hk}{h+1}$, and $b \in C^\infty(U, \mathbb{S}_h^{k'})$ be semi-homogeneous of degree $m' - \dfrac{hk'}{h+1}$. Denote by $\partial_{\xi''}^\alpha a((x'', \xi''), x', D_{x'})$, $\partial_{x''}^\alpha b((x'', \xi''), x', D_{x'})$ the pseudodifferential operators of the type* (8) *depending on a parameter (x'', ξ''), whose symbols are $\partial_{\xi''}^\alpha a((x'', \xi''), x', \xi')$, $\partial_{x''}^\alpha b((x'', \xi''), x', \xi')$, respectively. Then, one has that $a(x, D) \in \mathrm{OP}S_h^{m, k}$, $b(x, D) \in \mathrm{OP}S_h^{m', k'}$*

and for any integer $N>0$,

$$\sigma(a(x,D)\circ b(x,D))-$$

$$-\sum_{|\alpha|<N}\frac{i^{-|\alpha|}}{\alpha!}\sigma(\partial_{\xi''}^{\alpha}a(x',x'',D_{x'},\xi'')\#\partial_{x''}^{\alpha}b(x',x'',D_{x'},\xi''))\in S_h^{m+m'-N,k+k'}.$$

PROOF. We can repeat, step by step, the arguments used to prove Proposition 1.4, by constructing a new class of symbols related to $\mathbb{S}_h^k$ in the same way the class $QS_h^{m,k,l}$ is related to $S_h^{m,k}$. ■

1.6. *The main result.*

Let $P=p(x,D)$ be a properly supported operator as in Section 1.4 with $m\in\mathbb{R}_+$ and $k\in\mathbb{Z}_+$. We suppose that its characteristic set is the cone $\Sigma=\Sigma_1\cap\Sigma_2$ previously defined and, moreover, that in a small conic neighborhood of Σ

$$|p_m(x,\xi)|\gtrsim|\xi|^m\left(|x'|^h+\frac{|\xi'|}{|\xi|}\right)^k. \tag{12}$$

When (12) holds, we shall say that $p_m(x,\xi)$ is transversally elliptic. In view of (12), P is elliptic of degree m outside Σ; thus, in order to construct a parametrix, it is enough to study P in a small conic neighborhood of Σ. For this reason, we can assume that $|\xi'|\leqslant C|\xi''|$ for some convenient constant $C>0$.

Now we can state the main result of this paper.

THEOREM 1.12. *If* $p(x,D)$ *satisfies the above conditions, then the following statements are equivalent:*

(I) $p(x,D)$ *has a left (resp. right) parametrix* B *in* $\mathrm{OP}S_h^{-m,-k}$;

(II) *For any fixed* $\varrho\in\Sigma$, *the operator* $P_\varrho(x',D_{x'})$ *(described in Section* 1.4*) is injective from* $\mathcal{S}(\mathbb{R}^\nu)$ *to* $\mathcal{S}(\mathbb{R}^\nu)$ *(resp. surjective from* $\mathcal{S}'(\mathbb{R}^\nu)$ *to* $\mathcal{S}'(\mathbb{R}^\nu)$*)*;

(III) $p(x,D)$ *(resp. its formal adjoint* $p^*(x,D)$*) is hypoelliptic with loss of* $hk/(h+1)$ *derivatives.*

PROOF. We show that (I) ⇒ (III) ⇒ (II) ⇒ (I) for the operator $p(x,D)$. From Lemma 1.7, it clearly follows that (I) ⇒ (III). In order to see that (III) implies (II), by the Closed Graph Theorem we prove that, for any compact set $K \subset \mathbb{R}^n$ and for any $s, s' \in \mathbb{R}$ with $s' < s + m - \frac{hk}{h+1}$, there exists a positive constant $C = C(K, s, s')$ for which

$$\|u\|_{s+m-\frac{hk}{h+1}} \leqslant C(\|Pu\|_s + \|u\|_{s'}), \qquad \forall u \in C_0^\infty(K). \tag{13}$$

We now show that (13) implies (II). Take $\varrho = (0, x_0'', 0, \xi_0'')$; since $p_\Sigma(x', x'', \xi', \xi'')$ is semi-homogeneous of order $m - hk/(h+1)$, we can suppose $|\xi_0''| = 1$ without loss of generality. Fix two compact neighborhoods $K'' \subset \mathbb{R}^{n-\nu}$ of x_0'' and $K' \subset \mathbb{R}^\nu$ of the origin $0 \in \mathbb{R}^\nu$. Define $K = K' \times K'' \subset \mathbb{R}^n$ and $\widetilde{K} \subset \mathbb{R}^n$ a compact such that $\operatorname{supp}(Pu) \subsetneq \widetilde{K}$ if $u \in C_0^\infty(K)$. Choose $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi \equiv 1$ near $\widetilde{K}$, so that $Pu = \chi P(u)$, for $u \in C_0^\infty(K)$. Hence, we can assume that $p(x, \xi)$ is compactly supported in x. Let now $v' \in C_0^\infty(\mathbb{R}^\nu)$ and $v'' \in C_0^\infty(\mathbb{R}^{n-\nu})$ such that $\int |v''(x'')|^2 dx'' = 1$. Define $v(x', x'') = v'(x')\, v''(x'') \in C_0^\infty(\mathbb{R}^n)$. Put (with $t \geqslant 1$)

$$u_t(x) = e^{it^{h+1}\langle x'', \xi_0''\rangle} v(tx', t(x'' - x_0'')).$$

We observe that for t large, $u_t \in C_0^\infty(K)$ and, after a few computations, by (13) we get

$$\|Pu_t\|^2_{L^2(\mathbb{R}^n)} =$$

$$= t^{2(h+1)(m-\frac{hk}{h+1})-n} \|p_\Sigma(x', x_0'', D_{x'}, \xi_0'')\, v'\|^2_{L^2(\mathbb{R}^\nu)} + O(t^{2(h+1)(m-\frac{hk}{h+1}-\frac{1}{2(h+1)})-n}) \geqslant$$

$$\geqslant \widetilde{C}^2 t^{2(h+1)(m-\frac{hk}{h+1})-n} (\|v'\|^2_{L^2(\mathbb{R}^\nu)} + o(1)) - t^{2(h+1)s'-n} (\|v'\|^2_{L^2(\mathbb{R}^\nu)} + o(1))$$

as $t \to +\infty$. Dividing by $t^{2(h+1)(m-\frac{hk}{h+1})-n}$, and letting $t \to +\infty$, gives

$$\|p_\Sigma(x', x_0'', D_{x'}, \xi_0'')\, v'\|_{L^2(\mathbb{R}^\nu)} \geqslant \widetilde{C} \|v'\|_{L^2(\mathbb{R}^\nu)} \tag{14}$$

for every $v' \in C_0^\infty(\mathbb{R}^\nu)$ and, by density, for every $v' \in \mathcal{S}(\mathbb{R}^\nu)$. In view of relation (5), this completes the first part of the proof.

