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Regularity of the Free Boundary for non Degenerate Phase Transition Problems of Parabolic Type.

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ABSTRACT - In this work we give results on the regularity of viscosity solutions and of their free boundaries for a class of parabolic phase transition in which the dependence on the point (x, t) of free boundary is introduced in the phase transition relation.

1. Introduction.

Recently, some geometrical ideas of the minimal surface theory (see [3], [4]) and general results on the positive solutions of the heat equation by Harnack inequality (see [7], [8]) made a satisfactory improvement in the understanding of evolution free boundary problem with two phases. In the papers [3] and [4] Caffarelli considers elliptic free boundary problems, but the main strategy used is adaptable to the parabolic case even if the duality between the parabolicity of heat equation and the hyperbolicity of the free boundary condition produces some difficulties in studying this kind of problems. Therefore, in general, the regularity results in the parabolic case are weaker than the corresponding elliptic ones. In the paper [2] the authors consider viscosity solutions of a class of evolutionary free boundary problems (including the classical Stefan problem). One of the main results is that under suitable non–degeneracy conditions, Lipschitz free boundary regularize instantaneously and that viscosity solutions are indeed classical ones.

The condition on the free boundary expresses the fact that its speed

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depends on the heat fluxes and on the normal unit vector on the free boundary itself. We consider the situation in which the speed depends also on the point (x, t) of the free boundary.

In this case extracare in the scaling properties of the problem is required. The notion of viscosity solution, considered in this paper, have been used also in other important situations. In particular, in the works [9], [11], where free boundary problems arising in combustion theory are treated, and in [6], in which the regularity for the free boundary in the porus media equation is dealt with. The existence of viscosity solutions for these free boundary problems is an interesting open problem since the presence of free boundaries makes it impossible to use the standard approach with respect to the usual viscosity theory for second-order elliptic and parabolic equations.

2. Definitions, preliminary results.

Let us now introduce the class of free boundary problems with which we are going to deal with and the concept of viscosity solution. Let $Q_1 = B_1(0) \times (-1, 1)$, where $B_1(0)$ is the unit ball in $\mathbb{R}^n$, centered at 0.

First of all we give the definition of classical solution.

DEFINITION 1. Let u be a continuous function in Q_1. Then u is called a classical subsolution (supersolution) to a free boundary problem if, for $a_1 > 0$, $a_2 > 0$, we have: $\Delta u - a_1 u_t \geqslant 0$ ($\leqslant 0$) in $\Omega^+ := Q_1 \cap \{u > 0\}$; $\Delta u - a_2 u_t \geqslant 0$ ($\leqslant 0$) in $\Omega^- := Q_1 \cap \{u < 0\}$; $u \in \mathcal{C}^1(\overline{\Omega^+}) \cap \mathcal{C}^1(\overline{\Omega^-})$; for any $(x, t) \in \partial\Omega^+ \cap Q_1$, $\nabla_x u^+(x, t) \neq 0$ and

$$(2.1)\quad -V_\nu = \frac{u_t^+(x, t)}{u_\nu^+(x, t)} \leqslant G((x, t), \nu(x, t), u_\nu^+(x, t), u_\nu^-(x, t)) \quad (\geqslant)$$

where V_ν is the speed of the surface $F_t := \partial\Omega^+ \cap \{t\}$ in the direction $\nu := \frac{\nabla_x u^+}{|\nabla_x u^+|}$. Moreover, there exist two real constants α e c^* with $0 < \alpha \leqslant 1$ and $c^* > 0$ such that:

1) $G = G((x, t), \nu, d, e) : \mathbb{R}^{n+1} \times \partial B_1 \times \mathbb{R}^2 \to \mathbb{R}$ be a function Hölder continuous of esponent α, with Hölder constant $H > 0$ respect to (x, t) and $\overline{L} > 0$ respect to the other three arguments;

2) $G((x, t), \nu, d_1, e) - G((x, t), \nu, d_2, e) > c^*(d_1 - d_2)^\alpha$ if $d_1 > d_2$;

3) $G((x,t),\nu,d,e_1)-G((x,t),\nu,d,e_2)< -c^*(e_1-e_2)^\alpha$ if $e_1>e_2$.

If u is both classical subsolution and classical supersolution, then u is called a classical solution to a free boundary problem. The set $F=\partial\Omega^+\cap Q_1$ is called the free boundary.

DEFINITION 2. A function u continuous in Q_1 is called a viscosity subsolution (supersolution) to a free boundary problem if, for any sub-cylinder Q of Q_1 and for every classical supersolution (subsolution) v in Q, $u\leqslant v$ ($u\geqslant v$) on $\partial_p Q$ implies that $u\leqslant v$ ($u\geqslant v$) in Q.

We say that u is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

It is easy to show that classical solutions are viscosity solutions.

A famous example of evolution free boundary problem with two phases is the Stefan problem which is a simplified model of phase transition in solid-liquid systems (see [14], [15], [16]). In this case we have that the general interface condition (2.1) is of the kind: $G((x,t),\nu,u_\nu^+,u_\nu^-)=\frac{u_t^+}{u_\nu^+}=u_\nu^+-u_\nu^-$, and so the hypotheses 1), 2) and 3) are satisfied with $\alpha=1$.

In [12], it is proved existence of the classical solutions in a small time intervall. The global–in–time existence of the classical solution may fail. This leads us to construct for all times weak solutions [10] which are continuos [5]. Last result allow to show that weak solutions of the two–phase Stefan problem are viscosity solutions, according to the above definition.

We recall that a point $(x_0,t_0)\in F$ is regular from the right (or from the left) if there exists a $(n+1)$-dimensional ball $B^{(n+1)}\subset\Omega^+$ (Ω^-) such that $\overline{B}^{(n+1)}\cap F=\{(x_0,t_0)\}$.

In this paper we consider viscosity solutions u of a free boundary problem in a cylinder $Q_2=B_2(0)\times(-2,2)$, whose free boundary F is given by the graph of a Lipschitz function.

