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Globally Invertible Differentiable or Holomorphic Maps.

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0. Introduction.

The following problem was studied in [8]. Let M be a connected C^∞ manifold and $f\colon M \to M$ a locally invertible surjective C^∞ function. Is every such f invertible? The answer was that this is the case if the fundamental group $\pi_1(M, P)$ of M is finite (where P is any point of M). Of course, sometimes the answer is negative, e.g. for the circle. If instead of a locally invertible map $f\colon M \to M$ we consider a locally invertible map $f'\colon N \to M$ with M fixed but with fixing the domain N, then the answer is obviously negative for every M with $\pi_1(M, P) \neq 0$ (see [8, Cor. 4] for a particular case). But we want to consider the original problem in which the domain and the target are the same, i.e. we do not want to change our «universe». We consider only compact manifolds. We will give several examples of compact manifolds for which the answer is negative, but we will show that very often, «usually», the answer is positive. The motivation behind [8] was explained at the end of the introduction of [8] and in [8, sections 4.2, 4.3 and 4.4]; key words: Market Equilibrium, Limited Arbitrage and Uniqueness with Short Sales.

In the first section we will give several remarks on this topic and prove the following result.

PROPOSITION 0.1. *Fix an integer $n \geq 5$. Let M_{top} be a compact connected topological n-dimensional manifold which admits a differentiable structure and with $\pi_i(M_{\text{top}}, P) = 0$ for every $i \geq 2$. There exists a differentiable structure M on M_{top} and a locally invertible differentiable*

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map $f: M \to M$ *with* $\deg(f) \geqslant 2$ *if and only if* $\pi_1(M_{\text{top}}, P)$ *contains a proper subgroup* H *of finite index with* $H \cong \pi_1(M_{\text{top}}, P)$ *as abstract groups.*

It is well-known (see the exercise on page 180 of [14]) that for every integer $n \geqslant 4$ any finitely presented group may be realized as the fundamental group of a compact connected n-manifold. If $\dim(M) = 2$ we give a complete classification: the answer is negative if and only if M is either a 2-torus or a Klein bottle (Proposition 1.11). We stress that 0.1 follows very easily from standard properties of $K(\pi, 1)'s$ and that all results of section 1 are just easy exercises. However, we believe that the problem is nice and that it is reasonable to study it in several different categories.

In section 2 we consider the same problem for compact complex manifolds and holomorphic maps. Here is the main result of this paper.

THEOREM 0.2. *Let* X *be a compact complex surface such that there exists a holomorphicl ocally invertible* map $\pi: X \to X$ *with* $\deg(\pi) > 1$. *Then* X *belongs to one of the following classes*:

(i) $X \cong E \times B$ *with* E *elliptic curve and* B *smooth curve of genus* $\geqslant 2$;

(ii) X *is a torus*;

(iii) X *is a hyperelliptic surface*;

(iv) X *is a minimal ruled surface over an elliptic curve*;

(v) X *is one of the non-kähler surfaces without curves and with* $b_1(X) = 1$ *constructed by Inoue in* [12].

Every product $E \times B$ *with* E *elliptic curve, every torus, every hyperelliptic surface and every surface as in* (v) *has such a non-trivial covering. Some but not all the minimal ruled surfaces over an elliptic curves have such a non-trivial covering.*

For a complete description of the minimal ruled surfaces over an elliptic curve with such a non-trivial covering, see 2.3.

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1. Differentiable maps.

Unless otherwise stated we will use the following notations and conventions. Let M be an n-dimensional differentiable connected compact manifold and $f: M \to M$ a locally invertible differentiable map. Since M is compact, the continuous map f is a covering map. For any $P \in M$, set $d := \operatorname{card}(f^{-1}(P))$. Since M is connected, this integer d is independent from the choice of P and will be called the degree $\deg(f)$ of f. To avoid trivialities we always assume $d \geq 2$. Let X be a compact connected complex manifold and $\pi: X \to X$ a holomorphic locally invertible map. Hence π is a covering map; again we will set $d := \deg(\pi)$ but we will call n the complex dimension $\dim_C(X)$ of X; hence X is a $2n$-dimensional compact differentiable manifold.

