RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

FEI-TSEN LIANG

Level sets of Gauss curvature in surfaces of constant mean curvature

Rendiconti del Seminario Matematico della Università di Padova, tome 104 (2000), p. 1-26

<http://www.numdam.org/item?id=RSMUP_2000__104__1_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 104 (2000)

Level Sets of Gauss Curvature in Surfaces of Constant Mean Curvature.

FEI-TSEN LIANG (*)

ABSTRACT - For an embedded surface M in $\mathbb{R}^3$ with nonempty boundary ∂M, constant mean curvature H and Gauss curvature K, we consider the sets $E_0(k) = \{x \in M : K(x) < k\}$, $E_1(k) = \{x \in M : K(x) > k\}$ and $\Gamma(k) = \{x \in \overline{M} : K(x) = k\}$, (where $\overline{M} = M \cup \partial M$.) If k is not a critical value of K, such that $\Gamma(k)$ is not empty, then one and *only one* of the following cases occurs: (1) At least one component of $\Gamma(k)$ is a Jordan arc with two distinct endpoints on ∂M; (2) $E_1(k)$ is simply-connected, enclosed by $\Gamma(k)$ and containing a unique umbilical point. $\Gamma(k)$ is a simple closed curve, on the other side of which $E_0(k)$ is situated. $E_0(k)$ is of the same connectivity with that of M and enclosed by $\Gamma(k)$ together with ∂M; (3) Each component of $\Gamma(k)$ is a simple closed curve inside M, adjacent to which and on two sides of which a component of $E_0(k)$ and a component of $E_1(k)$ are situated, respectively; each of them either is diffeomorphic to a planar annulus or is of the same connectivity with that of M and is enclosed by this component of $\Gamma(k)$, together with either another component of $\Gamma(k)$ or components of ∂M. This result then yields the convexity of M if $\inf\{K(x), x \in \partial M\}$ is positive and M is simply-connected.

For an embedded surface M in $\mathbb{R}^3$ with boundary ∂M and Gauss curvature K, we may consider the sets

(1) $$E_0(k) = \{x \in M : K(x) < k\},$$

(2) $$E_1(k) = \{x \in M \},$$

and

(3) $$\Gamma(k) = \{x \in M \cup \partial M : K(x) = k\}.$$

(*) Indirizzo dell'A.: Institute of Mathematics, Academia Sinica, Taipei, Taiwan. E-mail: liang@math.sinica.edu.tw

The main purpose of this paper is to characerize the topological properties of the leval sets $E_0(k)$, $E_1(k)$, and $\Gamma(k)$ in case M is *of constant mean curvature* by means of Gauss-Bonnet theorem and some applications of the Hopf differential. We shall show that *one and only one* of the cases listed in Main Theorem 2 below occurs. From this, we draw the important conclusion that *M is convex, in case* $\inf\{K(x): x \in \partial M\}$ *is positive and M is simply-connected.* Indeed, it is the attempt to verify the last result which motivates the investigations made in this paper. We note also that famous examples, (e.g. Delaunay surfaces), show the necessity of the conditions of *simply-connectedness* in this result.

This convexity result is related to Hopf's conjecture which in fact is a consequence of the intuition that closed immersed surfaces in $\mathbb{R}^3$ of constant mean curvature are necessarily convex. Wente's couterexample [11] and subsequent discovery of ab undant examples of such immersions violate this intuition on convexity. In contrast, Huang and Lin prove in [8] via a *variational* approach that for a *nonparametric* surface M, the *negatively curved set*, (that is, the set of points at which K is negative), if exiats, must extend to the boundary ∂M of M. From this, the convexity of M follows in case $\inf\{K(x): x \in \partial M\}$ is positive and M is *nonparametric*, (which is a special case of the above mentioned result obtained in this paper via an entirely irrelevant approach). Furthermore, Huang and Lin introduce in [8] the notion of *extremal domains* , and prove that for *parametric* surfaces immersed in $\mathbb{R}^3$, the negatively curved set, if exists, must be at least as large as an *extremal domain.* This result has not been surpassed via the approach used in this paper.

For *nonparametric* surfaces of constant mean curvature with *capillary* boundary conditions, the convexity problem has been investigated in Finn [4] [5], Korevarr [9], and Chen-Huang [3]. On the other hand, Brascamp-Lieb [1] and Cafferelli-Friedman [2] obtain some convexity properties of solutions to certain linear elliptic Dirichlet boundary value problem.

Related results for higher dimentional parametric surfaces can be found in Korevarr-Lewis [10] and Huang [7].

Introduction.

Gauss-Bonnet theorem says that, for $E_0(k)$ diffeomorphic to the unit disk or a connected plane bounded domain by m_1 circles ($m_1 \geqslant 2$), there holds

$$\int_{E_0(k)} K dA + \int_{\partial E_0(k)} K_g ds = 2\pi \text{ or } 2(2-m_1)\pi, \tag{3.1}$$

respectively, where K_g is the geodesic curvature along $\partial E_0(k)$, dA is the area element of M and s is the arc length of $\partial E_0(k)$. Likewise, we have

$$\int_{E_1(k)} K dA + \int_{\partial E_1(k)} K_g ds = 2\pi \text{ or } 2(2-m_2)\pi, \tag{3.2}$$

provided $E_1(k)$ is diffeomorphic to the unit disk or a connected plane domain bounded by m_2 circles. In this connection, we note that, if M has nonempty boundary ∂M, we may set

$$K_0 = \inf\{K(x), x \in \partial M\}, \tag{4}$$

and

$$K_1 = \sup\{K(x), x \in \partial M\};$$

then, if k is *not* a *critical value* of K such that

$$k \leqslant K_0 \quad \text{or} \quad k \geqslant K_1,$$

then the boundary of $E_0(k)$ or $E_1(k)$ coincides with $\Gamma(k)$.

Recall also that, at *non-umbilic* points, the Gauss curvature K of a surface M of *constant mean curvature* H satisfies the partial differential equation

$$\triangle \log(H^2 - K) = 4K, \tag{5}$$

where $\triangle$ is the Laplace-Beltrami operator on M.