Without essential changes in the proofs, most results proven in [1] may be extended to our case where G depends on the point $(x,t)\in F$. Now, we list in the Theorem below such regularity results. The symbol $\langle\cdot,\cdot\rangle$ will denote the inner product in $\mathbb{R}^n$ or $\mathbb{R}^{n+1}$.

THEOREM A ([1], Theorems 2, 3, 4 and Corollaries 4, 5). *Suppose u is a viscosity solution to a free boundary problem in Q_2, whose free boundary F, contains the origin and is given by the graph $x_n=f(x',t)$,*

$(x', t) \in \mathbb{R}^{n-1} \times \mathbb{R}$, *of the Lipschitz function* f, *with Lipschitz constant* L. *Moreover* $M = \sup_{Q_2} u$ *and* $u\left(e_n, -\frac{3}{2}\right) = 1$, *where* e_n *is the unit vector in the* x_n *direction. Then, in* Q_1

i) *there exists a* $(n+1)$*-dimensional cone* $\Gamma(e_n, \theta)$, *with axis* e_n *and opening* $\theta = \theta(n, L, M, a_1, a_2)$, *such that, along every direction* $\nu \in \Gamma(e_n, \theta)$, u *is monotone increasing*;

ii) *there exists* $c = c(n, L, M, a_1, a_2)$ *such that:*

$$c^{-1} \frac{|u(x, t)|}{d_{x,t}} \leqslant |\nabla u(x, t)| \leqslant c \frac{|u(x, t)|}{d_{x,t}},$$

where $d_{x,t}$ denotes the distance between (x, t) and F, while $\nabla = (\nabla_x, D_t)$;

iii) *u is Lipschitz continuous.*

iv) (*Asymptotic development near regular points from the right or the left*).

Let $(x_0, t_0) \in F$ *be a regular point from the right,* ν *the inward spatial normal at* (x_0, t_0) *of* $B^{(n+1)} \cap \{t = t_0\}$ *and* $d(x, t)$ *be the distance between* (x, t) *and* (x_0, t_0). *Then there exist numbers* $\alpha_+, \alpha_-, \beta_+, \beta_-$ *such that near* (x_0, t_0),

$$(a) \qquad u(x, t) \geqslant (\alpha_+ \langle x - x_0, \nu\rangle + \beta_+ (t - t_0))^+ + \\ - (\alpha_- \langle x - x_0, \nu\rangle + \beta_- (t - t_0))^- + o(d(x, t))$$

with $\alpha_+ > 0$, $\alpha_- \geqslant 0$ *and equality holding on the hyperplane* $t = t_0$;

(*b*) $\beta_+ \geqslant \alpha_+ G((x_0, t_0), \nu, \alpha_+, \alpha_-)$, $\quad \beta_- \geqslant \alpha_- G((x_0, t_0), \nu, \alpha_+, \alpha_-)$.

If $B^{(n+1)} \subset \Omega^-$ ((x_0, t_0) *is a regular point from the left*) *the inequalities in* (*a*) *and* (*b*) *are reversed,* $\alpha_+ \geqslant 0$, $\alpha_- > 0$ *and* ν *is the outward spatial normal.*

v) (*Asymptotic development at «good points»*).

Near almost all points (x_0, t_0) *of differentiability of* F (*with respect to surface or caloric measures*) u *has the asymptotic behavior in* (*a*) *with equality sign in both* (*a*) *and* (*b*). *In this case* $\alpha_+ \geqslant 0$, $\alpha_- \geqslant 0$ *and* ν *is the normal to the tangent plane to* F_{t_0} *at* (x_0, t_0).

We note that in general the free boundary may not regularize instantaneously as the counterexample in the two-phase Stefan problem shows (see section 10 of [2]). Therefore we assume the following non-degeneracy condition which prevents simultaneous vanishing of the two fluxes from both sides of the free boundary: there exists $m > 0$ such that, if $(x_0, t_0) \in F$ is a regular point from the right or from the left, then, for any small r,

$$\frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |u| \geq mr .$$

From global considerations (see [4]) it follows, in some cases, this non-degeneracy condition, which allows us to conclude that Lipschitz free boundary are actually $\mathcal{C}^1$ graphs and therefore viscosity solutions are indeed classical.

For the reader's convenience, we state:

THEOREM B ([2], Main Theorem). *Let u be a viscosity solution of a free boundary problem in Q_2, whose free boundary, F, is given by the graph $x_n = f(x', t)$, $(x', t) \in \mathbb{R}^{n-1} \times \mathbb{R}$, of the Lipschitz function f, with Lipschitz constant L. Assume that $M = \sup_{Q_2} u$, $u\left(e_n, -\frac{3}{2}\right) = 1$, $(0, 0) \in F$, the non-degeneracy condition holds and that $G = G(\nu, d, e) : \partial B_1 \times \mathbb{R}^2 \to \mathbb{R}$ is a Lipschitz function in all its arguments, with Lipschitz constant L_G and, for some $c^* > 0$: $D_d G \geq c^*$, $D_e G \leq -c^*$. Then, the following conclusions hold:*

i) *in Q_1 the free boundary is a $\mathcal{C}^1$ graph in space and time. Moreover, for any $\eta > 0$, there exists a positive constant $C_1 = C_1(n, L_G, M, L, c^*, m, a_1, a_2, \eta)$ such that, for every (x', x_n, t), $(y', y_n, s) \in F$*

$$|\nabla_{x'} f(x', t) - \nabla_{x'} f(y', t)| \leq C_1(-\log|x' - y'|)^{-\frac{3}{2}+\eta}$$

$$|D_t f(x', t) - D_t f(x', s)| \leq C_1(-\log|t - s|)^{-\frac{1}{2}+\eta}.$$

ii) $\quad u \in \mathcal{C}^1(\overline{\Omega^+}) \cap \mathcal{C}^1(\overline{\Omega^-})$ *and on* $F \cap Q_1 \quad u_\nu^+ \geq C_2 > 0,$

with $C_2 = C_2(n, L_G, M, L, c^*, m, a_1, a_2, \eta)$. Therefore u is a classical solution.