In order to see that (I) is a consequence of (II), we follow the approach of Helffer in [8]. If we define $q_1(x, \xi) = \overline{p_m(x, \xi)}/(|p_m(x, \xi)|^2 + |\xi|^{2m-\frac{2hk}{h+1}})$, then the operator $Q_1 = q_1(x, D) \in \mathrm{OP}S_h^{-m, -k}$ and by Proposition 1.4 we get

$$Q_1 P = I + R_1$$

with $R_1 \in \mathrm{OP}S_h^{-\frac{1}{h+1}, -\frac{1}{h}}$. Using the asymptotic expansion $\sum(-1)^j R_1^j$ (see

Proposition 1.2.2) we deduce the existence of $Q_2 \in \mathrm{OP}S_h^{-m,\,-k}$ such that

$$Q_2 P = I + R_2$$

with $R_2 \in \mathrm{OP}\mathcal{H}_h^0$.

Now, we observe that $P_\Sigma^* \# P_\Sigma$ is a parameter-dependent pseudodifferential operator, whose symbol is

$$\sigma(P_\Sigma^* \# P_\Sigma) = \sum_{\alpha \in \mathbb{Z}_+^\nu} \frac{i^{-|\alpha|}}{\alpha!} \partial_{\xi'}^\alpha p_\Sigma^* \, \partial_{x'}^\alpha p_\Sigma = \sum_{\substack{l \in \mathbb{Z}_+ \\ 2k - l\frac{h+1}{h} \geq 0}} \sigma_l(P_\Sigma^* \# P_\Sigma)$$

with

$$\sigma_l(P_\Sigma^* \# P_\Sigma) = \sum_{\substack{j,\, s \in \mathbb{Z}_+;\; \alpha \in \mathbb{Z}_+^\nu \\ j + s + |\alpha| = l}} \frac{i^{-|\alpha|}}{\alpha!} \partial_{\xi'}^\alpha p_{\Sigma,\,j}^* \, \partial_{x'}^\alpha p_{\Sigma,\,s} \in C^\infty\left(U,\, \mathbb{S}_h^{2k - l\frac{h+1}{h}}\right).$$

Therefore $\sigma(P_\Sigma^* \# P_\Sigma) \in C^\infty(U,\, \mathbb{S}_h^{2k})$ and is semi-homogeneous of degree $2m - 2hk/(h+1)$. Furthermore, the principal symbol $\sigma_0(P_\Sigma^* \# P_\Sigma)$ is real non negative for any $(x'', \xi'') \in U$. By (12) it follows that for any compact $H \subset U$ there exists a constant $C > 0$ for which

$$\sigma_0(P_\Sigma^* \# P_\Sigma) = |p_{\Sigma,\,0}|^2 \geq C^2(|x'|^h + |\xi'|)^{2k}$$

whenever $(x'', \xi'') \in H$. Finally, by Hypothesis (II) $P_\Sigma^* \# P_\Sigma \colon \mathcal{S}(\mathbb{R}^\nu) \to \mathcal{S}(\mathbb{R}^\nu)$ is injective for any $(x'', \xi'') \in U$. Thus, using Proposition 1.10 we get a left inverse Q of $P_\Sigma^* \# P_\Sigma$ depending on a parameter $(x'', \xi'') \in U$ so that the operator $Q \# P_\Sigma^*$ is a left pseudodifferential inverse of P_Σ, whose symbol $c((x'', \xi''), x', \xi')$ belongs to $C^\infty(U,\, \mathbb{S}_h^{-k})$ and is semi-homogeneous of degree $-m + hk/(h+1)$. Using Lemma 1.8, it immediately follows that $C = c(x, D) \in \mathrm{OP}S_h^{-m,\,-k}$ and by Proposition 1.11, we get

$$CP_\Sigma(x, D) = I + R_3$$

with $R_3 \in \mathrm{OP}S_h^{-1,\,0} \subset \mathrm{OP}S_h^{0,\,\frac{h+1}{h}}$. Moreover, by (6) we have

$$C(P - P_\Sigma(x, D)) = R_4$$

where $R_4 \in \mathrm{OP}S_h^{0,\,\frac{1}{h}}$. Therefore, we deduce that

$$CP = I + R_5$$

with $R_5 \in \mathrm{OP}S_h^{0,\,\frac{1}{h}}$. Now, using the asymptotic expansion $\sum(-1)^j R_5^j$ (see

Proposition 1.2.1) we get the existence of $Q_3 \in \mathrm{OP}S_h^{-m, -k}$ such that

$$Q_3 P = I + R_6$$

where $R_6 \in \mathrm{OP}S_h^{0, \infty}$. Finally, if we define $B = -R_6 Q_2 + Q_3$, we obtain

$$BP = -R_6 - R_6 R_2 + I + R_6 = I - R_6 R_2$$

where $R_6 R_2 \in \mathrm{OP}S^{-\infty}$ by Corollary 1.5.

It remains to prove the theorem for the formal adjoint $P^* = p^*(x, D)$. First of all, we observe that $p^*(x, D)$ is a classical pseudo-differential operator which belongs to $\mathrm{OP}S_h^{m, k}$ by Proposition 1.6; its principal symbol is $\overline{p_m(x, \xi)}$ and by (12) satisfies $|\overline{p_m(x, \xi)}| \geqslant |\xi|^m \left(|x'|^h + \frac{|\xi'|}{|\xi|} \right)^k$. Now if B is a right parametrix for $p(x, D)$ in $\mathrm{OP}S_h^{-m, -k}$, then, from Proposition 1.6, its formal adjoint B^* is a left parametrix for $p^*(x, D)$ in $\mathrm{OP}S_h^{-m, -k}$. By the classical theory of PDO we get that $p^* = e^{i\langle D_\xi, D_x \rangle} \overline{p}$, and hence

$$(p^*)^W = e^{-i\langle D_\xi, D_x \rangle/2} p^* = \overline{p^W}. \tag{15}$$

Thus, for any $\varrho \in \Sigma$

$$(P^*)_\varrho(x', D_{x'}) = \mathrm{Op}^W(p_\varrho^*) = (\mathrm{Op}^W(p_\varrho))^* = (P_\varrho(x', D_{x'}))^*. \tag{16}$$