In Section 1, we shall first present some preliminary results on the Hopf differential ϕ with $|\phi|^2 = H^2 - K$, by means of which (5) is easily derived in the end of Section 1 and the geodesic curvature K_g along $\Gamma(k)$ is calculated in Section 2. In particular, for each component $\tilde{\Gamma}(k)$ of $\Gamma(k)$, we obtain precise formulae for $\int_{\tilde{\Gamma}(k)} K_g ds$ in terms of k, $H^2 - k$, $|\nabla K|$ and $\int_{\tilde{\Gamma}(k)} \frac{\partial}{\partial s} \log \phi$, where ∇ denotes the covariant differentiation on M. Namely, we shall prove in Section 2 the following

MAIN THEOREM 1. *Suppose M is an embedded surface in $\mathbb{R}^3$ with constant mean curvature H and suppose k is* not *a critical value of K*

on M. For a component $\tilde{\Gamma}(k)$ of $\Gamma(k)$ which is a simple closed curve inside M, denote n as the unit outward normal with respect to the domain at the left of $\tilde{\Gamma}(k)$ when $\tilde{\Gamma}(k)$ is traversed counterclockwise. If $\tilde{\Gamma}(k)$ is described counterclockwise so that a component $\tilde{E}_0(k)$ of $E_0(k)$ is at the left of $\tilde{\Gamma}(k)$, then there holds

$$\int_{\tilde{\Gamma}(k)} K_g\, ds = -\frac{1}{4} \int_{\tilde{\Gamma}(k)} (\nabla \log |\phi|^2)\cdot n\, ds ,$$

while, if a component $\tilde{E}_1(k)$ of $E_1(k)$ is at the left of $\tilde{\Gamma}(k)$ instead,

$$\int_{\tilde{\Gamma}(k)} K_g\, ds = -\frac{1}{4} \int_{\tilde{\Gamma}(k)} (\nabla \log |\phi|^2)\cdot n\, ds + \int_{\tilde{\Gamma}(k)} \frac{\partial}{\partial s} \log \phi\, ds .$$

Here ∇ denotes the covariant differentiation on M.

In consideration of the fact that $|\phi|^2$ takes the constant value $H^2 - k$ along $\tilde{\Gamma}(k)$, we may reformulate previous results as follows.

Main Theorem 1*. *Assume k and M satisfy the same conditions as in Main Theorem 1. For a component $\tilde{\Gamma}(k)$ of $\Gamma(k)$ which is a simple closed curve, there holds*

$$\int_{\tilde{\Gamma}(k)} K_g\, ds = \frac{1}{4} \int_{\tilde{\Gamma}(k)} \frac{|\nabla K|}{H^2 - k} ds ,$$

if the former case in Main Theorem 1 *occurs, while*

$$\int_{\tilde{\Gamma}(k)} K_g\, ds = \frac{1}{4} \int_{\tilde{\Gamma}(k)} \frac{|\nabla K|}{H^2 - k} ds + \int_{\tilde{\Gamma}(k)} \frac{\partial}{\partial s} \arg \phi\, ds ,$$

if the latter case in Main Theorem 1 *occurs.*

Suppose the boundary of M consists of m components, namely

$$\partial M = \partial_1 M \cup \partial_2 M \cup \ldots \cup \partial_m M .$$

For each i, $1 \leqslant i \leqslant m$, if there exists a number k_i, for which a component $\tilde{\Gamma}^*(k_i)$ of $\Gamma^*(k_i)$ is a simple closed curve intersecting $\partial_i M$ at a finite number of points and not enclosing a simply-connected region inside M, then we set $\partial_i^* M = \cup \tilde{\Gamma}^*(k_i)$; if, however, for $\partial_i M$, no such a number k_i exist,

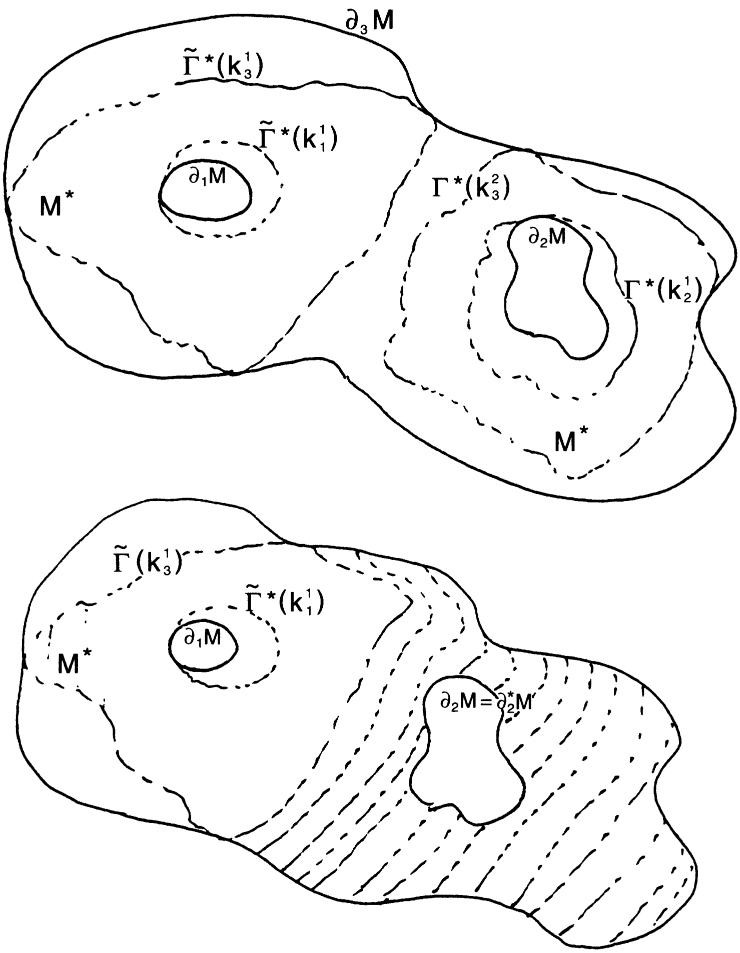

Figure 1

then we set $\partial_i^* M = \partial_i M$. Denote M^* as the domain contained in M and enclosed by $\partial_1^* M \cup \ldots \cup \partial_m^* M$. (See Figure 1). We then set

$$E_0^*(k) = \{x \in M^* \mid K(x) < k\},$$

and

$$E_1^*(k) = \{x \in M^* \mid K(x) > k\}.$$

Delete small neighborhoods of *umbilical points* on M first and then after a simple limiting procedure, we obtain the following results by virtue of Main Theorem 1, the partial differential equation (5) and Green's identity.