REMARK 1.1. Fix $P \in M$. Since f is a covering map there is a subgroup H of $\pi_1(M, P)$ with H of index d and $H \cong \pi_1(M, P)$. In particular $\pi_1(M, P)$ has a proper subgroup of finite index isomorphic to $\pi_1(M, P)$. If the universal covering of M is contractible, i.e. if M is a $K(\pi_1(M, P), 1)$ (or, equivalently, if $\pi_i(M, P) = 0$ for every $i \geq 2$), then this condition is also a sufficient condition for the existence of a continuous degree d covering map $m: M_{\text{top}} \to M_{\text{top}}$: just use the universal defining property of $K(\pi, 1)$-spaces.

PROPOSITION 1.2. *Let M be a connected differentiable manifold such that $\pi_1(M, P)$ has a proper subgroup, H, of finite index isomorphic to $\pi_1(M, P)$. Assume that the topological space M_{top} is a $K(\pi, 1)$, i.e. assume $\pi_i(M, p) = 0$ for every integer $i \geq 2$. Assume that the topological space M_{top} has only finitely many differentiable structures. Then there exists a differentiable structure, say M_{diff}, on M_{top}, and an integer $d \geq 2$ such that there is a degree d differentiable covering map $f: M_{\text{diff}} \to M_{\text{diff}}$.*

PROOF. Let x be the index of H in $\pi_1(M, P)$. By Remark 1.1 there is a continuous degree x covering map $m: M_{\text{top}} \to M_{\text{top}}$. Fix on the target M_{top} the differentiable structure corresponding to M. Then the covering map m induces a differentiable structure of M_{top} seen as the domain of the map m and for which the map m is differentiable and locally invertible. The same is true if we take instead of m any iteration of the map m. Since M_{top} has only finitely many differentiable structures, we will find a differentiable structure M_{diff} on M_{top} and an integer $t \geq 1$ such that the map $m \circ \ldots \circ m$ (t times) induces a differentiable covering map from M_{diff} onto itself.

REMARK 1.3. With the notations of 1.2, call x the index of H in $\pi_1(M, P)$. The proof of 1.2 shows the existence of a positive integer t such that we may find a differentiable structure M_{diff} on M_{top} and f with $\deg(f) = x^t$.

PROOF OF PROPOSITION 0.1. For any compact topological n-manifold Z, let $M^n(Z)$ be the set of all smooth structures on a topological manifold homotopic to Z. Since $n \geqslant 5$ the set $M^n(M_{\mathrm{top}})$ is finite ([13, p. 2]). By assumption we have $M^n(M_{\mathrm{top}}) \neq \varnothing$. Hence we conclude by Remark 1.3.

EXAMPLE 1.4. Let $M = (S^1)^n$ be a compact real torus, i.e. the quotient space of $\boldsymbol{R}^n$ (seen as an additive group) by a discrete subgroup of translations Γ with $\boldsymbol{R}^n/\Gamma$ compact; we may take $\Gamma := \boldsymbol{Z}^n$. We see easily directly that for every integer $d > 1$ there is a degree d differentiable covering map $f: M \to M$. This follows also from Remark 1.1 or from the case $n = 1$ (the circle) and Example 1.6 below.

EXAMPLE 1.5. Let X be an n-dimensional complex compact torus, i.e. a compact complex manifold isomorphic to $\boldsymbol{C}^n/\Gamma$, where Γ is a rank $2n$ subgroup of the abelian group $\boldsymbol{C}^n$ acting on $\boldsymbol{C}^n$ by translations. Fix an integer $t \geqslant 2$. The multiplication by t on $\boldsymbol{C}^n$ induces a covering holomorphic map $X \to X$ of degree t^{2n}.

EXAMPLE 1.6. Let D be any differentiable manifold. Let M be a differentiable manifold such that there exists a locally invertible differentiable map $f: M \to M$ of degree $d > 1$. The map f induces a differentiable map $g: M \times D \to M \times D$, $g((x, y)) := (f(x), y)$, which is locally invertible and with $\deg(g) = \deg(f) > 1$. Furthermore, g is proper or a covering map if and only if f has the same property. Hence from any example, M, we obtain in a trivial way a huge number of higher dimensional examples. The same is true in the category of complex manifolds and holomorphic maps.