Finally, we observe that if $P_\varrho(x', D_{x'})$ is surjective in $\mathcal{S}'(\mathbb{R}^\nu)$, then $(P^*)_\varrho(x', D_{x'})$ is injective in $\mathcal{S}(\mathbb{R}^\nu)$ by virtue of (16). Thus, in view of the previous arguments, P^* has a left parametrix in $\mathrm{OP}S_h^{-m, -k}$ and, hence, P has a right parametrix in $\mathrm{OP}S_h^{-m, -k}$. This concludes the proof. ■

2. Hypoellipticity for a class of pseudodifferential operators with double characteristics.

2.1. *A class of pseudodifferential operators with double characteristics.*

Let X be an open set of $\mathbb{R}^n$ and Σ_1, Σ_2 be two involutive closed cones of codimension ν in $T^*X \backslash 0$, with transversal and symplectic intersection Σ. There locally exist some smooth functions $u_{1,1}, \ldots, u_{1,\nu}$ and $u_{2,1}, \ldots, u_{2,\nu}$ homogeneous of degree 0 and of degree 1, respectively, in $T^*X \backslash 0$ such that their differentials $du_{s,j} (s = 1, 2; j = 1, \ldots, \nu)$ are linearly independent and Σ_s is defined by $u_{s,1} = \ldots = u_{s,\nu} = 0$ (with s=1,2). In the following of the paper, it will be useful to observe that the equations $u_{s,j}$ $(s = 1, 2; j = 1, \ldots, \nu)$ can be extended to a complete system of local coordinates of $T^\star X \backslash 0$ near Σ. Furthermore, since Σ_1, Σ_2 are involutive, we locally have

$$\text{(17)}\quad \{u_{s,i}, u_{s,j}\} = \sum_{k=1}^{n} \left(\frac{\partial u_{s,i}}{\partial \xi_k} \frac{\partial u_{s,j}}{\partial x_k} - \frac{\partial u_{s,i}}{\partial x_k} \frac{\partial u_{s,j}}{\partial \xi_k} \right) = 0 \quad \text{on } \Sigma_s (s=1,2)$$

and, since $\Sigma = \Sigma_1 \cap \Sigma_2$ is symplectic, the matrix

$$\text{(18)}\qquad \begin{pmatrix} (\{u_{1,i}, u_{1,j}\})_{i,j=1,\ldots,\nu} & (\{u_{1,i}, u_{2,j}\})_{i,j=1,\ldots,\nu} \\ (\{u_{2,i}, u_{1,j}\})_{i,j=1,\ldots,\nu} & (\{u_{2,i}, u_{2,j}\})_{i,j=1,\ldots,\nu} \end{pmatrix}$$

is invertible at every point of Σ (described by the local equations $u_{i,j}$).

The following proposition shows that, by using a canonical diffeomorphism, we can locally reduce $\Sigma_1, \Sigma_2, \Sigma$ to the flat case.

PROPOSITION 2.1. *Let $\Sigma_1, \Sigma_2, \Sigma$ be as above. Then, for every point ϱ of Σ, there exist a conic neighborhood U of ϱ in $T^\star X \backslash 0$, a conic neighborhood V in $T^\star \mathbb{R}^n \backslash 0$ and a canonical symplectomorphism (i.e., homogeneous of degree one in the fibers) $\chi : U \to V$ for which $\chi(U \cap \Sigma_1) = \{(y, \eta) \in V | y_1 = \ldots = y_\nu = 0\}$ and $\chi(U \cap \Sigma_2) = \{(y, \eta) \in V | \eta_1 = \ldots = \eta_\nu = 0\}$. Such a map χ will be called a local canonical flattening of Σ_1 and Σ_2 near ϱ.*

PROOF. This result is a consequence of Theorem 21.2.4 [13] (see also Lemma 4.1 [14]). ■

At this point we would like to define a set of symbols in $T^\star X \backslash 0$ which is invariant under change of coordinates and extends the class $S_h^{m,k}$ of the flat case. Unfortunately, the construction of a pseudodifferential calculus associated with these new symbols is not a trivial adjustment of the methods used in [1], [9]; more precisely, the substitution of $1/r^{\frac{1}{2}}$ with $1/r^{\frac{h}{h+1}}$ in our weight d_Σ involves some difficulties in the proof of the stability under composition and of the invariance under canonical symplectomorphisms, which we do not know how to overcome in the general case, i.e. for arbitrary k. However, if we restrict to the study of classical pseudodifferential operators with double characteristics (i.e., the ones in

$\mathrm{OP}S_h^{m,2}$ in the flat case), then we are able to treat our hypoellipticity problem proceeding in a different way from [1], [9].

Let $P = p(x, D)$ be a classical pseudodifferential operator, with asymptotic expansion $p \sim \sum_{j \geq 0} p_{m-j}$. We denote by $p^s_{m-1}(x, \xi)$ the subprincipal symbol of P

$$p^s_{m-1}(x, \xi) = p_{m-1}(x, \xi) + \frac{i}{2} \sum_{j=1}^{n} \frac{\partial^2 p_m}{\partial x_j \partial \xi_j}(x, \xi).$$

We shall say that P satisfies the vanishing conditions (H) if:

$$(\mathrm{H}) \quad \begin{cases} \forall \alpha, \beta \in \mathbb{Z}^\nu_+ \text{ with } |\alpha|/h + |\beta| < 2: & \partial^\alpha_{u_1} \partial^\beta_{u_2} p_m |_\Sigma \equiv 0 \\ \forall \alpha \in \mathbb{Z}^\nu_+ \text{ with } |\alpha| < h - 1: \partial^\alpha_{u_1} p^s_{m-1} |_\Sigma \equiv 0 \end{cases}$$

where $\partial^\alpha_{u_s} = \partial^{\alpha_1}_{u_{s,1}} \partial^{\alpha_2}_{u_{s,2}} \dots \partial^{\alpha_\nu}_{u_{s,\nu}}$ with $s = 1, 2$.

REMARK 2.2. *We point out that the conditions* (H) *do not depend on the particular equations $u_{s,j}$ chosen to described locally Σ_1, Σ_2. Furthermore, we observe that if Σ_1, Σ_2 are flat as in Section* 1, *then every classical operator $p(x, D)$ verifying* (H) *belongs to* $\mathrm{OP}S_h^{m,2}$ *as a consequence of Proposition* 1.1.