COROLLARY 1. *Assume k and M satisfy the same conditions as in Main Theorem* 1. *Consider a component $\tilde{E}_0^*(k)$ of $E_0^*(k)$ or a component $\tilde{E}_1^*(k)$ of $E_1^*(k)$ whose boundary consists of q components of ∂M^*, $q \geqslant 0$, together with p components of $\Gamma(k)$, $p \geqslant 1$, each of which is a* simple closed curve. *If, for such a component $\tilde{E}_0^*(k)$, its boundary $\partial \tilde{E}_0^*(k)$ is described counterclockwise so that $\tilde{E}_0^*(k)$ is at the left of $\partial \tilde{E}_0^*(k)$, then we have*

$$\int_{\partial \tilde{E}_0^*(k)} K_g \, ds = - \int_{\tilde{E}_0^*(k)} K dA \; ;$$

if, on the other hand, for such a component $\tilde{E}_1^*(k)$, its boundary $\partial \tilde{E}_1^*(k)$ is described counterclockwise so that $\tilde{E}_1^*(k)$ is at the left of $\partial \tilde{E}_1^*(k)$, we then have

$$\int_{\partial \tilde{E}_1^*(k)} K_g \, ds = - \int_{\tilde{E}_1^*(k)} K dA + \int_{\partial \tilde{E}_1^*(k)} \frac{\partial}{\partial s} \arg \phi \, ds \, .$$

Corollary 1 and the Gauss-Bonnet formula (3.1) yield the following result.

THEOREM 1. *Assume k and M satisfy the same conditions as in Main Theorem* 1. *Then a component $\tilde{E}_0^*(k)$ of $E_0^*(k)$ which is of the type indicated in Corollary* 1 *must be diffeomorphic to a planar annulus; and hence the component $\tilde{E}_0(k)$ of $E_0(k)$ (defined in* (1.1)) *containing $E_0^*(k)$ either is diffeomorphic to a planar annulus or is of the same connectivity with that of M.*

Also, by Lemma 2 and Lemma 3 in Section 1 below, together with the fact that ϕ vanishes precisely at *umbilical* points, we have

LEMMA 1. *For each simple closed curve Γ on M traversed counter-clockwise, we have*

$$\int_{\Gamma} \frac{\partial}{\partial s} \arg\phi = 0 \text{ or } 2m\pi ,$$

where m is a positive integer.

Lemma 1, Corollary 1 and the Gauss-Bonnet formula (3.2) enable us to infer the following

THEOREM 2. *Assume k and M satisfy the same conditions as in Main Theorem 1. Then a component $\tilde{E}_1^*(k)$ of $E_1^*(k)$ which is of the type indicated in Corollary 1 must be diffeomorphic to either a unit disk or a plan nar annulus. In the former case, $\tilde{E}_1^*(k)$ coicides with a component $\tilde{E}_1(k)$ of $E_1(k)$, contains a unique umbilical point and $\int_{\partial\tilde{E}_1(k)} \frac{\partial}{\partial s}\arg\phi = 2\pi$. In the latter case,* the *component $\tilde{E}_1(k)$ of $E_1(k)$ containing $\tilde{E}_1^*(k)$ either is diffeomorphic to a planar annulus or is of the same connectivity with that of M.*

Theorem 1 and Theorem 2 then yield

THEOREM 3. *Suppose M is simply-connected and either a complete surface of constant mean curvature H without umbilical points or a minimal surface, i.e. $H = 0$. Suppose k is not a critical value of K on M. Then each component $\tilde{E}_0(k)$ of $E_0(k)$ or component $\tilde{E}_1(k)$ of $E_1(k)$ which contains a component $\tilde{E}_0^*(k)$ of $E_0^*(k)$ or a component $\tilde{E}_1^*(k)$ of $E_1^*(k)$ of the type indicated in Corollary 1 must either be diffeomorphic to a planar annulus or is of the same connectivity with that of M.*

Also we obtain from Theorem 1 the following result.

THEOREM 4. *Assume k and M satisfy the same conditions as in Main Theorem 1. If no component of $\Gamma(k)$ is a Jordan arc with two distinct endpoints on ∂M and if a simply-connected component of $E_1(k)$ is enclosed by a component $\tilde{\Gamma}(k)$ of $\Gamma(k)$, then, adjacent to $\tilde{\Gamma}(k)$ and on the other side of $\tilde{\Gamma}(k)$, there situates a component of $E_0(k)$ which either is diffeomorphic to a planar annulus or of the same connectivity wit h that of M.*

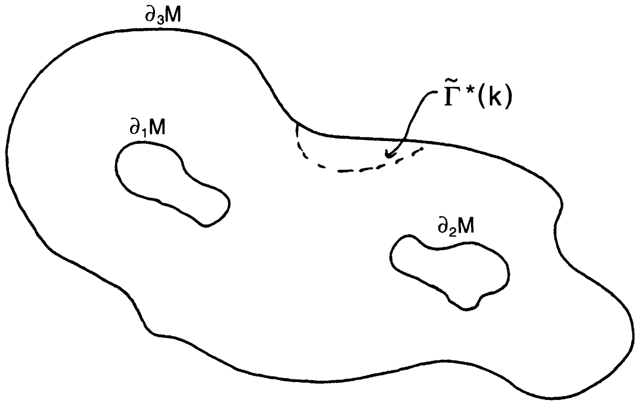

Figure 2

We shall, however, prove in Section 3 the following result.

THEOREM 5. *In Theorem* 4, *the component of* $E_0(k)$ *indicated there cannot be enclosed by two components of* $\Gamma(k)$; *in other words, this component of* $E_0(k)$ *is enclosed by a component of* $\Gamma(k)$ *and one or more component of* ∂M.

Assuming the truth of Theorem 5, we have, by virtue of Theorem 1, Theorem 2, Theorem 4 and Theorem 5, the following

MAIN THEOREM 2. *Assume k and M satisfy the same conditions as in Main Theorem* 1. *If* $\Gamma(k)$ *is not empty, one and* only one *of the following cases occurs.*

(1) *At least one component of* $\Gamma(k)$ *is a Jordan arc with two distinct endpoints on* ∂M. (*See Figure* 2).

(2) $E_1(k)$ *is simply-connected, enclosed by* $\Gamma(k)$ *and containing a unique umbilical point.* $\Gamma(k)$ *is a simple closed curve, on the other side of which* $E_0(k)$ *is situated.* $E_0(k)$ *is of the same connectivity with that of* M *and enclosed by* $\Gamma(k)$ *together with* ∂M. (*See Figure* 3).

(3) *Each component of* $\Gamma(k)$ *is a simple closed curve inside* M, *adjacent to which and on two sides of which a component of* $E_0(k)$ *and a component of* $E_1(k)$ *are situated, respectively; each of them either is diffeomorphic to a planar annulus or is of the same connectivity with that of* M *and is enclosed by this component of* $\Gamma(k)$, *together with either*

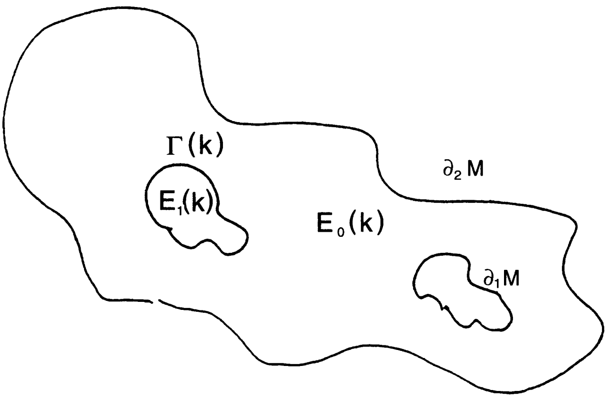

Figure 3

another component of $\Gamma(k)$ or one or more components of ∂M. (See Figure 4).