REMARK 1.7. Since the topological Euler characteristic is multiplicative for finite coverings, we have $e(M) = 0$. The same is true for a complex compact manifold X and for the complex Euler characteristic $\chi(\boldsymbol{O}_X)$.

REMARK 1.8. Since f is locally invertible, we have $f^*(TM) \cong TM$. Since $d \neq 1$, this implies that all Pontryagin numbers of M are zero ([14, § 16]). For a compact complex manifold X as above the same is true for its Chern numbers related to its holomorphic tangent bundle ([14, § 16]). In particular we have $c_1(X)^n = 0$ and $c_n(X) = 0$.

LEMMA 1.9. *Assume M not orientable and let $u : M' \to M$ be the orientation covering. Hence u is a double covering and M' is connected and orientable. Let $f : M \to M$ be a degree d covering. Then there exists a degree d covering $f' : M' \to M'$ such that $f \circ u = u \circ f'$.*

PROOF. Let (M'', u', f') be the cartesian product of the maps f and u. Hence M'' is a differentiable manifold and $u' : M'' \to M, f' : M'' \to M'$ are covering maps with $\deg(u') = 2$, $\deg(f') = d$ and $f \circ u' = u \circ f''$. It is sufficient to check that M'' and M' are diffeomorphic. Since M' is orientable, every connected component of M'' is orientable. First we assume that M'' is connected. By definition of orientable covering the map u' factors through u, say $u' = g \circ u$. Since both u' and u are degree two covering maps, g is a diffeomorphism, as wanted. Now assume M'' not connected. Since $\deg(u'') = 2$, this implies that M'' has two connected components, each of them mapped by u'' diffeomorphically onto M. Since M is assumed to be not orientable, we obtained a contradiction.

REMARK 1.10. Let M be a compact 2-dimensional manifold with a covering map of degree $d \geqslant 2$. By Remark 1.7 we have $e(M) = 0$. Hence if M is orientable, then it is a torus. Viceversa, by Example 1.4 every torus has such a non-trivial covering. Now assume M not orientable. By the first part of the proof and Lemma 1.9 the orientable covering M' of M is a torus. Hence M is a Klein bottle ([15, bottom of p. 75]). We see M' as $\boldsymbol{R}^2/\boldsymbol{Z}^2$ and M as the quotient of M' by the involution $\sigma : M' \to M'$ induced by the affine map $m : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ with $m((x, y)) = (x + 1/2, -y)$. Fix an odd integer $n \geqslant 3$ and set $d := n^2$. Let $f' : M' \to M'$ be the degree d local diffeomorphism induced by the linear map $f'' : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ with $f''((x, y)) = (nx, ny)$. Since n is odd we have $f'' \circ m((x, y)) = (nx + n/2, -ny)$, $m \circ f''((x, y)) = (nx + 1/2, -ny)$ and hence $f' \circ \sigma = \sigma \circ f'$ induces a degree d differentiable map $f : M \to M$. Since f' is locally invertible, f is locally invertible. Hence the Klein bottle is a solution of our problem and the only non-orientable one. Hence we have proved the following result.

PROPOSITION 1.11. *Let M be a compact 2-dimensional manifold with a covering map of degree $d \geqslant 2$. Then M is either a torus or a Klein bottle. Viceversa, for every integer $d \geqslant 2$ the torus has a degree d differentiable covering map and for every odd integer $n \geqslant 3$ the Klein bottle has a degree n^2 differentiable covering map.*

2. Compact complex surfaces.

In this section we prove Theorem 0.2. To prove 0.2 we will analyze several cases. In some of the cases we will obtained very precise informations which go much further than just the statement of 0.2. Let X be a complex compact smooth surface such that there exists a locally biholomorphic map $\pi : X \to X$ with $d := \deg(\pi) > 1$. Since the fundamental group of X is infinite, X is not a rational surface. Hence X contains at most finitely many exceptional curves of the first kind. Assume the existence of an exceptional curve D of the first kind. Since $D \cong \boldsymbol{P}^1$ is simply connected and each connected component, T, of $f^{-1}(D)$ has normal bundle $N_{T/X}$ isomorphic to $N_{D/X}, f^{-1}(D)$ is the disjoint union of d exceptional curves of the first kind. Hence there is no such D, i.e. X is a minimal surface. Since $\pi^*(\omega_X) \cong \omega_X$, we obtain $\omega_X^2 = d(\omega_X^2)$, i.e. $\omega_X^2 = 0$. Furthermore, $e(X) = \chi(\boldsymbol{O}_X) = 0$ (Remark 1.7). From the classification of surfaces (see [5, table on bottom of p. 402] and exclude the elliptic surfaces which are not hyperelliptic or a torus or have an elliptic curve as a factor) we will obtain that X belongs to one of the following 6 classes:

1) a $\boldsymbol{P}^1$-bundle over an elliptic curve;

2) a complex torus;

3) a hyperelliptic surface;

4) an analytic surface with $\boldsymbol{C}(X) \cong \boldsymbol{C}$, i.e. such that the only meromorphic functions on X are the constant ones;

5) a product $E \times B$ with E elliptic curve and B smooth curve with $p_a(B) \geqslant 2$;

6) a non-algebraic surface with algebraic dimension $a(X) = 0$.

To check if one of the surfaces in cases 1), ..., 6) has a self-map, we need to consider the case $\kappa(X) = 1$ and the case $a(X) = 1$. Since $\chi(\boldsymbol{O}_X) = \omega_X^2 = 0$, in both cases X must be an elliptic fibration $f : X \to B$ with B smooth curve and $p_a(B) + 1 = h^1(X, \boldsymbol{O}_X) \geqslant 2$. By [4, Lemme at p. 345] X

is birationally equivalent to $E \times B$ with E elliptic curve. Since X is minimal, we have $X \cong E \times B$. Any surface $E \times B$ gives a solution to our problem by 1.6.

For every complex torus X and every integer $t \geqslant 2$ there is such map π with $d := \deg(\pi) = t^4$ (Example 1.5).

(2.1) Here we assume that $\boldsymbol{C}(X) \cong \boldsymbol{C}$, i.e. that the only meromorphic functions on X are the constant ones. In particular X is not algebraic. By [7, Th. 2.16] X contains only finitely many irreducible complex curves. Fix an irreducible curve $A \subset X$ (if any). Notice that for every irreducible component, B, of $\pi^{-1}(A)$ we have $p_a(B) \geqslant p_a(A)$. Furthermore, if $p_a(A) \geqslant 2$, then $p_a(B) > p_a(A)$, unless $\pi|B : B \to \pi^{-1}(A)$ is an isomorphism; in this case, since $\deg(\pi) > 1$, there is another irreducible component of $\pi^{-1}(A)$. By iterating π we see that the finiteness of the set of curves in X implies that for every such curve A the algebraic set $\pi^{-1}(A)$ is irreducible and $p_a(A) \leqslant 1$. Notice that $\pi^{-1}(A)\cdot\pi^{-1}(A) = \deg(\pi)(A\cdot A)$. Since X has only finitely many curves, we see that the set of all possible self-intersection numbers $A\cdot A$ is bounded. Since $\deg(\pi) > 1$, we obtain that for every curve $A \subset X$ we have $A\cdot A = 0$. We claim that if $p_a(A) = 0$, i.e. if $A \cong \boldsymbol{P}^1$, this implies that X is a ruled surface. To check the claim use for instance that the normal bundle N_A of A in X is trivial, $h^0(A, N_A) = 1$, $h^1(A, N_A) = 0$ and hence that by deformation theory A moves inside X is a one-dimensional family. In particular if $p_a(A) = 0$, then X is algebraic, contradiction. Now assume $p_a(A) = 1$. By the adjunction formula and the assumption $A\cdot A = 0$, we have $\omega_X\cdot A = 0$, i.e. A is an irreducible curve of canonical type in the sense of [5, Def. 1.6 of part 2]. By [5, Th. 4.2 of part 3] X is an elliptic surface. Hence X contains infinitely many complex curves, contradiction. It remains the case in which X contains no complex curve. In summary, we have excluded all compact complex surfaces X with $\boldsymbol{C}(X) \cong \boldsymbol{C}$ except the ones without any complex curve. For this case, see 2.4, 2.5 and 2.6.