The next result is crucial to prove the invariance of conditions (H) under homogeneous symplectomorphisms and related elliptic Fourier integral operators.

PROPOSITION 2.3. *Let $P = p(x, D)$ be as above described. The following statements are equivalent:*

1) *P satisfies the vanishing conditions* (H);

2) *there exist classical pseudodifferential operators $U_{1,j}$, $U_{2,j}$ with principal symbols $u_{1,j}$, $u_{2,j}$ $(j = 1, \dots, \nu)$ such that P admits the following decomposition*

$$(19) \quad P = \sum_{\substack{\alpha, \beta \in \mathbb{Z}^\nu_+ \\ \frac{|\alpha|}{h} + |\beta| = 2}} A_{\alpha,\beta} U_1^\alpha U_2^\beta + \\ + \sum_{\substack{\alpha \in \mathbb{Z}^\nu_+ \\ |\alpha| = h-1}} B_\alpha U_1^\alpha + \sum_{\substack{\beta \in \mathbb{Z}^\nu_+ \\ |\beta| = 1}} C_\beta U_2^\beta \quad (\mathrm{mod.}\, \mathrm{OP}S^{m-2})$$

where $U_s^\alpha = U_{s,1}^{\alpha_1} U_{s,2}^{\alpha_2} \dots U_{s,\nu}^{\alpha_\nu}$ $(s = 1, 2)$ and $A_{\alpha,\beta}$, B_α, C_β are classical *pseudodifferential operators of degree $m-|\beta|$, $m-1$, $m-2$, respectively.*

PROOF. It suffices to observe that assertions 1) and 2) are each other equivalent to the following result about the structure of $p_m(x, \xi)$ and $p^s_{m-1}(x, \xi)$; there exist some smooth functions $a_{\alpha,\beta}(x, \xi)$, $b_\alpha(x, \xi)$, $c_\beta(x, \xi)$ homogeneous of degree $m-|\beta|$, $m-1$, $m-2$, respectively, such that

$$(20) \qquad p_m(x, \xi) = \sum_{\substack{\alpha, \beta \in Z^\nu_+ \\ \frac{|\alpha|}{h} + |\beta| = 2}} a_{\alpha,\beta}(x, \xi)\, u_1^\alpha(x, \xi)\, u_2^\beta(x, \xi)$$

$$(21) \qquad p^s_{m-1}(x, \xi) = \sum_{\substack{\alpha \in Z^\nu_+ \\ |\alpha| = h-1}} b_\alpha(x, \xi)\, u_1^\alpha(x, \xi) + \sum_{\substack{\beta \in Z^\nu_+ \\ |\beta| = 1}} c_\beta(x, \xi)\, u_2^\beta(x, \xi)$$

with the usual notation $u_s^\alpha(x, \xi) = u_{s,1}^{\alpha_1}(x, \xi)\, u_{s,2}^{\alpha_2}(x, \xi) \dots u_{s,\nu}^{\alpha_\nu}(x, \xi)$ $(s = 1, 2)$.

This concludes the proof. ■

Let $X, Y \subset \mathbb{R}^n$ be open set and let

$$\chi : T^*X \backslash 0 \rightarrow T^*Y \backslash 0$$

be a smooth homogeneous (of degree one in the fibers) canonical transformation. Let $\Lambda_\chi \subset (T^*Y\backslash 0) \times (T^*X\backslash 0)$ (resp. $\Lambda_{\chi^{-1}} \subset (T^*X\backslash 0) \times \times (T^*Y\backslash 0)$) be the canonical relation associated with χ (resp. χ^{-1}) and finally denote by

$$F \in I^0(Y \times X, \Lambda_\chi) \qquad (\text{resp. } F^{-1} \in I^0(X \times Y, \Lambda_{\chi^{-1}}))$$

an elliptic Fourier integral operator of order 0, associated with Λ_χ (resp. $\Lambda_{\chi^{-1}}$) (see [12] or [22]), with $FF^{-1} \equiv I$, $F^{-1}F \equiv I$. As it is well-known, we have that $\widetilde{U}_{s,j} = FU_{s,j}F^{-1}$ is also a classical pseudodifferential operator with principal symbol $u_{s,j} \circ \chi^{-1}$ for any $s = 1, 2$ and $j = 1, 2, \dots, \nu$. Furthermore, $\widetilde{A}_{\alpha,\beta} = FA_{\alpha,\beta}F^{-1}$ (with $|\alpha|/h + |\beta| = 2$), $\widetilde{B}_\alpha = FB_\alpha F^{-1}$ (with $|\alpha| = h-1$) and $\widetilde{C}_\beta = FC_\beta F^{-1}$ (with $|\beta| = 1$) are classical pseudodifferential operators of the same order of $A_{\alpha,\beta}$, B_α and C_β respectively. Hence

(22) $\quad \tilde{P} = FPF^{-1} =$

$$= \sum_{\substack{\alpha,\beta\in\mathbb{Z}_+^\nu \\ \frac{|\alpha|}{h}+|\beta|=2}} \tilde{A}_{\alpha,\beta}\tilde{U}_1^\alpha\tilde{U}_2^\beta + \sum_{\substack{\alpha\in\mathbb{Z}_+^\nu \\ |\alpha|=h-1}} \tilde{B}_\alpha\tilde{U}_1^\alpha + \sum_{\substack{\beta\in\mathbb{Z}_+^\nu \\ |\beta|=1}} \tilde{C}_\beta\tilde{U}_2^\beta \qquad (\mathrm{mod.\,OP}S^{m-2}).$$

Therefore, similarity preserves the structure shown in (19), whence the invariance of the vanishing conditions (H) under homogeneous symplectomorphisms immediately follows from Proposition 2.3. Moreover, by Propositions 2.1 and 1.1, we see that every classical pseudodifferential operator satisfying (H) belongs, microlocally, to the class $\mathrm{OP}S_h^{m,2}$, modulo a suitable canonical symplectomorphism.