As a consequence of Main Theorem 2 and Theorem 3, we have

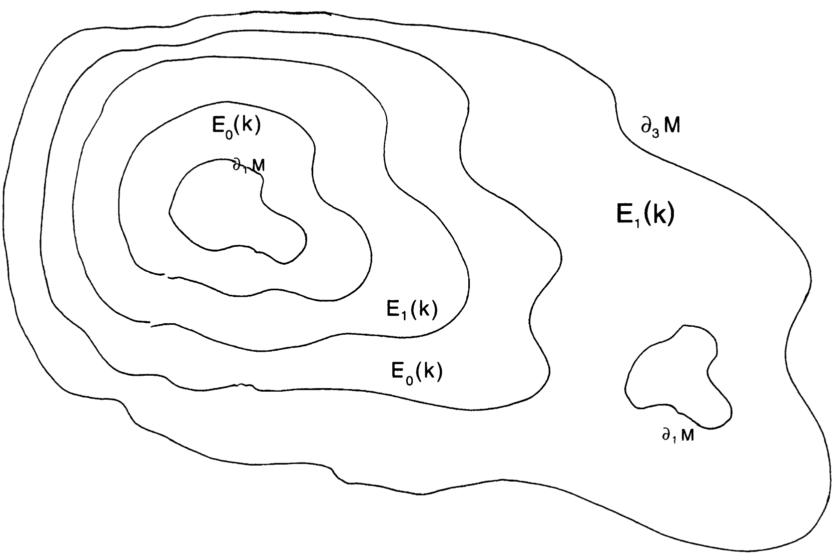

Figure 4

MAIN THEOREM 3. *Suppose M is simply-connected and is either a complete surface of constant mean curvature without umbilical points or a minimal surfaces. Suppose k is not a critical value of K and M. Then either* (1) *or* (3) *in Main Theorem* 2 *must occur.*

We may observe that Main Theorem 2 yields the following

MAIN THEOREM 4. *If M is simply-connected and if K_0 defined in* (4) *is positive, then $K>0$ inside M; i.e. M is convex.*

Indeed, if K_0 is positive, for each number $k<0$, case (1) in Main Theorem 2 cannot occur. Moreover, as M is simply connected, case (3) in Main Theorem 2 cannot occur for any number k. Thus, case (2) in Main Theorem 2 occurs for some number $k \geqslant 0$. This amounts to the truth of Main Theorem 4.

1. Preliminary results.

Let M be an oriented two-dimensional connected surface and $x: M \rightarrow \mathbb{R}^3$ be an isometric immersion of M into $\mathbb{R}^3$. At a neighborhood of any point of M, we may adapt an isothermal coordinate $z=\xi_1+i\xi_2$ in such a way that the first fundamental form is, for some scalar $\lambda>0$,

$$ds^2=\lambda|dz|^2.$$

Let us set

$$e_i=\left(\frac{1}{\lambda}\right)\frac{\partial x}{\partial \xi_i}, \qquad i=1,2,$$

and define the unit normal vector field by

$$e_3=e_1\times e_2.$$

We may define 1-forms w_i, w_{12}, w_{i3}, $i=1,2$, by the following formulae

$$dx=w_1 e_1+w_2 e_2,$$

$$de_1=w_{12}e_2+w_{13}e_3,$$

$$de_2=-w_{12}e_1+w_{23}e_3.$$

Then, writing, for $i = 1, 2$,

$$w_{i3} = h_{i1} w_1 + h_{i2} w_2,$$

we have $h_{12} = h_{21}$, and the mean curvature of M is given by

$$H = \frac{1}{2}(h_{11} + h_{22}).$$

Furthermore, setting

$$\phi = \frac{1}{2}(h_{11} - h_{22}) - ih_{12},$$

there holds

$$K = H^2 - |\phi|^2,$$

where K is the Gauss curvature of M.

The following result is a consequence of the Codazzi equation.

LEMMA 2. (Hopf [1], page 37) There holds

$$\frac{\partial(\lambda^2 \phi)}{\partial \bar{z}} = \lambda^2 \frac{\partial H}{\partial z}. \tag{6}$$

From Lemma 2, we immediately obtain the following result.

LEMMA 3. (Hopf [1], page 38) M has constant mean curvature if and only if the function $\lambda^2 \phi$ is an analytic function of z.

Hence, at a non-umbilic point of the surface M of constant mean curvature, we have

$$0 = \triangle \log(\lambda^4 |\phi|^2) = 4 \triangle \log \lambda + \triangle \log(H^2 - K).$$

On the other hand, by the Gauss equation

$$K = - \triangle \log \lambda .$$

These last two identities can be combined to yield (5).

2. Geodesic curvature K_g along the level curve $\Gamma(k)$ of K; Proof of Main Theorem 1.

Set $\xi_1 = u$, $\xi_2 = v$, and choose k such that $\Gamma(k)$ is a *smooth* curve

$$x(u(s), v(s)) = x(s),$$

where s is the arc length of $\Gamma(k)$ as a curve on a surface M of *constant* mean curvature H. Let, henceforth, subscripts denote the variables with respect to which partial differentiation is taken and

$$n_M = \frac{x_u \times x_v}{|x_u \times x_v|}$$

stand for the unit normal to M. Also, throughout this section, let dot denote the partial differentiation with respect to s, and then, along $\Gamma(k)$, the geodesic curvature

$$K_g = \lambda^2(\dot{u}\ddot{v} - \dot{v}\ddot{u}) + \frac{1}{2}(\lambda^2)_u \dot{v}(\dot{u}^2 + \dot{v}^2) - \frac{1}{2}(\lambda^2)_v \dot{u}(\dot{u}^2 + \dot{v}^2) \tag{7}$$

$$= \frac{\dot{u}\ddot{v} - \dot{v}\ddot{u}}{\dot{u}^2 + \dot{v}^2} + \frac{1}{2}\frac{(\lambda^2)_u}{\lambda^2}\dot{v} - \frac{1}{2}\frac{(\lambda^2)_v}{\lambda^2}\dot{u},$$

where the last equality follows from the fact that

$$\lambda^2(\dot{u}^2 + \dot{v}^2) = 1\,. \tag{8}$$

Along $\Gamma(k)$, $|\phi|^2 \equiv$ constant $H^2 - k$, which yields

$$(|\phi|^2)_u \dot{u} + (|\phi|^2)_v \dot{v} = 0\,. \tag{9}$$

This identity enables us to express the right hand side of (7) in terms of $|\phi|^2$ and its differentiation with respect to z and $\bar{z}$.