The purpose of this article is to prove a generalization of the Theorem B in the case in which the function G depends on the point (x, t), and can be stated in the following way:

THEOREM 1. *Let u be a viscosity solution of a free boundary problem in Q_2, whose free boundary, F, is given by the graph $x_n = f(x', t)$, $(x', t) \in \mathbb{R}^{n-1} \times \mathbb{R}$, of the Lipschitz function (in space and time) f, with Lipschitz constant L. Assume that $M = \sup_{Q_2} u$, $u\left(e_n, -\frac{3}{2}\right) = 1$, $(0, 0) \in \in F$ and that the hypotheses* 1), 2), 3) *of pages* 2 *and* 3 *are satisfied by the function G. Moreover we suppose that non-degeneracy condition holds. Then:*

a) *in Q_1 the free boundary is a $\mathcal{C}^1$ graph in space and time. Moreover, $\forall \eta > 0$, there exists a positive constant $C_1 = C_1(n, L, M, H, \overline{L}, c^*, m, a_1, a_2, \eta, \alpha)$ such that, for every (x', x_n, t), $(y', y_n, s) \in F$ we have:*

$$|\nabla_{x'} f(x', t) - \nabla_{x'} f(y', t)| \leq C_1(-\log|x' - y'|)^{-\frac{3}{3-\alpha}+\eta}$$

$$|D_t f(x', t) - D_t f(x', s)| \leq C_1(-\log|t - s|)^{-\frac{\alpha}{3-\alpha}+\eta};$$

b) $\quad u \in \mathcal{C}^1(\overline{\Omega^+}) \cap \mathcal{C}^1(\overline{\Omega^-})$ *and on* $F \cap Q_1 \quad u_\nu^+ \geq C_2 > 0$,

with $C_2 = C_2(n, L, M, H, \overline{L}, c^*, m, a_1, a_2, \eta, \alpha)$. Therefore u is a classical solution.

To prove the Theorem 1 we adopt the strategy of the Theorem B. We recall that u is monotone increasing along any direction ν in a cone $\Gamma(e, \theta) := \{\nu : |\nu| = 1, e \cdot \nu \geq \cos\theta\}$, with $|e| = 1$, is equivalent to saying that for any small $\varepsilon > 0$

$$u(x) \geq \sup_{y \in B_{\varepsilon \sin\theta}(x - \varepsilon e)} u(y).$$

If $u(x) \geq \sup_{y \in B_\varepsilon(x - \varepsilon e)} u(y)$, then the level surfaces of u are hyperplanes.

Furthermore, $v(x) = \sup_{y \in B_\delta(x - \varepsilon e)} u(y)$ for any ε, δ small enough, is a subsolution of the same free boundary problem as u. Using such remarks, the regularity results will be achieved by improving the opening θ of the cone of monotonicity, that is by showing that θ converges to $\pi/2$ on a sequence of dyadically contracting cylinders around a free boundary point. The starting point is the existence of a monotonicity cone $\Gamma(e, \theta)$ in space and time, such that in a neighborhood of F, $D_\tau u \geq 0$ for each $\tau \in \Gamma(e, \theta)$ (Theorem A i)). The next step consists in increasing the opening of cone of monotonicity in space and in time away from the free boundary. Then we carry such interior gain to the free boundary in a small cylinder. Finally, an appropriate rescaling and iteration of the above steps gives the results. In this final iteration the opening in space and time behave differently so that it is convenient to introduce the following angles: $\delta = \pi/2 - \theta$ (defect angle in space) and $\mu = \pi/2 - \theta^t$ (defect angle in time), where θ is the opening of the spatial section of the monotonicity cone $\Gamma_x(e_n, \theta)$, while θ^t is the opening of the section of the cone of monotonicity in the (e_n, e_t)-plane. To get an enlargement of the cone in space away from the free boundary, we may use the techniques contained in the sections 2 and 4 of [2] that exploit Harnack's inequality. In fact some of the Lemmas in that paper hold also in our situation, in particular all the results obtained away from free boundary in which the free boundary condition does not enter. Therefore to prove the Theorem 1, we must show that, in this new situation, it is again possible to have an interior gain process in time (section 3), the propagation to the free boundary by Propagation Lemma (section 4), the regularization in space and in space–time (section 5).

The various constant c's, which will appear in the sequel, may vary from formula to formula and, unless explicitly stated, will depend on some or all of the relevant constants $n, L, M, H, \overline{L}, c^*, m, a_1, a_2, \eta, \alpha$.

3. Interior gain process in time.

We suppose that e_n was the projection in space of the axis of the monotonicity cone, whose defect angle in time is $\mu = \pi/2 - \theta^t$. This means that, there exist $A, B \in \mathbb{R}$, $A \leqslant B$ with $B - A \approx \mu$ and $A \leqslant \leqslant \frac{-D_t u^+(x,t)}{D_n u^+(x,t)} \left(\frac{-D_t u^-(x,t)}{D_n u^-(x,t)} \right) \leqslant B$ for (x, t) everywhere not on the free boundary F and almost everywhere on F. The enlargement of the monotonicity cone in time away from and in both sides the free boundary, that is equivalent to increase A or lower B, requires the following result:

LEMMA 2. *Let u be a viscosity solution to a free boundary problem in*

$$Q_1 = B_1 \times (-1, 1), \qquad (0, 0) \in F(u),$$

$$G((0,0), e_n, \alpha_+, \alpha_-) \geqslant -(A+B)/2 \quad (G((0,0), e_n, \alpha_+, \alpha_-) \leqslant -(A+B)/2),$$

$$|G((x,t),\nu,d,e)-G((y,s),\nu,d,e)|\leqslant$$

$$\leqslant\delta^{\frac{\alpha}{3}}\cdot|(x,t)-(y,s)|^{\alpha}\qquad\forall(\nu,d,e)\in\partial B_1\times\mathbb{R}^2.$$