2.2. *The localized polynomial and its invariant meaning.*

We shall introduce an invariant «naturally» attached to the pseudodifferential operators in the class considered above, which generalizes the notion of localized polynomial described in Section 1.4. We shall give an intrinsic definition of the «new» localized polynomial strictly related to the geometry of the cone Σ (see Parenti-Parmeggiani [18], for example). From now on we suppose that $P = p(x, D)$ be a classical pseudodifferential operator with principal symbol $p_m(x,\xi)$ and subprincipal symbol $p_{m-1}^s(x,\xi)$. In order to make more readable the following exposition, we fix some notations. Let $\mathfrak{I} = \{1, 2, \ldots, \nu\}$ be a set of indexes; let $p\in\mathbb{Z}_+$, $\gamma\in\mathfrak{I}^p$ ($\mathfrak{I}^p$ denotes the p-fold Cartesian product of $\mathfrak{I}$ if $p\in\mathbb{N}$ and the empty set if $p=0$) and define $(H_{u_s})_\gamma = H_{u_{s,\gamma_1}}H_{u_{s,\gamma_2}}\ldots H_{u_{s,\gamma_p}}$ with $s = 1, 2$, where $H_{u_{s,j}}$ denotes the Hamiltonian field of $u_{s,j}$ in $T^*X\backslash 0$ ($s = 1, 2$ and $j = 1, 2, \ldots, \nu$). Moreover, for any multi-index $\alpha\in\mathbb{Z}_+^\nu$ we define $(H_{u_s})^\alpha = H_{u_{s,1}}^{\alpha_1}H_{u_{s,2}}^{\alpha_2}\ldots H_{u_{s,\nu}}^{\alpha_\nu}$.

The next lemma will be useful in the following of the paper:

LEMMA 2.4. *If $p_m(x,\xi)$, $p_{m-1}^s(x,\xi)$ satisfy the vanishing conditions* (H), *then one has*:

1) *for every $p, q\in\mathbb{Z}_+$ with $p/h+q<2$ and for every $\gamma\in\mathfrak{I}^p, \theta\in\mathfrak{I}^q$*

$$(H_{u_2})_\gamma(H_{u_1})_\theta p_m|_\Sigma \equiv 0\,, \qquad (H_{u_1})_\theta(H_{u_2})_\gamma p_m|_\Sigma \equiv 0$$

2) *for every* $p \in \mathbb{Z}_+$ *with* $p < h - 1$ *and for every* $\gamma \in \mathfrak{I}^p$

$$(H_{u_2})_\gamma p^s_{m-1} |_\Sigma \equiv 0$$

PROOF. As in the proof of Proposition 2.3, we can show that $p_m(x, \xi)$ and $p^s_{m-1}(x, \xi)$ have the structure described in (20) and (21). Moreover, from (17) we have $H_{u_{2,s}} u_{2,j} = \{u_{2,s}, u_{2,j}\} \equiv 0$ on Σ_2. A computation completes the proof. ■

For any $\varrho \in \Sigma$, we consider the symplectic vector space $(T_\varrho \Sigma)^\sigma$ (i.e., the symplectic orthogonal of $T_\varrho \Sigma$ with respect to the canonical 2-form $\sum d\xi_j \wedge dx_j$ of $T^* X \backslash 0$). Since Σ_1 and Σ_2 have a transversal intersection, it follows that $(T_\varrho \Sigma)^\sigma = (T_\varrho \Sigma_1)^\sigma \oplus (T_\varrho \Sigma_2)^\sigma$. Whence, any $v \in T_\varrho T^* X$ can be uniquely decomposed as $v = v_1 + v_2$ with $v_1 \in (T_\varrho \Sigma_1)^\sigma$ and $v_2 \in (T_\varrho \Sigma_2)^\sigma$. Let V_1, V_2 be two smooth sections of $TT^* X$ defined in a neighborhood U of ϱ such that, for any $\varrho' \in \Sigma_s \cap U$, $V_s(\varrho') \in (T_{\varrho'} \Sigma_s)^\sigma$ and $V_s(\varrho) = v_s$ (with $s = 1, 2$). We can now introduce the main invariant attached to the operator $P = p(x, D)$.

DEFINITION 2.1. *The localized polynomial* $p_\varrho(v)$ *of* $P = p(x, D)$ *in* $\varrho \in \Sigma$ *with* $v \in (T_\varrho \Sigma)^\sigma$ *is defined as*

$$(23) \quad p_\varrho(v) = \sum_{\substack{p, q \in \mathbb{Z}_+ \\ p/h + q = 2}} \frac{1}{p! q!} (V_2^p V_1^q p_m)(\varrho) + \frac{1}{(h-1)!} (V_2^{h-1} p^s_{m-1})(\varrho)$$

We now want to prove that the above definition is independent of the extensions V_1, V_2 of v_1, v_2. Suppose that $\Sigma_1 \cap U = \{(x, \xi) \in U \mid u_{1,1}(x, \xi) = \ldots = u_{1,\nu}(x, \xi) = 0\}$ and $\Sigma_2 \cap U = \{(x, \xi) \in U \mid u_{2,1}(x, \xi) = \ldots = u_{2,\nu}(x, \xi) = 0\}$ with the same assumptions on $u_{s,j}$ of Section 2.1. There exist some smooth functions $c_{1,1}, \ldots, c_{1,\nu}, c_{2,1}, \ldots, c_{2,\nu}$ and two smooth vector fields W_1, W_2 vanishing on Σ_1, Σ_2, respectively, which are defined near ϱ, such that (with $s = 1, 2$)

$$V_s = \sum_{j=1}^{\nu} c_{s,j} H_{u_{s,j}} + W_s.$$

If we use these relations in (23), by Lemma 2.4 we can write the localized polynomial in a more explicit form

$$(24)\qquad p_\varrho(v) = \sum_{\substack{\alpha,\,\beta \in \mathbb{Z}_+^\nu \\ \frac{|\alpha|}{h} + |\beta| = 2}} \frac{1}{\alpha!\beta!} (H_{u_2}^\alpha H_{u_1}^\beta p_m)(\varrho)\, c_1^\beta(\varrho)\, c_2^\alpha(\varrho) + $$

$$+ \sum_{\substack{\alpha \in \mathbb{Z}_+^\nu \\ |\alpha| = h-1}} \frac{1}{\alpha!} (H_{u_2}^\alpha p_{m-1}^s)(\varrho)\, c_2^\alpha(\varrho)$$

(with the usual convention $c_s^\alpha(\varrho) = c_{s,1}^{\alpha_1}(\varrho) c_{s,2}^{\alpha_2}(\varrho) \dots c_{s,\nu}^{\alpha_\nu}(\varrho)$ if $s = 1, 2$). It is easy to see that the r.h.s. of (24) depends on v_1 and v_2 only, hence, on v only. Moreover, the l.h.s. is independent of the local equations $u_{s,j}$ chosen to describe Σ_1 and Σ_2 near ϱ.