2.1. To motivate our calculation, we assume, first of all, that

$$(|\phi|^2)_u \neq 0 \quad \text{and} \quad \dot{v} \neq 0\,. \tag{10}$$

Then, by (9), along $\Gamma(k)$,

$$\frac{\dot{u}}{\dot{v}} = -\frac{(|\phi|^2)_v}{(|\phi|^2)_u} = \frac{i((|\phi|^2)_{\bar{z}} - (|\phi|^2)_z)}{(|\phi|^2)_z + (|\phi|^2)_{\bar{z}}};$$

i.e.

$$\dot{u} = \left[\frac{i((|\phi|^2)_{\bar{z}} - (|\phi|^2)_z}{(|\phi|^2)_z + (|\phi|^2)_{\bar{z}}}\right]\dot{v}.$$

Thus

(11) $\quad \dot{u}\ddot{v} - \dot{v}\ddot{u}$

$$= -\left[i\frac{\partial}{\partial s}\left(\frac{(|\phi|^2)_{\bar{z}} - (|\phi|^2)_z}{(|\phi|^2)_z + (|\phi|^2)_{\bar{z}}}\right)\right](\dot{v})^2$$

$$= -\left[i\frac{\partial}{\partial s}((|\phi|^2)_{\bar{z}} - (|\phi|^2)_z)\right]\frac{\dot{v}^2}{(|\phi|^2)_z + (|\phi|^2)_{\bar{z}}}$$

$$+ i\frac{(|\phi|^2)_{\bar{z}} - (|\phi|^2)_z}{(|\phi|^2)_{\bar{z}} + (|\phi|^2)_z}\left[\frac{\partial}{\partial s}((|\phi|^2)_z + (|\phi|^2)_{\bar{z}})\right]\frac{\dot{v}^2}{(|\phi|^2)_z + (|\phi|^2)_{\bar{z}}}$$

$$= -\left[i\frac{\partial}{\partial s}((|\phi|^2)_{\bar{z}} - (|\phi|^2)_z)\right]\frac{\dot{v}^2}{(|\phi|^2)_z + (|\phi|^2)_{\bar{z}}}$$

$$+ \frac{\dot{u}}{\dot{v}}\left[\frac{\partial}{\partial s}((|\phi|^2)_z + (|\phi|^2)_{\bar{z}})\right]\frac{\dot{v}^2}{(|\phi|^2)_z + (|\phi|^2)_{\bar{z}}},$$

by (10), where, as

$$\frac{\partial}{\partial s} = \left[\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\right)\dot{u} + i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)\dot{v}\right],$$

we have

(12) $\quad i\dfrac{\partial}{\partial s}((|\phi|^2)_{\bar{z}} - (|\phi|^2)_z)$

$$= i\left[((|\phi|^2)_{\bar{z}\bar{z}} - (|\phi|^2_{zz})\,\dot{u} + i(2(|\phi|^2)_{z\bar{z}} - (|\phi|^2)_{zz} - (|\phi|^2)_{\bar{z}\bar{z}})\,\dot{v}\right],$$

and

$$(13)\qquad \frac{\partial}{\partial s}((|\phi|^2)_z+(|\phi|^2)_{\bar z})$$

$$= ((|\phi|^2)_{zz} + (|\phi|^2)_{\bar z\bar z} + 2(|\phi|^2)_{z\bar z})\,\dot u + i((|\phi|^2)_{zz} - (|\phi|^2)_{\bar z\bar z})\,\dot v\,.$$

Inserting (12) and (13) into (11), we obtain

$$(14)\qquad \dot u\ddot v - \dot v\ddot u$$

$$= \left[2(|\phi|^2)_{z\bar z}\left(\dot v + \frac{\dot u^2}{\dot v}\right) + ((|\phi|^2)_{zz} + (|\phi|^2)_{\bar z\bar z})\left(-\dot v + \frac{\dot u^2}{\dot v}\right) + 2i((|\phi|^2)_{zz} - (|\phi|^2)_{\bar z\bar z})\dot u\right]$$

$$\frac{(\dot v)^2}{(|\phi|^2)_z + (|\phi|^2)_{\bar z}}$$

$$= [2(|\phi|^2)_{z\bar z}(\dot u^2 + \dot v^2) + ((|\phi|^2)_{zz} + (|\phi|^2)_{\bar z\bar z})(\dot u^2 - \dot v^2) + 2i((|\phi|^2)_{zz} - (|\phi|^2)_{\bar z\bar z})\,\dot u\dot v]$$

$$\frac{(\dot v)^2}{(|\phi|^2)_z + (|\phi|^2)_{\bar z}}\,.$$

2.1.1. To proceed further, we note that, along $\Gamma_0(k)$, as $|\phi|^2 \equiv$ constant,

$$(15)\qquad (|\phi|^2)_z\frac{\dot z}{|\dot z|} + (|\phi|^2)_{\bar z}\frac{\dot{\bar z}}{|\dot z|} = 0\,.$$

We emphasize here that the calculations performed in this subsection is valid no matter whether (10) *holds.* We aim at deriving (27) below, which, together with the discussion made in 2.1.2 and the beginning of 2.1.3 will yield Proposition and hence Theorem 6 in the end of 2.1.3.

Suppose first that, along $\Gamma(k)$, there holds,

$$\left(\dot{v}\frac{\partial}{\partial u}-\dot{u}\frac{\partial}{\partial v}\right)(|\phi|^2)<0\,; \tag{16}$$

that is, $\Gamma(k)$ is described in such a way that a component $\widetilde{E}_0(k)$ of $E_0(k)$ (defined in (1.1)) situates at the left of $\Gamma(k)$. Then, observing that

$$\dot{v}\frac{\partial}{\partial u}-\dot{u}\frac{\partial}{\partial v}=i\left(-z\frac{\partial}{\partial z}+\bar{z}\frac{\partial}{\partial \bar{z}}\right),$$

we have, by (15) and (16),

$$\frac{\dot{z}}{|\dot{z}|}=\frac{-i(|\phi|^2)_{\bar{z}}}{((|\phi|^2)_z(|\phi|^2)_{\bar{z}})^{1/2}}=-i\left(\frac{(|\phi|^2)_{\bar{z}}}{(|\phi|^2)_z}\right)^{1/2}, \tag{17}$$

and

$$\frac{\dot{\bar{z}}}{|\dot{z}|}=\frac{i(|\phi|^2)_z}{((|\phi|^2)_z(|\phi|^2)_{\bar{z}})^{1/2}}=i\left(\frac{(|\phi|^2)_z}{(|\phi|^2)_{\bar{z}}}\right)^{1/2}, \tag{18}$$

which gives

$$\frac{\partial}{\partial z}\left(\frac{\dot{\bar{z}}}{|\dot{z}|}\right)=-\frac{i}{2}\frac{((|\phi|^2)_z)^{1/2}}{((|\phi|^2)_{\bar{z}})^{3/2}}(|\phi|^2)_{z\bar{z}}+\frac{i}{2}\frac{1}{((|\phi|^2)_z(|\phi|^2)_{\bar{z}})^{1/2}}(|\phi|^2)_{zz}. \tag{19}$$