Then there exists $c_1, C>0$ *and* $c_2\in(0,1)$ *such that if* δ *is small,* $\delta\leqslant\leqslant c_2\mu^{\frac{3}{\alpha}}$, *we have*

$$-\frac{D_t u}{D_n u}\leqslant B-c_1\mu\qquad\left(-\frac{D_t u}{D_n u}\geqslant A+c_1\mu\right)$$

$$\forall(x,t)\in(B_{1/8}(3e_n/4)\cup B_{1/8}(-3e_n/4))\times(-C\delta/\mu,C\delta/\mu).$$

PROOF. For almost every $(x_0,t_0)\in F$ with respect to surface measure, we have

$$\frac{D_t u^+(x_0,t_0)}{D_n u^+(x_0,t_0)}=\frac{D_t u^+(x_0,t_0)}{D_\nu u^+(x_0,t_0)}\cdot(1+O(\delta))=$$

$$=G((x_0,t_0),\nu(x_0,t_0),D_\nu u^+(x_0,t_0),D_\nu u^-(x_0,t_0))\cdot(1+O(\delta)).$$

By Lemma 7 in [2], with $\alpha_\pm=D_n u^\pm\left(\pm\frac{3}{4}e_n,0\right)$ and $R_t=\Omega^+\cap(-a_1++t,t)$, it follows that for every $t\in(-C\delta/\mu,C\delta/\mu)$:

$$\frac{1}{|F_t\cap B_{1/8}(0)|}\int_{F_t\cap B_{1/8}(0)}|D_n u^+-\alpha_+|^2\,ds\leqslant\alpha_+^2\cdot O(\delta/\mu).$$

Let $\tilde{\Sigma}_t=\{p\in F\cap\overline{R}_t:\ \alpha_\pm(1-k\delta^{\frac{1}{3}})\leqslant D_n u^\pm(p)\leqslant\alpha_\pm(1+k\delta^{\frac{1}{3}})\}$, then $|\tilde{\Sigma}_t|\geqslant\frac{1}{2}|F\cap\overline{R}_t|\ \ \forall t\in(-C\delta/\mu,C\delta/\mu)$.

Now, for each $(x,t)\in\tilde{\Sigma}_t$, with $t\in(-C\delta/\mu,C\delta/\mu)$, we deduce

$$|G((x,t),\nu(x,t),D_\nu u^+(x,t),D_\nu u^-(x,t))-G((0,0),e_n,\alpha_+,\alpha_-)|\leqslant$$

$$\leqslant\delta^{\frac{\alpha}{3}}|(x,t)|^{\alpha}+\overline{L}|\nu(x,t)-e_n|^{\alpha}+$$

$$+\overline{L}|D_\nu u^+(x,t)-\alpha_+|^{\alpha}+\overline{L}|D_\nu u^-(x,t)-\alpha_-|^{\alpha}\leqslant$$

$$\leqslant\delta^{\frac{\alpha}{3}}\{r+\overline{L}[\sqrt{2}+(c^{\alpha}+c^{\alpha}k+k^{\alpha})\cdot(\alpha_+^{\alpha}+\alpha_-^{\alpha})]\}=\delta^{\frac{\alpha}{3}}l,$$

where $l=r+\overline{L}[\sqrt{2}+(c^{\alpha}+c^{\alpha}k+k^{\alpha})(\alpha_+^{\alpha}+\alpha_-^{\alpha})]$.

By $G((0,0), e_n, \alpha_+, \alpha_-) \geq -(A+B)/2$ we obtain

$$G((x,t), \nu(x,t), D_\nu u^+(x,t), D_\nu u^-(x,t)) \geq G((0,0), e_n, \alpha_+, \alpha_-) - l\delta^{\frac{\alpha}{3}} \geq$$

$$\geq -(A+B)/2 - l\delta^{\frac{\alpha}{3}} \geq -B + c_0\mu - l\delta^{\frac{\alpha}{3}} \geq -B + c^*\mu \quad (c^* > 0).$$

Since the rest of the proof is the same as that of Lemma 8 in [2], the Lemma is proved. ■

4. Propagation to the free boundary.

In this section we use the Lemmas 9, 10, 11 in [2], that hold in our situation, to obtain the Propagation Lemma. These Lemmas exploit a powerful topological method introduced by Caffarelli in [3]. In the next Lemma we carry to the free boundary the interior gain in the aperture of the monotonicity cone using a family of subsolutions able to «measure» this opening. This is one of the most delicate points where we use hypotheses 2) and 3) satisfied by G. We observe that first, we must prove regularity in space and then in time because in Lemma 2 we require that defect angle was much smaller in space than in time.

LEMMA 3. (*Propagation Lemma*). *Let $u_1 \leq u_2$ be two viscosity solutions of the free boundary problem in $Q_2 = B_2 \times (-2, 2)$, with $F(u_2)$ Lipschitz continuous and $(0,0) \in F$. Assume that:*

$$\text{(i)}\quad v_\varepsilon(x,t) := \sup_{B_\varepsilon^{(n+1)}(x,t)} u_1 \leq u_2(x,t) \quad \textit{if } (x,t) \in B_1 \times (-T, T)$$

and, for some h small:

(ii) $u_2(x,t) - v_{(1+h\sigma)\varepsilon}(x,t) \geq P\sigma\varepsilon u_2(3e_n/4, 0)$
$\forall (x,t) \in B_{1/8}(3e_n/4) \times (-T, T) \subset \{u_2 > 0\}$

(iii) $u_2(x,t) - v_{(1+h\sigma)\varepsilon}(x,t) \geq -P\sigma\varepsilon u_2(-3e_n/4, 0)$
$\forall (x,t) \in B_{1/8}(-3e_n/4) \times (-T, T) \subset \{u_2 < 0\}$