The next lemma puts in evidence the behaviour of the localized polynomial under symplectic change of coordinates.

LEMMA 2.5. *If F is a classical properly supported elliptic Fourier operator of order 0 associated with a homogeneous symplectomorphism $\chi : T^*X\backslash 0 \to T^*Y\backslash 0$ and* $\mathrm{Op}(\tilde p) = F\,\mathrm{Op}(p) F^{-1}$, then for any $\varrho \in \Sigma$, *for any $v \in (T_\varrho \Sigma)^\sigma$,*

$$\tilde p_{\chi(\varrho)}(d\chi(\varrho)\, v) = p_\varrho(v).$$

PROOF. Let V_1, V_2 as defined above; notice that $\chi(\Sigma_s \cap U)$ is a neighborhood of $\chi(\varrho)$ in $T^*Y\backslash 0$ (with $s = 1, 2$). If we define, for every $\varrho' \in \chi(\Sigma_s \cap U)$, $\tilde V_s(\varrho') = d\chi(\chi^{-1}(\varrho'))(V_s(\chi^{-1}(\varrho')))$ $(s = 1, 2)$, then $\tilde V_1$, $\tilde V_2$ are two smooth sections of TT^*Y by means of which we can construct the localized polynomial of $\tilde P$ in $\chi(\varrho)$ and evalued in $d\chi(\varrho)(v)$ as previously done in (23). Let $\tilde p_m$, $\tilde p_{m-1}^s$ be the principal and subprincipal symbols of $\mathrm{Op}(\tilde p)$. As it is well-known, one has $p_m = \tilde p_m \circ \chi$ so that $(V_2^p V_1^q p_m)(\varrho) = (\tilde V_2^p \tilde V_1^q \tilde p_m)(\chi(\varrho))$ (with $p/h + q = 2$). Moreover, if one considers the structure of $\mathrm{Op}(p)$ and $\mathrm{Op}(\tilde p)$ shown in (19), (22) and writes the expression of $p_{m-1}^s(x, \xi)$, $\tilde p_{m-1}^s(y, \eta)$, then one can easily see that $(V_2^{h-1} p_{m-1}^s)(\varrho) = (V_2^{h-1}(\tilde p_{m-1}^s \circ \chi))(\varrho) = (\tilde V_2^{h-1} \tilde p_{m-1}^s)(\chi(\varrho))$, since Σ_2 is an involutive cone. In fact, the terms in $p_{m-1}^s(x, \xi)$ (resp., in $\tilde p_{m-1}^s(y, \eta)$), which do not vanish in ϱ (resp., in $\chi(\varrho)$) under the iterated action of the vector field V_2 (resp., $\tilde V_2$), are the only ones that involve the principal symbols of the operators $A_{\alpha,\beta}$, B_α, C_β, $U_{s,j}$ used in (19) (resp., of the operators $\tilde A_{\alpha,\beta}$, $\tilde B_\alpha$, $\tilde C_\beta$, $\tilde U_{s,j}$ used in (22)). This completes the proof. ∎

In order to «quantize» the polynomials $p_\varrho(.)$, fix any $\varrho \in \Sigma$ and let

$$\psi : T^*\mathbb{R}^\nu \simeq \mathbb{R}^\nu_{x'} \times \mathbb{R}^\nu_{\xi'} \to (T_\varrho \Sigma)^\sigma$$

be a linear symplectomorphism. Put

$$p_{\varrho,\psi}(x', \xi') = p_\varrho(\psi(x', \xi'))$$

and denote by $P_{\varrho,\psi} = P_{\varrho,\psi}(x', D_{x'})$ the operator

$$\mathrm{Op}^W(p_{\varrho,\psi})(x', D_{x'}) : \mathcal{S}(\mathbb{R}^\nu) \to \mathcal{S}(\mathbb{R}^\nu)$$

defined as

$$Op^W(p_{\varrho,\psi})(x', D_{x'})\, f(x') =$$

$$= (2\pi)^{-\nu} \int\int e^{i\langle x'-y', \eta'\rangle} p_{\varrho,\psi}\left(\frac{x'+y'}{2}, \eta'\right) f(y')\, dy'\, d\eta', \qquad f \in \mathcal{S}(\mathbb{R}^\nu).$$

The next lemma will be useful in the sequel of the paper.

LEMMA 2.6. *One has:*

1) *If $\psi, \psi' : T^*\mathbb{R}^\nu \to (T_\varrho\Sigma)^\sigma$ are two linear symplectic maps, then the related operators $P_{\varrho,\psi}(x', D_{x'})$ and $P_{\varrho,\psi'}(x', D_{x'})$ are unitarily equivalent.*

2) *If the operators $P = \mathrm{Op}(p)$ and $\tilde{P} = \mathrm{Op}(\tilde{p})$ are defined as in Lemma 2.5, then*

$$\tilde{P}_{\chi(\varrho),\, d\chi(\varrho) \circ \psi} = P_{\varrho,\psi}$$

for any $\varrho \in \Sigma$ and any linear symplectic map $\psi : T^*\mathbb{R}^\nu \to (T_\varrho\Sigma)^\sigma$.

PROOF. In order to prove 1), we observe that $p_{\varrho,\psi'} = p_{\varrho,\psi} \circ (\psi^{-1} \circ \psi')$, whence, as it is well-known (see [13], Vol. III, thm. 18.5.9), there exists an unitary operator $Q : L^2(\mathbb{R}^\nu) \to L^2(\mathbb{R}^\nu)$ (uniquely determined up to a complex factor of modulus 1), which is also an automorphism of $\mathcal{S}(\mathbb{R}^\nu)$ and $\mathcal{S}'(\mathbb{R}^\nu)$, such that

$$P_{\varrho,\psi'} = Q^{-1} P_{\varrho,\psi} Q\,. \tag{25}$$

Finally, the assertion 2) is a trivial consequence of Lemma 2.5. ∎

Without loss of generality, from now on we assume that the matrix $(\{u_{2,i}, u_{1,j}\}(\varrho))_{i,j=1,\dots,\nu}$ is the identity. Infact, since Σ is a symplectic cone and Σ_1, Σ_2 are involutive cones, the matrix $(\{u_{2,i}, u_{1,j}\}(\varrho))_{i,j=1,\dots,\nu}$ is invertible; if we denote by $(a_{i,j})_{i,j=1,\dots,\nu}$ its inverse matrix, it suffices to replace the local equations $u_{2,1}, \dots, u_{2,\nu}$ by $u'_{2,i}(x, \xi) = \sum_{j=1}^{\nu} a_{i,j} u_{2,j}(x, \xi)$ (with $i = 1, 2, \dots, \nu$).