(2.2) Here we assume that X is a hyperelliptic surface. We want to prove that for every integer t prime to 12 there is an unramified covering $\pi : X \to X$ with $\deg(\pi) = t$. By [5, p. 37], there are two elliptic curves E_1 and E_0 and a finite abelian group G acting on $E_1 \times E_0$ and such that $X \cong E_1 \times E_0/G$. Furthermore, all such groups G and all such actions are completely classified (see [6, p. 37]; as remarked there, the subcase a3) does not occur in characteristic 0). In all these cases the action of G on $E_1 \times E_0$ is induced by an action of G on E_1 and an action of G on E_0 and for every

$h \in G$ the corresponding action of h on E_1 is given by the translation with a point $P(h) \in E_1$; $P(h)$ is a torsion point of E_1; we need to know $P(h)$ only for the generators of G. Here we consider subcase a1); we have $G \cong \boldsymbol{Z}/2\boldsymbol{Z}$ and if $h \in G$ is not the identity $P(h)$ is any torsion point $a \in E_1$ whose order is 2. Take any odd integer z and consider the map $\alpha : E_1 \times E_0 \to E_1 \times \times E_0$ given by $\alpha((x, y)) = (zx, y)$; α is a degree z unramified covering; since $2a = 0$ and z is odd, we have $zP(h) = P(h)$; thus the action of G commutes with α and hence it induces a degree z unramified covering $f: X \to X$, as wanted. The same proof works in subcases b1), c1 and d), i.e. in the other subcases in which G is cyclic; we have $\mathrm{card}(G) = 3$ (resp. 4, resp. 6) in subcase b1) (resp. c1), resp. d)); here we take a generator h of G and take as $P(h)$ any torsion point of order $\mathrm{card}(G)$; we take as z any integer prime to $\mathrm{card}(G)$. Now we consider subcase a2); we have $G \cong (\boldsymbol{Z}/2\boldsymbol{Z}) \times (\boldsymbol{Z}/2\boldsymbol{Z})$; if h, h' are generators of G, $P(h)$ and $P(h')$ are distinct torsion points of E_1 with order 2; again we take the same map α with as z any odd integer. Now we consider subcase b2); here $G \cong \cong (\boldsymbol{Z}/3\boldsymbol{Z}) \times (\boldsymbol{Z}/3\boldsymbol{Z})$ and we may use the same map α with as z any integer such that $z \equiv 1 \bmod (3)$. Now we consider subcase c2); here $G \cong (\boldsymbol{Z}/2\boldsymbol{Z}) \times \times (\boldsymbol{Z}/5\boldsymbol{Z})$; just take $z \equiv 1 \bmod (4)$ and conclude in the usual way.