(iv) $|G((x,t), \nu, d, e) - G((y,s), \nu, d, e)| \leq H|(x,t) - (y,s)|^\alpha$

$\forall (\nu, d, e) \in \partial B_1 \times \mathbb{R}^2$ *where $H < C^\alpha \sigma^\alpha m^\alpha c^*$. Then, if $\varepsilon > 0$ and $h > 0$ are small enough, there exists $c \in (0,1)$ such that in $B_{1/2} \times (-T/2, T/2)$ it results*

$$v_{(1+ch\sigma)\varepsilon}(x,t) \leq u_2(x,t).$$

PROOF. We construct a continuous family of function $\overline{v}_\eta = \tilde{v}_\eta + P\sigma\varepsilon w$ such that $\overline{v}_\eta \leqslant u_2$ $\forall \eta \in [0, 1]$, with $\overline{v}_1 \geqslant v_{(1+ch\sigma)\varepsilon}$ in $B_{1/2} \times (-T/2, T/2)$, where $\tilde{v}_\eta(x, t) := \sup\limits_{B^{(n+1)}_{\varepsilon\varphi\sigma\eta}(x, t)} u_1$ and w is a continuous function non negative in

$$D := \left\{B_{9/10}(0) \setminus \left[B_{1/8}\left(\frac{3}{4}e_n\right) \cup B_{1/8}\left(-\frac{3}{4}e_n\right)\right]\right\} \times \left(-\frac{9T}{10}, \frac{9T}{10}\right)$$

such that

$$\begin{cases} \Delta w - a_1 D_t w = 0 & \text{in } D \cap \{u_2 > 0\} \\ \Delta w - a_2 D_t w = 0 & \text{in } D \cap \{u_2 < 0\} \\ w = 0 & \text{on } D \cap \{u_2 = 0\} \\ w = u_2\left(\frac{3}{4}e_n, 0\right) & \text{on } \partial B_{1/8}\left(\frac{3}{4}e_n\right) \times \left(-\frac{9T}{10}, \frac{9T}{10}\right) \\ w = -u_2\left(-\frac{3}{4}e_n, 0\right) & \text{on } \partial B_{1/8}\left(-\frac{3}{4}e_n\right) \times \left(-\frac{9T}{10}, \frac{9T}{10}\right) \\ w = 0 & \text{on } \partial_p D \setminus \left\{\partial\left[B_{1/8}\left(\frac{3}{4}e_n\right) \cup B_{1/8}\left(-\frac{3}{4}e_n\right)\right] \times \left(-\frac{9T}{10}, \frac{9T}{10}\right)\right\}. \end{cases}$$

Now, we show that $S := \{\eta \in [0, 1] : \overline{v}_\eta(x, t) \leqslant u_2(x, t),\ \forall (x, t) \in B_{1/2} \times \times (-T/2, T/2)\}$ is both open and closed in $[0, 1]$. Exploiting hypotheses, maximum principle and the continuity of the functions considered, we deduce that $0 \in S$, so $S \neq \emptyset$, and S is closed.

We show that S is open. We assume that $\overline{v}_{\eta_0} \leqslant u_2$ for some $\eta_0 \in [0, 1]$. Supposing that there exists (x_0, t_0) such that $u_2(x_0, t_0) = \overline{v}_{\eta_0}(x_0, t_0) = 0$ $(u_1(y_0, s_0) = 0)$, by Lemma 10 in [2] we get that

$$v_{\eta_0}(x_0, t_0) \geqslant \alpha_+^* \langle x - x_0, \nu^* \rangle^+ - \alpha_-^* \langle x - x_0, \nu^* \rangle^- + o(|x - x_0|),$$

where $\alpha_+^* = \alpha_+^{(1)} |\tau^*|$, $\alpha_-^* = \alpha_-^{(1)} |\tau^*|$, $\nu^* = \tau^*/|\tau^*|$,

$$\tau^* := \nu^{(1)} + \frac{\varepsilon^2 \varphi_{\sigma\eta_0}(x_0, t_0)}{|y_0 - x_0|} \nabla_x \varphi_{\sigma\eta_0}(x_0, t_0), \qquad \nu^{(1)} = \frac{y_0 - x_0}{|y_0 - x_0|}, \quad \text{and}$$

$$\frac{s_0 - t_0}{|y_0 - x_0|} = \frac{\beta_+^{(1)}}{\alpha_+^{(1)}} \leqslant G((y_0, s_0), \nu^{(1)}, \alpha_+^{(1)}, \alpha_-^{(1)}).$$

Now, by Theorem A, we deduce:

$$u_2(x, t_0) = \alpha_+^{(2)}\langle x - x_0, \nu^{(2)}\rangle^+ + \alpha_-^{(2)}\langle x - x_0, \nu^{(2)}\rangle^- + o(|x - x_0|),$$

$$\nu^{(2)} = \nu^* \quad \text{and} \quad G((x_0, t_0), \nu^{(2)}, \alpha_+^{(2)}, \alpha_-^{(2)}) \leqslant \frac{\beta_+^{(2)}}{\alpha_+^{(2)}} :=$$

$$= |\tau^*|^{-1} \cdot \left(\frac{s_0 - t_0}{|y_0 - x_0|} + \frac{\varepsilon^2 \varphi_{\sigma\eta_0}(x_0, t_0)}{|y_0 - x_0|} D_t \varphi_{\sigma\eta_0}(x_0, t_0) \right).$$

By Corollary 1 in [1], since $F = F(u_2)$ is Lipschitz, we obtain $w/|u_2| \geqslant N$ in $\{u_2 \neq 0\}$ strictly away from the parabolic boundary of D. Near x_0, for $t = t_0$, we have:

$$\overline{v}_{\eta_0}(x, t_0) \geqslant \overline{\alpha}_+ \langle x - x_0, \nu^* \rangle^+ - \overline{\alpha}_- \langle x - x_0, \nu^* \rangle^- + o(|x - x_0|)$$

where $\overline{\alpha}_+ = \alpha_+^* + C\sigma\varepsilon\alpha_+^{(2)}$, $\overline{\alpha}_- = \alpha_-^* - C\sigma\varepsilon\alpha_-^{(2)}$ and $C = PN$ is the constant that appears in the hypothesis (iv).