We may observe that, as (15) yields

$$\frac{(|\phi|^2)_z}{(|\phi|^2)_{\bar{z}}}=-\frac{\dot{\bar{z}}}{\dot{z}}\quad\text{and}\quad\frac{(|\phi|^2)_{\bar{z}}}{(|\phi|^2)_z}=-\frac{\dot{z}}{\dot{\bar{z}}},$$

we have

$$\frac{((|\phi|^2)_{\bar{z}})^{1/2}}{((|\phi|^2)_z)^{3/2}}=\frac{1}{((|\phi|^2)_z(|\phi|^2)_{\bar{z}})^{1/2}}\frac{(|\phi|^2)_{\bar{z}}}{(|\phi|^2)_z}=\frac{1}{((|\phi|^2)_z(|\phi|^2)_{\bar{z}})^{1/2}}\left(-\frac{\dot{z}}{\dot{\bar{z}}}\right) \tag{20.1}$$

$$=\frac{-1}{((|\phi|^2)_z(|\phi|^2)_{\bar{z}})^{1/2}}\left(\frac{\dot{z}^2}{|\dot{z}|^2}\right),$$

and

$$(20.2)\qquad \frac{((|\phi|^2)_z)^{1/2}}{((|\phi|^2)_{\bar z})^{2/3}} = \frac{-1}{((|\phi|^2)_z((|\phi|^2)_{\bar z})^{1/2}}\left(\frac{\dot{\bar z}^2}{|\dot z|^2}\right).$$

By (19), we have

$$\frac{\partial}{\partial z}\left(\frac{\dot{\bar z}}{|\dot z|}\right) - \underline{\frac{\partial}{\partial \bar z}\left(\frac{\dot z}{|\dot z|}\right)}$$

$$= \frac{\partial}{\partial z}\left(\frac{\dot{\bar z}}{|\dot z|}\right) - \frac{\partial}{\partial z}\left(\frac{\dot{\bar z}}{|\dot z|}\right)$$

$$= \frac{i}{2}\left[-\frac{((|\phi|^2)_z)^{1/2}}{((|\phi|^2)_{\bar z})^{2/3}}(|\phi|^2)_{z\bar z} + \frac{1}{((|\phi|^2)_z(|\phi|^2)_{\bar z})^{1/2}}(|\phi|^2)_{zz}\right]$$

$$+\frac{i}{2}\left[-\frac{((|\phi|^2)_{\bar z})^{1/2}}{((|\phi|^2)_z)^{3/2}}(|\phi|^2)_{z\bar z} + \frac{1}{((|\phi|^2)_z(|\phi|^2)_{\bar z})^{1/2}}(|\phi|^2)_{\bar z\bar z}\right].$$

Thus, in consideration of (20.1) and (20.2), we obtain

$$(21)\qquad \frac{\partial}{\partial z}\left(\frac{\dot{\bar z}}{|\dot z|}\right) - \frac{\partial}{\partial \bar z}\left(\frac{\dot z}{|\dot z|}\right)$$

$$= \frac{i}{2}\left[(|\phi|^2)_{zz} + (|\phi|^2)_{\bar z\bar z} + \left(\frac{\dot z^2 + \dot{\bar z}^2}{|\dot z|^2}\right)(|\phi|^2)_{z\bar z}\right]((|\phi|^2)_z(|\phi|^2)_{\bar z})^{-1/2}.$$

Likewise, we have

$$(22)\qquad \frac{\partial}{\partial z}\left(\frac{\dot{\bar z}}{|\dot z|}\right) + \frac{\partial}{\partial \bar z}\left(\frac{\dot z}{|\dot z|}\right)$$

$$= \frac{i}{2}\left[((|\phi|^2)_{zz} - (|\phi|^2)_{\bar z\bar z} + \left(\frac{\dot{\bar z}^2 - \dot z^2}{|\dot z|^2}\right)(|\phi|^2)_{z\bar z}\right]((|\phi|^2)_z(|\phi|^2)_{\bar z})^{-1/2}.$$

We may observe

$$(2.31)\qquad \frac{\partial}{\partial z}\left(\frac{\dot{\bar z}}{|\dot z|}\right)=\frac{|\dot z|}{\dot{\bar z}}\left(\frac{\dot{\bar z}}{|\dot z|}\frac{\partial}{\partial z}\left(\frac{\dot{\bar z}}{|\dot z|}\right)\right)$$

$$=\frac{1}{2}\frac{\dot z}{|\dot z|}\left(\frac{\partial}{\partial z}\left(\frac{\dot{\bar z}^2}{|\dot z|^2}\right)\right)$$

$$=\frac{1}{4|\dot z|}\left(\dot z\frac{\partial}{\partial z}\left(\frac{\dot{\bar z}^2+\dot{\bar z}^2}{|\dot z|^2}\right)-\dot z\frac{\partial}{\partial z}\left(\frac{\dot z^2-\dot{\bar z}^2}{|\dot z|^2}\right)\right),$$

and also

$$(23.2)\qquad \frac{\partial}{\partial \bar z}\left(\frac{\dot z}{|\dot z|}\right)=\overline{\frac{\partial}{\partial z}\left(\frac{\dot{\bar z}}{|\dot z|}\right)}$$

$$=\frac{1}{4|\dot z|}\left(\dot{\bar z}\frac{\partial}{\partial \bar z}\left(\frac{\dot z^2+\dot{\bar z}^2}{|\dot z|^2}\right)+\dot{\bar z}\frac{\partial}{\partial \bar z}\left(\frac{\dot z^2-\dot{\bar z}^2}{|\dot z|^2}\right)\right).$$