(2.3) Here we assume that X is a $\boldsymbol{P}^1$-bundle over an elliptic curve C. This implies the existence of a rank 2 vector bundle E on C such that $X \cong \cong \boldsymbol{P}(E)$ ([3, Th. III.10]). We want to classify all such vector bundles E and hence all such surfaces X. If E, E' are rank 2 vector bundles on C we have $\boldsymbol{P}(E) \cong \boldsymbol{P}(E')$ if and only if $E' \cong E \otimes L$ for some $L \in \mathrm{Pic}(C)$ ([9, Prop. V. 2.2]). since $\deg(E \otimes L) = \deg(E) + 2\deg(L)$ ([8, p. 408]), to achieve such classification we may assume that either $\deg(E) = 0$ or $\deg(E) = = -1$. Since every fiber of g is simply connected, the unramified degree d covering π is induced by an unramified degree d covering $v: C \to C$. Hence there is a $L \in \mathrm{Pic}(C)$ such that $v^*(E) \cong E \otimes L$. We have $\deg(v^*(E)) = \deg(v) \cdot \deg(E)$ and hence $\deg(L) = (d-1)\deg(E)/2$. M. Atiyah gave a complete classification of all vector bundles on an elliptic curve ([2, Part II]). First assume $\deg(E) = 0$. If E is indecomposable, up to a twist by a line bundle, E is uniquely determined and it is a non-trivial extension, $\boldsymbol{e}$, of $\boldsymbol{O}_C$ by $\boldsymbol{O}_C$. The set of all extensions of $\boldsymbol{O}_C$ by $\boldsymbol{O}_C$ is parametrized by the abelian group $H^1(C, \boldsymbol{O}_C) \cong \boldsymbol{C}$ which has no torsion. In this case $v^*(E)$ is still semi-stable and given by the extension $d\boldsymbol{e}$ of $\boldsymbol{O}_C$ by $\boldsymbol{O}_C$. Hence $v^*(E)$ is indecomposable and we have $v^*(E) \cong E \otimes L$ for some $L \in \mathrm{Pic}^0(C)$. Thus in this case $\boldsymbol{P}(E)$ is a solution of our problem. Now assume E decomposable, say $E \cong A \oplus B$ with $A \in \mathrm{Pic}^a(C)$, $B \in \mathrm{Pic}^{-a}(C)$ and $a \geqslant 0$. We have $v^*(A) \in \mathrm{Pic}^{da}(C)$ and $v^*(B) \in \mathrm{Pic}^{-da}(C)$; by Krull-Remak-Schmidt theorem ([1]) the indecomposable factors of a vector bundle on C are uniquely determined, up to a permutation; hence a necessary condition for the existence of $L \in \mathrm{Pic}(C)$ such that $v^*(E) \cong E$ is that $a=0$; indeed if $a>0$ we should have either $A \otimes L \cong v^*(A)$ (forcing $\deg(L) = (d-1)\,a > 0$) and $B \otimes L \cong B$ (forcing $\deg(L) = = (1-d)\,a < 0$) or $A \otimes L \cong v^*(B)$ (forcing $\deg(L) = -(d+1)\,a < 0$) and $B \otimes L \cong v^*(A)$ (forcing $\deg(L) = (d+1)\,a > 0$). Now we assume $a=0$. Up to a twist by a line bundle we may assume $A \cong \boldsymbol{O}_X$. Hence $v^*(A) \cong \boldsymbol{O}_X$. Every $B \in \mathrm{Pic}^0(C)$ with $v^*(B) \cong B$ gives a solution. Since $\mathrm{Pic}^0(C)$ is an elliptic curve, it has exactly d^2 torsion points of order dividing d. Hence there are exactly d^2 line bundles $B \in \mathrm{Pic}^0(B)$ with $v^*(B) \cong B$ and each of these line bundles gives a solution of our problem. Viceversa, any other solution of our problem is given by a pair (B, v), $B \in \mathrm{Pic}^0(C)$, such that there exists $L \in \mathrm{Pic}^0(C)$ with $L \cong v^*(B)$ and with$B \otimes L \cong \boldsymbol{O}_C$. Hence $L \cong \cong B^*$ and for fixed v we need to find all $B \in \mathrm{Pic}^0(C)$ with $v^*(B) \cong B^*$. If v is just the multiplication by an integer $t > 0$ (and in this case $d = t^2$) $v^*(B) \cong B^{\otimes n}$ and we are looking for all the line bundles, B, such that $B^{\otimes(n+1)} \cong \boldsymbol{O}_C$, i.e. we are looking for the torsion points of $\mathrm{Pic}^0(C)$ with order dividing $n+1$. On a general elliptic curve C the only endomorphisms of C are induced by the multiplication by an integer, up to a translation and on any elliptic curve the group of all endomorphisms is known ([11, Ch. 12, § 4]). In each case for each elliptic curve we may find all the possible line bundles B. We just warn the reader that the case $B \cong \boldsymbol{O}_C$ is counted twice, because $\boldsymbol{O}_X$ is considered to have order 1, i.e. order dividing both d and $n+1$. Now we assume $\deg(E) = -1$ and use again Atiyah's classification of vector bundles on C. First assume E indecomposable; by Atiyah's classification ([2, Th. 7 at p. 434]), up to a twist by a line bundle there is a unique such E and such E is semistable; furthermore, for every endomorphism v the bundle $v^*(E)$ is semistable, i.e. it is either indecomposable or the direct sum of two line bundles with the same degree; if d is even $v^*(E)$ cannot be stable, because no rank 2 vector bundle on E with even degree is stable; hence if d is even the pair (E, v) does not give a solution. Assume d odd; since $v^*(E)$ is semistable and $\deg(V^*(E)) = -d$ is odd, $v^*(E)$ is stable; hence there is $L \in \mathrm{Pic}(C)$ with $v^*(E) \cong E \otimes L$ ([2, Th. 7 at p. 434]); hence in this case (E, v) gives a solution. Now assume E decomposable, say $E \cong A \oplus B$ with $A \in \mathrm{Pic}^a(C)$, $B \in \mathrm{Pic}^{-a-1}(C)$ and $a \geqslant 0$; since $v^*(A) \in \mathrm{Pic}^{da}(C)$, $v^*(B) \in \mathrm{Pic}^{-da-d}(C)$