Again by Theorem A we obtain:

$$G((x_0, t_0), \nu^{(2)}, \alpha_+^{(2)}, \alpha_-^{(2)}) \leqslant \frac{\beta_+^{(2)}}{\alpha_+^{(2)}} \leqslant \left(\frac{\beta_+^{(1)}}{\alpha_+^{(1)}} + C\sigma\varepsilon h \right)(1 + \tilde{C}\sigma\varepsilon h) \leqslant$$

$$\leqslant G((y_0, s_0), \nu^{(1)}, \alpha_+^{(1)}, \alpha_-^{(1)}) + R\sigma\varepsilon h.$$

Since $\alpha_+^{(1)} \leqslant \overline{\alpha}_+ - C\sigma\varepsilon\alpha_+^{(2)} + \tilde{C}\sigma\varepsilon h$, $\alpha_-^{(1)} \geqslant \overline{\alpha}_- + C\sigma\varepsilon\alpha_-^{(2)} - \tilde{C}\sigma\varepsilon h$, considering the initial hypotheses satisfied by G, and $\alpha_+^{(2)} + \alpha_-^{(2)} \geqslant m > 0$, we get

$$G((y_0, s_0), \nu^{(1)}, \alpha_+^{(1)}, \alpha_-^{(1)}) \leqslant$$

$$\leqslant G((y_0, s_0), \nu^{(1)}, \overline{\alpha}_+ - C\sigma\varepsilon\alpha_+^{(2)} + \tilde{C}\sigma\varepsilon h, \overline{\alpha}_- + C\sigma\varepsilon\alpha_-^{(2)} - \tilde{C}\sigma\varepsilon h) \leqslant$$

$$\leqslant G((x_0, t_0), \nu^{(2)}, \overline{\alpha}_+, \overline{\alpha}_-) + H|(y_0, s_0) - (x_0, t_0)|^\alpha + \overline{L}|\nu^{(1)} - \nu^{(2)}|^\alpha +$$

$$+ \overline{L}(\tilde{C}\sigma\varepsilon h)^\alpha - c^*(C\sigma\varepsilon\alpha_+^{(2)})^\alpha + \overline{L}(\tilde{C}\sigma\varepsilon h)^\alpha - c^*(C\sigma\varepsilon\alpha_-^{(2)})^\alpha \leqslant$$

$$\leqslant G((x_0, t_0), \nu^{(2)}, \overline{\alpha}_+, \overline{\alpha}_-) + H\varepsilon^\alpha - C^\alpha\sigma^\alpha\varepsilon^\alpha c^* m^\alpha + (H + \overline{C})\,\sigma^\alpha\varepsilon^\alpha h^\alpha.$$

But $H < C^\alpha\sigma^\alpha m^\alpha c^*$ by hypothesis, thus, choosing $h^\alpha < \dfrac{C^\alpha\sigma^\alpha c^* m^\alpha - H}{\sigma^\alpha(H + \overline{C} + R)}$, we have

$$G((x_0, t_0), \nu^{(2)}, \alpha_+^{(2)}, \alpha_-^{(2)}) < G((x_0, t_0), \nu^{(2)}, \overline{\alpha}_+, \overline{\alpha}_-). \tag{4.1}$$

Since $u_2 - \overline{v}_{\eta_0} \geqslant 0$ and it is a supercaloric function in $\{\overline{v}_{\eta_0} > 0\}$, it follows that $\alpha_-^{(2)} \leqslant \overline{\alpha}_-$. By Hopf maximum principle we have $\alpha_+^{(2)} > \overline{\alpha}_+$ and so

$$G((x_0, t_0), \nu^{(2)}, \alpha_+^{(2)}, \alpha_-^{(2)}) > G((x_0, t_0), \nu^{(2)}, \overline{\alpha}_+, \overline{\alpha}_-).$$

This inequality and (4.1) give us a contradiction, and so the Theorem is proved. ■

Now, we show the regularization in space that is the defect angle in space is as small as we prefer and so the free boundary is a $\mathcal{C}^1$ domain in space. Operating as in the Lemma 13 in [2], and supposing that $H_1 \leqslant \leqslant C^\alpha \sigma^\alpha m^\alpha c^*$ to may apply the Lemma 3, we have the following:

LEMMA 4. *Let u be a viscosity solution to a free boundary problem in $B_1 \times (-1, 1)$, monotone increasing in every direction $\tau \in \Gamma(e_n, \theta, \theta^t)$ (elliptic cone with axis e_n and aperture θ in space and θ_t in time) where $0 < \theta_0 \leqslant \theta^t \leqslant \theta < \pi/2$. Then there exist $c, \overline{c} > 0$ and a unit vector ν_1 such that in $B_{1/2} \times (-1, 1)$, if $H_1 \leqslant C^\alpha \sigma^\alpha m^\alpha c^*$, the function $u_1(x, t) := = u(x, \overline{c}\delta^2 t)$ is monotone increasing along every direction $\tau \in \in \Gamma(\nu_1, \theta_1, \theta_1^t)$ with*

$$\delta_1 := \pi/2 - \theta_1 \leqslant \delta - c\delta^3 \quad \text{and} \quad \theta_1^t \geqslant \theta_0.$$

We note that H_1 is the Hölder constant of $G_1((x, t), \nu(x, t), u_\nu^+(x, t), u_\nu^-(x, t)) = \overline{c}\delta^2 G((x, \overline{c}\delta^2 t), \nu(x, \overline{c}\delta^2 t), u_\nu^+(x, \overline{c}\delta^2 t), u_\nu^-(x, \overline{c}\delta^2 t))$.