Therefore, if we define for any $\varrho \in \Sigma$ the map $\varphi : T^* \mathbb{R}^\nu \simeq \mathbb{R}^\nu_{x'} \times \mathbb{R}^\nu_{\xi'} \to \to (T_\varrho \Sigma)^\sigma$ as

$$\varphi(x', \xi') = \sum_{j=1}^{\nu} (x_j H_{u_{2,j}}(\varrho) - \xi_j H_{u_{1,j}}(\varrho))$$

we get that φ is a linear simplectomorphism. Now, define the *test operator* associated with P as

$$P_\varrho(x', D_{x'}) = P_{\varrho, \varphi}(x', D_{x'}).$$

Some remarks are in order.

I) In the flat case, i.e. when $u_{1,j}(x, \xi) = x_j$, $u_{2,j}(x, \xi) = \xi_j$ (with $j = 1, \dots, \nu$), the polynomial $p_{\varrho, \varphi}(x', \xi')$ coincides with the localized polynomial described in (3) of Section 1.4. Therefore the operator $P_\varrho(x', D_{x'})$ defined above extends the one described in Section 1.4 when we treated the flat case.

II) Denote by $P^* = \mathrm{Op}(p^*)$ the formal adjoint of $P = p(x, D)$ and observe that $(p^*)^W = \overline{p^W}$ as we have already seen in (15). Hence, P^* satisfies the vanishing conditions (H) iff P does; moreover, for any $\varrho \in \Sigma$ and any $v \in (T_\varrho \Sigma)^\sigma$ one has $p^*_\varrho(v) = \overline{p_\varrho(v)}$. As a consequence, for any $\varrho \in \Sigma$, we have

$$(P^*)_\varrho(x', D_{x'}) = (\mathrm{Op}^W(p^*_\varrho \circ \varphi)) = (\mathrm{Op}^W(p_\varrho \circ \varphi))^* = (P_\varrho(x', D_{x'}))^*.$$

III) In the same hypothesis of Lemma 2.5, let $\tilde{P} = FPF^{-1}$. By Lemma 2.6 the operators $P_\varrho(x', D_{x'})$ and $\tilde{P}_{\chi(\varrho)}(y', D_{y'})$ are unitarily equivalent so that they have the same «spectral properties». Thus, in particular $P_\varrho(x', D_{x'})$ is injective in $L^2(\mathbb{R}^\nu)$ iff $\tilde{P}_{\chi(\varrho)}(y', D_{y'})$ is so.

IV) From Proposition 2.1 it follows that there exists a local canonical flattening χ of Σ_1 and Σ_2 near every point ϱ of Σ. Therefore, if we denote by F (resp., F^{-1}) an elliptic classical Fourier integral operator of order 0, associated with χ (resp., χ^{-1}), then we have that $\tilde{P} = FPF^{-1}$ belongs, microlocally, to $\mathrm{OP}S_h^{m,2}$, since the operator $\tilde{P}$ satisfies the vanishing conditions (H) when $\chi(\Sigma_1)$ and $\chi(\Sigma_2)$ are flat near $\chi(\varrho)$ (see Remark 2.2). Moreover, using I) it is immediate to check that the operator $\widetilde{P}_{\chi(\varrho)}(y', D_{y'})$ coincides with the one described in the Section 1.4.

Since the Fourier integral operators F and F^{-1} described in *IV*) preserve Sobolev spaces, we can give a necessary and sufficient condition for the hypoellipticity of the operator P with loss of $2h/(h+1)$ derivatives, whose proof immediately follows from Theorem 1.12 by virtue of the previous remarks.

THEOREM 2.7. *Let $P = p(x, D)$ be a classical properly supported pseudodifferential operator satisfying the vanishing conditions* (H) *and the cone Σ, described above, be its characteristic set. Moreover, suppose that the principal symbol $p_m(x, \xi)$ of P is transversally elliptic, namely $p_m(x, \xi)$ satisfies, in a small conic neighborhood of Σ, the estimate*

$$|p_m(x, \xi)| \gtrsim |\xi|^m (\mathrm{dist}^h_{\Sigma_1}(x, \xi) + \mathrm{dist}_{\Sigma_2}(x, \xi))^2.$$

Then the following statements are equivalent:

(*I*) *the operator P (resp. its formal adjoint P^*) is hypoelliptic with loss of $2h/(h+1)$ derivatives;*

(*II*) *given any fixed $\varrho \in \Sigma$, the operator $P_\varrho(x', D_{x'})$ is injective from $S(\mathbb{R}^\nu)$ to $S(\mathbb{R}^\nu)$ (resp. surjective from $S'(\mathbb{R}^\nu)$ to $S'(\mathbb{R}^\nu)$).*

2.3. *Some examples.*

In this Section we show some examples for which we can reduce the condition of hypoellipticity with loss of $2h/(h+1)$ derivatives to an explicit algebraic condition by virtue of Theorem 2.7.

Assume that the cone Σ_1 and Σ_2 have codimension 1, namely, with the above notations

$$\Sigma_1 = \{(x, \xi) \in T^*X \backslash 0 \mid u_1(x, \xi) = 0\},$$

$$\Sigma_2 = \{(x, \xi) \in T^*X \backslash 0 \mid u_2(x, \xi) = 0\}.$$

We consider a classical pseudodifferential operator $P = p(x, D)$ with asymptotic expansion $p \sim \sum_{j \geq 0} p_{m-j}$ satisfying the hypotheses of Theorem 2.7. Thus the hypoellipticity with loss of $2h/(h+1)$ derivatives of P depends on the injectivity of the ordinary differential operator $P_\varrho(t, D_t)$

(where $\varrho \in \Sigma$), which is the Weyl-quantization in $(t, \tau) \in \mathbb{R} \times \mathbb{R}$ of

$$p_\varrho(t, \tau) = a(\varrho)\,\tau^2 + b(\varrho)\,t^h \tau + c(\varrho) t^{2h} + d(\varrho) t^{h-1}$$

where:

$$a(\varrho) = \frac{1}{2}(H_{u_1}^2 p_m)(\varrho),$$

$$b(\varrho) = -\frac{1}{h!}(\{u_2, u_1\}(\varrho))^{-h}(H_{u_2}^h H_{u_1} p_m)(\varrho),$$

$$c(\varrho) = \frac{1}{(2h)!}(\{u_2, u_1\}(\varrho))^{-2h}(H_{u_2}^{2h} p_m)(\varrho),$$

$$d(\varrho) = \frac{1}{(h-1)!}(\{u_2, u_1\}(\varrho))^{1-h}(H_{u_2}^{h-1} p_{m-1}^s)(\varrho).$$