and $d \geq 2$, by Krull-Remak-Schmidt theorem ([1]) there is no line bundle $L \in \mathrm{Pic}(C)$ with $v^*(E) \cong E \otimes L$ (just look again at $\deg(L)$).

(2.4) By [17] and [12, Theorem in § 5], every smooth complex compact surface without curves is an Inoue surface, i.e. belongs to one of the three classes of surfaces constructed in [12]. Here we will consider the case in which X is an Inoue surface S_M. We will show that for every such X and every integer $t^3 \geq 2$ there is a locally biholomorphic map $f: X \to X$ with $\deg(f) = t^3$. Fix any such X and an integer $t \geq 2$. X is associated to a matrix $M = (m_{ij}) \in SL(3, \boldsymbol{Z})$ with eigenvalues α real, β and $\overline{\beta}$ with $\beta \neq \overline{\beta}$ and $\alpha > 1$ ([12, § 2]). There are generators g_i, $0 \leq i \leq 3$, of $\pi_1(X)$ such that $g_i g_j = g_j g_i$ if $1 \leq i \leq j \leq 3$ and $g_0 g_i g_0^{-1} = g_1^{m_{i1}} g_2^{m_{i2}} g_3^{m_{i3}}$ for $1 \leq i \leq 3$. Let H be the subgroup of $\pi_1(X)$ generated by g_0, g_1^t, g_2^t and g_3^t. Let $f: X' \to X$ be unramified covering associated to H. Since $g_0 g_i^t g_0^{-1} = (g_1^t)^{m_{i1}} (g_2^t)^{m_{i2}} (g_3^t)^{m_{i3}}$ and $H = \pi_1(X')$, we obtain that H has index t^3 in $\pi_1(X)$ (i.e. that $\deg(f) = t^3$) and that X' is associated to M. Hence $X' \cong X$.

(2.5) Here we fix the complex compact surface X constructed in [12, § 3] with respect to the parameters N, p, q, r, t with $N \in SL(2, \boldsymbol{Z})$ with two real eigenvalues, p, q and r are integers with $r \neq 0$ and $t \in \boldsymbol{C}$. Here we will show that for every integer $x \geq 2$ there exists a degree x^2 locally biholomorphic morphisms $f: X \to X$. For the construction of X Inoue fixed real eigenvectors (a_1, b_1) and (a_2, b_2) of N and then defines X as the quotient of $\boldsymbol{H} \times \boldsymbol{C}$ ($\boldsymbol{H} \subset \boldsymbol{C}$ the upper half plane) with respect to a group G of biholomorphic transformation of $\boldsymbol{H} \times \boldsymbol{C}$ onto itself with certain generators g_0, g_1, g_2 and g_3 (see [12, pp. 273-274]). We define a degree x^2 locally biholomorphic morphism $f: X' \to X = \boldsymbol{H} \times \boldsymbol{C}/G$ taking $X' := \boldsymbol{H} \times \boldsymbol{C}/H$ with H subgroup of G generated by g_0, $(g_1)^x$, $(g_2)^x$ and $(g_3)^{x^2}$; here we use that the commutator of $(g_1)^x$ and $(g_2)^x$ is $(g_3)^{rx^2}$. We have $X' \cong X$ because H corresponds to the same parameters N, p, q, r and t just taking (xa_1, xb_1) and (xa_2, xb_2) instead of (a_1, b_1) and (a_2, b_2) as eigenvectors of N.

(2.6) The proof of (2.5) shows that for every integer $x \geq 2$ and every complex compact surface X constructed in [12,§ 4] with respect to the parameters N, p, q and r there is a locally biholomorphic morphism $f: X \to X$ with $\deg(f) = x^2$.

The proof of Theorem 0.2 is over.

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