Applying Lemma 4 inductively to the function

$$u_{k+1}(x, t) := u_k(2^{-r_k}x, \overline{c}2^{-r_k}\delta_k^2 t)\cdot 2^{r_k}, \quad k \geqslant 1 \text{ with } u_1(x, t) = u(x, \overline{c}\delta^2 t),$$

($\overline{c}$ and δ like in the previous Lemma), we obtain:

THEOREM 5. *Let u be a viscosity solution in $B_1 \times (-1, 1)$. Then for each time level $t \in (-1, 1)$, the surface $F_t = F \cap \{t\}$ is a $\mathcal{C}^1$ surface.*

5. Regularization in space-time.

The results of previous section, in particular the fact that the defect angle in space can be made as small as we want, permit us to exploit the Lemma 2 and to obtain the regularization in space-time.

LEMMA 6. *Let u be a viscosity solution in $B_1 \times (-1, 1)$ of a free boundary problem such that:*

i) *u is monotone increasing in any direction of a space cone* $\Gamma_x(e_n, \theta)$, *with* $0 < \theta_0 \leqslant \theta < \frac{\pi}{2}$.

ii) *There exist constants* $\bar{c}_1 > 0$ *and* A, B *such that u is monotone increasing along the directions* $e_t + Be_n$ *and* $-e_t - Ae_n$ *with* $0 < B - A \leqslant \bar{c}_1 \mu$.

Then, if $\delta = \frac{\pi}{2} - \theta \ll \mu^{3/\alpha}$ *and* $H < \min(\delta^{\alpha/3}, c^* \delta^{\alpha-1} m^\alpha C^\alpha \mu)$, *exist constants* c_1, c_2, C *positive and* A_1, B_1 *depending only on* θ_0 *and* n, *and a spatial unit vector* ν_1, *such that, in* $B_{1/2} \times (-C\delta/2\mu, C\delta/2\mu)$

a) *u is monotone increasing in any direction* $\tau \in \Gamma_x(\nu_1, \theta_1)$, *with*

$$\delta_1 := \pi/2 - \theta_1 \leqslant \delta - c_1 \delta^2/\mu .$$

b) *u is monotone increasing along the directions* $e_t + B_1 \nu_1$ *and* $-e_t - A_1 \nu_1$ *with*

$$0 < B_1 - A_1 \leqslant \bar{c}_1 \mu_1 \quad \text{and} \quad \mu_1 \leqslant \mu - c_2 \delta .$$

PROOF. We recall that to apply Lemma 2, it is necessary that $H < \delta^{\alpha/3}$ and $\delta \ll \mu^{3/\alpha}$, and to apply Lemma 3 to the function $w(x, t) = u\left(x, \frac{\delta}{\mu} t\right)$, one needs to have $H < c^* \delta^{\alpha-1} m^\alpha C^\alpha \mu$. Then the proof is the same as that of the Lemma 14 in [2]. ■

PROOF OF THEOREM 1. If λ is very small, then $\tilde{u}(x, t) = \frac{u(\lambda x, \lambda t)}{\lambda}$ satisfies the hypotheses of the Lemma 6. In fact, by Lemma 4 and Theorem 5 it is possible to obtain $\delta_0 = c\mu_0^{3/\alpha}$ where $c \ll 1$ is a constant such that

$$c^{\frac{3-\alpha}{3}} \mu_0^{\frac{3-\alpha}{\alpha}} \ll 1 \tag{5.1}$$

and δ_0, μ_0 are the defect angle in space and in time respectively. By (5.1) we have that $\delta_0 \ll \mu_0^{3/\alpha}$, $\delta_0^{\frac{3-\alpha}{3}} \ll 1$ and $\delta_0 \ll 1$. Moreover, decreasing λ with δ_0 and μ_0 fixed, if necessary, we obtain $H_0 < \min(\delta_0^{\alpha/3}, c^* \delta_0^{\alpha-1} m^\alpha C^\alpha \mu_0)$ where H_0 is the Hölder constant of the function $\tilde{u}(x, t)$. We want to apply Lemma 6 inductively to the functions $u_k(x, t) := 2^k \cdot \tilde{u}(2^{-k} x, 2^{-k} t)$ with $k \in \mathbb{N}$. We need that for every $k \in \mathbb{N}$

$$\delta_k \ll \mu_k^{3/\alpha} \tag{5.2}$$

and

$$H_k < \min(\delta_k^{\alpha/3},\ c^*\delta_k^{\alpha-1} m^\alpha C^\alpha \mu_k) \tag{5.3}$$

where H_k is the Hölder constant of the G_k associates to the viscosity solution $u_k(x, t)$, δ_k and μ_k are the defect angle in space and in time respectively, to the step k.

We prove that (5.2) holds.

We consider the function $h(x) = \dfrac{1 - c_1 x}{(1 - (\alpha/3)\, c_1 x)^{3/\alpha}}$ where c_1 is the constant in the Lemma 6, such that $\delta_1 \leqslant \delta_0(1 - c_1\delta_0/\mu_0)$. We have $0 < h(x) < 1$ for each $x \in (0, 1/c_1)$.