Inserting (23.1) and (23.2) into (21) and (22), we obtain

$$(|\phi|^2)_{zz}+(|\phi|^2)_{\bar z\bar z}$$

$$=\left\{\frac{i}{2|\dot z|}\left[\left(-\dot z\frac{\partial}{\partial z}+\dot{\bar z}\frac{\partial}{\partial \bar z}\right)\left(\frac{\dot z^2+\dot{\bar z}^2}{|\dot z|^2}\right)\right]+\frac{i}{2|\dot z|}\left[\left(\dot z\frac{\partial}{\partial z}+\dot{\bar z}\frac{\partial}{\partial \bar z}\right)\left(\frac{\dot z^2-\dot{\bar z}^2}{|\dot z|^2}\right)\right]\right\}$$

$$(|\phi|^2)_z(|\phi|^2)_{\bar z})^{1/2}-\left(\frac{\dot z^2+\dot{\bar z}^2}{|\dot z|^2}\right)(|\phi|^2)_{z\bar z},$$

and also

$$(|\phi|^2)_{zz}-(|\phi|^2)_{z\bar z}$$

$$=\left\{\frac{i}{2|\dot z|}\left[\left(-\dot z\frac{\partial}{\partial z}+\dot{\bar z}\frac{\partial}{\partial \bar z}\right)\left(\frac{\dot{\bar z}^2-\dot z^2}{|\dot z|^2}\right)\right]+\frac{i}{2|\dot z|}\left[\left(\dot z\frac{\partial}{\partial z}+\dot{\bar z}\frac{\partial}{\partial \bar z}\right)\left(\frac{\dot z^2+\dot{\bar z}^2}{|\dot z|^2}\right)\right]\right\}$$

$$((|\phi|^2)_z(|\phi|^2)_{\bar z})^{1/2}+\left(\frac{\dot z^2-\dot{\bar z}^2}{|\dot z|^2}\right)(|\phi|^2)_{z\bar z},$$

And hence, as

$$\left(\frac{\dot{z}^2+\dot{\bar{z}}^2}{|\dot{z}|^2}\right)^2-\left(\frac{\dot{z}^2-\dot{\bar{z}}^2}{|\dot{z}|^2}\right)^2\equiv 4\,, \tag{24}$$

we have

$$(\dot{z}^2+\dot{\bar{z}}^2)((|\phi|^2)_{zz}+(|\phi|^2)_{\bar{z}\bar{z}})+(\dot{z}^2-\dot{\bar{z}}^2)((|\phi|^2)_{zz}-(|\phi|^2)_{\bar{z}\bar{z}})+4|\dot{z}|^2(|\phi|^2)_{z\bar{z}} \tag{25}$$

$$=\Bigg\{\Bigg[(\dot{z}^2+\dot{\bar{z}}^2)\left(\left(-\dot{z}\frac{\partial}{\partial z}+\dot{\bar{z}}\frac{\partial}{\partial\bar{z}}\right)\left(\frac{\dot{z}^2+\dot{\bar{z}}^2}{|\dot{z}|^2}\right)\right)$$

$$+(\dot{z}^2-\dot{\bar{z}}^2)\left(\left(-\dot{z}\frac{\partial}{\partial z}+\dot{\bar{z}}\frac{\partial}{\partial\bar{z}}\right)\left(\frac{\dot{\bar{z}}^2-\dot{z}^2}{|\dot{z}|^2}\right)\right)\Bigg]$$

$$+\Bigg[(\dot{z}^2+\dot{\bar{z}}^2)\left(\left(\dot{z}\frac{\partial}{\partial z}+\dot{\bar{z}}\frac{\partial}{\partial\bar{z}}\right)\left(\frac{\dot{z}^2-\dot{\bar{z}}^2}{|\dot{z}|^2}\right)\right)$$

$$-(\dot{z}^2-\dot{\bar{z}}^2)\left(\left(\dot{z}\frac{\partial}{\partial z}+\dot{\bar{z}}\frac{\partial}{\partial\bar{z}}\right)\left(\frac{\dot{z}^2+\dot{\bar{z}}^2}{|\dot{z}|^2}\right)\right)\Bigg]\Bigg\}\frac{i((|\phi|^2)_z(|\phi|^2)_{\bar{z}})^{1/2}}{2|\dot{z}|}\,.$$

For the first term in the bracket in the right hand side of (25), we have

$$(\dot{z}^2+\dot{\bar{z}}^2)\left(\left(-\dot{z}\frac{\partial}{\partial z}+\dot{\bar{z}}\frac{\partial}{\partial\bar{z}}\right)\left(\frac{\dot{z}^2+\dot{\bar{z}}^2}{|\dot{z}|^2}\right)\right) \tag{26}$$

$$+(\dot{z}^2-\dot{\bar{z}}^2)\left(\left(-\dot{z}\frac{\partial}{\partial z}+\dot{\bar{z}}\frac{\partial}{\partial\bar{z}}\right)\left(\frac{\dot{\bar{z}}^2-\dot{z}^2}{|\dot{z}|^2}\right)\right)$$

$$=|\dot{z}|^2\left[\left(-\dot{z}\frac{\partial}{\partial z}+\dot{\bar{z}}\frac{\partial}{\partial\bar{z}}\right)\left(\left(\frac{\dot{z}^2+\dot{\bar{z}}^2}{|\dot{z}|^2}\right)^2-\left(\frac{\dot{z}^2-\dot{\bar{z}}^2}{|\dot{z}|^2}\right)^2\right)\right]$$

$$=0\,,$$

by (24). Inserting (26) into (25), we obtain

$$(27)\qquad (\dot{z}^2+\dot{\bar{z}}^2)((|\phi|^2)_{zz}+(|\phi|^2)_{\bar{z}\bar{z}})+(\dot{z}^2-\dot{\bar{z}}^2)((|\phi|^2)_{zz}-(|\phi|^2)_{\bar{z}\bar{z}})$$

$$+4|\dot{z}|^2(|\phi|^2)_{z\bar{z}}$$

$$=\frac{i((|\phi|^2)_z(|\phi|^2)_{\bar{z}})^{1/2}}{2|\dot{z}|}\left[(\dot{z}^2+\dot{\bar{z}}^2)\left(\left(\dot{z}\frac{\partial}{\partial z}+\dot{\bar{z}}\frac{\partial}{\partial \bar{z}}\right)\left(\frac{\dot{z}^2-\dot{\bar{z}}^2}{|\dot{z}|^2}\right)\right)\right.$$

$$\left.-(\dot{z}^2-\dot{\bar{z}}^2)\left(\left(\dot{z}\frac{\partial}{\partial z}+\dot{\bar{z}}\frac{\partial}{\partial \bar{z}}\right)\left(\frac{\dot{\bar{z}}^2+\dot{z}^2}{|\dot{z}|^2}\right)\right)\right].$$