Therefore, we easily get

$$P_\varrho(t, D_t) = a(\varrho) D_t^2 + b(\varrho) t^h D_t + c(\varrho) t^{2h} + (d(\varrho) - i b(\varrho)\, h/2)\, t^{h-1}. \tag{26}$$

From the transversal ellipticity of $p_m(x, \xi)$ it immediately follows that there exists a positive constant C depending on $\varrho \in \Sigma$ for which

$$|a(\varrho)\,\tau^2 + b(\varrho)\,t^h \tau + c(\varrho) t^{2h}| \geq C(|\tau| + |t|^h)^2,$$

when $(t, \tau) \in \mathbb{R} \times \mathbb{R}$. In particular one gets $a(\varrho) \neq 0$ and, moreover, if $\lambda(\varrho)$ and $\mu(\varrho)$ are the complex roots of the equation

$$a(\varrho)\,\tau^2 + b(\varrho)\,\tau + c(\varrho) = 0,$$

then $\operatorname{Im}\lambda(\varrho) \neq 0$ and $\operatorname{Im}\mu(\varrho) \neq 0$. First of all, we observe that the injectivity of the operator $P_\varrho(t, D_t)$ does not depend on the term $d(\varrho)$ when $\operatorname{Im}\lambda(\varrho) \cdot \operatorname{Im}\mu(\varrho) > 0$. More precisely, in view of the results contained in Chapter 7 (Corollary 7.3.2) of Mascarello-Rodino [16] we have that

1) if $\operatorname{Im}\lambda(\varrho) < 0$ and $\operatorname{Im}\mu(\varrho) < 0$ then $P_\varrho(t, D_t)$ is injective in $\mathcal{S}(\mathbb{R})$;
2) if $\operatorname{Im}\lambda(\varrho) > 0$ and $\operatorname{Im}\mu(\varrho) > 0$ then

$$\dim(\operatorname{Ker} P_\varrho(t, D_t) \cap \mathcal{S}(\mathbb{R})) = \begin{cases} 2 & \text{when } h \text{ is odd} \\ 0 & \text{when } h \text{ is } \textit{even}. \end{cases}$$

In order to treat the case when h is an odd positive integer and $\operatorname{Im}\lambda(\varrho) \cdot \operatorname{Im}\mu(\varrho) < 0$ we refer to the results proved in the paper of Gilioli-Treves

[4] and Gilioli [5]. In fact, consider the following second order pseudodifferential operator in $\mathbb{R}_t \times \mathbb{R}_x$

$$Q = -a(\varrho)((\partial_t - i\lambda(\varrho)\, t^h\, |D_x|)(\partial_t - i\mu(\varrho) t^h\, |D_x|) + \gamma(\varrho) t^{h-1}\, |D_x|),$$

where $\gamma(\varrho) = -(i(\lambda(\varrho) - \mu(\varrho))\, h/2 + d(\varrho)/a(\varrho))$ and $|D_x|$ is the pseudodifferential operator with symbol $|\xi|$ (the behaviour of $|D_x|$ near $\xi = 0$ is irrelevant here, because we are interested to study Q near the cone $\tilde{\Sigma} = \{(t, x, \tau, \xi) \in T^*\mathbb{R}^2 \setminus 0 \mid t = \tau = 0\}$). It is easy to see that $Q \in \mathrm{OP}S_h^{2,2}$ with respect to the flat cones $\tilde{\Sigma}_1 = \{(t, x, \tau, \xi) \in T^*\mathbb{R}^2 \setminus 0 \mid t = 0\}$ and $\tilde{\Sigma}_2 = \{(t, x, \tau, \xi) \in T^*\mathbb{R}^2 \setminus 0 \mid \tau = 0\}$; furthermore, the principal symbol of Q

$$a(\varrho)\, \tau^2 + b(\varrho)\, t^h\, |\xi|\tau + c(\varrho) t^{2h} \xi^2$$

is transversally elliptic. The crucial observation is that the test operator in $(0, x, 0, \pm 1)$ associated with Q coincide with $P_\varrho(t, D_t)$. Hence the injectivity of $P_\varrho(t, D_t)$ in $\mathcal{S}(\mathbb{R})$ is equivalent to the hypoellipticity with loss of $2h/(h+1)$ derivatives of Q. Without loss of generality, we can assume that $\mathrm{Im}\, \lambda(\varrho) < 0$ and $\mathrm{Im}\, \mu(\varrho) > 0$. In view of the precise results given by Gilioli in [5], we obtain that $P_\varrho(t, D_t)$ is injective in $\mathcal{S}(\mathbb{R})$ if and only

$$\frac{\gamma(\varrho)}{i(\lambda(\varrho) - \mu(\varrho))} \neq \begin{cases} m(h+1) \\ m(h+1) + 1 \end{cases} \quad \text{for any } m \in \mathbb{Z}_+ .$$

Finally, if h is an even positive integer and $\mathrm{Im}\, \lambda(\varrho) \cdot \mathrm{Im}\, \mu(\varrho) < 0$, we refer to the paper of A. Menikoff [17]; thus, by an argument similar to the one above, it turns out that the test operator

$$P_\varrho(t, D_t) = a(\varrho)((D_t - \lambda(\varrho)\, t^h)(D_t - \mu(\varrho)\, t^h) - \gamma(\varrho)\, t^{h-1})$$

is injective in $\mathcal{S}(\mathbb{R}^\nu)$ if and only if

$$\frac{\gamma(\varrho)}{i(\lambda(\varrho) - \mu(\varrho))} \neq m(h+1) + 1/2 \quad \text{for any } m \in \mathbb{Z} .$$

Therefore, when the conditions above are satisfied for every $\varrho \in \Sigma$, the operator P is hypoelliptic with loss of $2h/(h+1)$ derivatives.

In [15], precise conditions are given for the injectivity of ordinary differential operators with polynomial coefficients, that hold also for operators of order higher than the order of $P_\varrho(t, D_t)$ in (26). We address the reader to [15] for more details.

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