Since $\delta_0 \ll \mu_0^{3/\alpha}$, choosing $\bar{c}_2 < (c_2, c_1\alpha/3)$ and $\mu_k = \mu_{k-1} - \bar{c}_2\delta_{k-1}$, we obtain

$$\frac{\delta_0}{\mu_0} \ll 1 \quad \Rightarrow \quad \frac{\delta_0}{\mu_0} < \frac{1}{c_1} \quad \Rightarrow$$

$$\Rightarrow \quad \frac{\delta_1}{\mu_1^{3/\alpha}} \leqslant \frac{\delta_0(1 - c_1\delta_0/\mu_0)}{\mu_0^{3/\alpha}(1 - \bar{c}_2\delta_0/\mu_0)^{3/\alpha}} \leqslant \frac{\delta_0}{\mu_0^{3/\alpha}} \ll 1\,.$$

Then proceeding in this way, for each $k > 1$ we have

$$\frac{\delta_{k-1}}{\mu_{k-1}^{3/\alpha}} \ll 1 \quad \Rightarrow \quad \frac{\delta_{k-1}}{\mu_{k-1}} < \frac{1}{c_1} \quad \Rightarrow \quad \frac{\delta_k}{\mu_k^{3/\alpha}} \leqslant \ldots \leqslant \frac{\delta_0}{\mu_0^{3/\alpha}} = c \ll 1\,.$$

Now we show (5.3).

We know that $H_0 < \min(\delta_0^{\alpha/3},\ c^*\delta_0^{\alpha-1} m^\alpha C^\alpha \mu_0)$.

We suppose that $H_{k-1} < \min(\delta_{k-1}^{\alpha/3},\ c^*\delta_{k-1}^{\alpha-1} m^\alpha C^\alpha \mu_{k-1})$; then to the step $k-1$, applying Lemma 6, we have $\delta_k = \delta_{k-1} - c_1\dfrac{\delta_{k-1}^2}{\mu_{k-1}}$.

Since $\delta_{k-1} = c'\mu_{k-1}^{3/\alpha}$, where $c' < c \ll 1$, we get:

$$\delta_k = \delta_{k-1}(1 - c_1 c'^{\alpha/3}\delta_{k-1}^{\frac{3-\alpha}{3}}) \qquad \mu_k = \mu_{k-1}(1 - \bar{c}_2 c'^{\alpha/3}\delta_{k-1}^{\frac{3-\alpha}{3}}),$$

and so $H_k = 2^{-\alpha}H_{k-1} < 2^{-\alpha}\delta_{k-1}^{\alpha-1}c^* m^\alpha C^\alpha \mu_{k-1} < 2^{-\alpha}\delta_k^{\alpha-1}c^* m^\alpha C^\alpha \mu_{k-1}$.

To obtain $H_k < c^*\delta_k^{\alpha-1} m^\alpha C^\alpha \mu_k$, it is sufficient that $2^{-\alpha}\mu_{k-1} \leqslant \mu_k$. Since, by (5.1), we have

$$c'^{\alpha/3}\delta_{k-1}^{\frac{3-\alpha}{3}} \leqslant c^{\alpha/3}\delta_0^{\frac{3-\alpha}{3}} \ll 1\,,$$

then

$$c'^{\alpha/3}\delta_{k-1}^{\frac{3-\alpha}{3}} \leqslant (1-2^{-\alpha})/\bar{c}_2 \Rightarrow 2^{-\alpha} \leqslant (1-\bar{c}_2 c'^{\alpha/3}\delta_{k-1}^{\frac{3-\alpha}{3}}) \Rightarrow 2^{-\alpha}\mu_{k-1} \leqslant \mu_k .$$

Finally, we show that $H_k < \delta_k^{\alpha/3}$.

We have: $H_k = 2^{-\alpha}H_{k-1} < 2^{-\alpha}\delta_{k-1}^{\alpha/3}$ and

$$2^{-\alpha}\delta_{k-1}^{\alpha/3} \leqslant \delta_k^{\alpha/3} \Leftrightarrow 2^{-\alpha} \leqslant (1-c_1 c'^{\alpha/3}\delta_{k-1}^{\frac{3-\alpha}{3}})^{\alpha/3} \Leftrightarrow c'^{\alpha/3}\delta_{k-1}^{\frac{3-\alpha}{3}} \leqslant (1-2^{-3})/c_1 .$$

By (5.1) last inequalities hold, in fact

$$c'^{\alpha/3}\delta_{k-1}^{\frac{3-\alpha}{3}} \leqslant c^{\alpha/3}\delta_0^{\frac{3-\alpha}{3}} \ll 1 .$$

Therefore (5.3) holds and so it is possible to apply Lemma 6 to the u_k.

In such way we define the sequences $\{A_k\}$, $\{B_k\}$, $\{\delta_k\}$, $\{\mu_k\}$ and $\Gamma_x(\nu_k, \theta_k)$ (spatial cones with axis ν_k and opening θ_k) which satisfy in $B_{2^{-k}} \times \left(-\frac{C\delta_k}{2^k\mu_k}, \frac{C\delta_k}{2^k\mu_k}\right)$ the following properties:

i) u is monotone increasing along the spatial directions $\tau \in \Gamma_x(\nu_k, \theta_k)$;

ii) u is monotone increasing along the direction $e_t + B_k\nu_k$ and $-e_t - A_k\nu_k$ where $0 < B_k - A_k \leqslant \bar{c}_1\mu_k$;

iii) the sequences $\{\delta_k\}$ and $\{\mu_k\}$ are such that

$$\delta_{k+1} = \delta_k - c_1\frac{\delta_k^2}{\mu_k} \qquad \mu_{k+1} = \mu_k - \bar{c}_2\delta_k .$$

Then, from iii) we deduce

$$\delta_k \sim \frac{c_1(\eta)}{k^{\frac{3}{3-\alpha}-\eta}} \qquad \mu_k \sim \frac{c_2(\eta)}{k^{\frac{\alpha}{3-\alpha}-\eta}}$$

for every $\eta > 0$ small enough. This asymptotic behaviors correspond exactly to the modulus of continuity of $|\nabla_{x'}f|$ and $D_t f$ in Theorem 1. Now applying the results of K. Widman [17], since u_t is bounded and for each $\bar{t} \in (-1, 1)$ the set $\Omega^{\pm} \cap \{t = \bar{t}\}$ is a Liapunov-Dini domain we obtain, at each level time, that $\nabla_x u^{\pm}$ are continuous up to the free boundary. Therefore, exploiting the free boundary condition the proof is easily completed. ■

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