2.1.2. Assume again that (10) holds and hence so does (14). Inserting (27) into (14), noting

$$\frac{\partial}{\partial s}=\dot{z}\frac{\partial}{\partial z}+\dot{\bar{z}}\frac{\partial}{\partial \bar{z}},$$

and observing that, by (17) and (18),

$$(28)\qquad \frac{\dot{v}}{(|\phi|^2)_z+(|\phi|^2)_{\bar{z}}}=\frac{i}{2}\frac{\dot{\bar{z}}-\dot{z}}{(|\phi|^2)_z+(|\phi|^2)_{\bar{z}}}=\frac{-1}{2}\frac{|\dot{z}|}{((|\phi|^2)_z(|\phi|^2)_{\bar{z}})^{1/2}},$$

we obtain

$$(29)\qquad \frac{\dot{u}\ddot{v}-\dot{v}\ddot{u}}{\dot{u}^2+\dot{v}^2}$$

$$=\frac{-i}{8}\left[\frac{(\dot{z}^2+\dot{\bar{z}}^2)}{|\dot{z}|^2}\left(\frac{\partial}{\partial s}\left(\frac{\dot{z}^2-\dot{\bar{z}}^2}{|\dot{z}|^2}\right)\right)-\frac{(\dot{z}^2-\dot{\bar{z}}^2)}{|\dot{z}|^2}\left(\frac{\partial}{\partial s}\left(\frac{\dot{z}^2+\dot{\bar{z}}^2}{|\dot{z}|^2}\right)\right)\right].$$

2.1.3. Under the assumption of (10), in case (16) fails to hold along $\Gamma(k)$, then there must hold along $\Gamma(k)$

$$(30)\qquad \left(\dot{v}\frac{\partial}{\partial u}-\dot{u}\frac{\partial}{\partial v}\right)(|\phi|^2)>0,$$

which will reverse the positive and negative signs in the right hand side of (17) and (18). Thus, the signs in (27) and (28) will both be reversed, and hence (29) is still valid in case (30) holds along $\Gamma(k)$.

2.2. Now assume that (10) fails to hold; i.e. at some point of $\Gamma(k)$,

$$(|\phi|^2)_u = 0 \qquad \text{or} \qquad \dot{v} = 0\,.$$

In case

$$(|\phi|^2)_u = 0 \qquad \text{and} \qquad \dot{v} \neq 0\,.$$

at some point of $\Gamma(k)$, then, by (9), there must also hold $(|\phi|^2)_v = 0$; hence $|\nabla K| = 0$ at this point of $\Gamma(k)$. This says that k is a critical value of K and $|\nabla K| = 0$ at *every* point of $\Gamma(k)$.

On the other hand, if $\dot{v} = 0$ and $(|\phi|^2)_u \neq 0$, then, by (9), there also holds $\dot{u} = 0$, contradicting (8). Thus, it remains to consider the case

$$\dot{v} = 0,\ (|\phi|^2)_u = 0 \qquad \text{and} \qquad \dot{u} \neq 0,\ (|\phi|^2)_v \neq 0\,. \tag{31}$$

Differentiating (9) with respect to s, we have

$$(|\phi|^2)_{uu}(\dot{u})^2 + 2(|\phi|^2)_{uv}\dot{u}\dot{v} + (|\phi|^2)_{vv}(\dot{v})^2 + (|\phi|^2)_u\ddot{u} + (|\phi|^2)_v\ddot{v} = 0\,.$$

In consideration of (31), this gives us

$$(|\phi|^2)_{uu}(\dot{u})^2 + (|\phi|^2)_u\ddot{v} = 0\,, \tag{32}$$

in which

$$\begin{aligned}(|\phi|^2)_{uu} &= ((|\phi|^2)_z + (|\phi|^2)_{\bar{z}})_u \\ &= (|\phi|^2)_{zz} + (|\phi|^2)_{\bar{z}\bar{z}} + 2(|\phi|^2)_{z\bar{z}}.\end{aligned} \tag{33}$$

Inserting (31) into (27), we obtain, by (33),

$$(|\phi|^2)_{uu} = \frac{i|(|\phi|^2)_v|}{8|\dot{u}|^3}\left[2\,\dot{u}\left(\dot{u}\,\frac{\partial}{\partial u}\left(\frac{4i\dot{u}\dot{v}}{|\dot{u}|^2}\right)\right)\right], \tag{34}$$

in case (16) holds. (We recall that, as emphasized in the beginning of 2.1.1, (27) holds without the assumption (10).) By (31) and (32), we have, however,

$$\frac{\dot{u}\ddot{v} - \dot{v}\ddot{u}}{\dot{u}^2 + \dot{v}^2} = -\,\frac{(|\phi|^2)_{uu}}{(|\phi|^2)_v}\,\dot{u}\,, \tag{35}$$

and, in case (16) holds, by (17) and (18),

$$\frac{\dot{u}}{(|\phi|^2)_v} = \frac{|\dot{u}|}{|(|\phi|^2)_v|}. \tag{36}$$

Inserting (34) and (36) into (35), we see that (29) is valid in case (16) holds. The same observation with that made in 2.1.3 shows that (29) is valid in case (30) holds instead. Thus, we may formulate

PROPOSITION 1. *If k is not a critical value of K, then setting*

$$f = \frac{1}{2}\,\frac{\dot{z}^2 + \dot{\bar{z}}^2}{|\dot{z}|^2} = \frac{\dot{u}^2 - \dot{\bar{z}}^2}{\dot{u}^2 + \dot{v}^2} \tag{37}$$

and

$$g = \frac{-i}{2}\,\frac{\dot{z}^2 - \dot{\bar{z}}^2}{|\dot{z}|^2} = \frac{2\,\dot{u}\dot{v}}{\dot{u}^2 + \dot{v}^2}, \tag{38}$$

there holds, at every point of $\Gamma(k)$

$$\frac{\dot{u}\ddot{v} - \dot{v}\ddot{u}}{\dot{u}^2 + \dot{v}^2} = \frac{1}{2}\left[f\left(\frac{\partial}{\partial s}g\right) - g\left(\frac{\partial}{\partial s}f\right)\right]. \tag{39}$$

We may note that, by (24), for f and g defined in (37) and (38), respectively, we have

$$f^2 + g^2 = 1 .$$

Also, we may define $\theta(s)$ by

$$\frac{\dot{u}}{(\dot{u}^2 + \dot{v}^2)^{1/2}} = \cos\theta(s) \quad \text{and} \quad \frac{\dot{v}}{(\dot{u}^2 + \dot{v}^2)^{1/2}} = \sin\theta(s). \tag{40}$$

Then $f(s) = \cos 2\theta(s)$ and $g(s) = \sin 2\theta(s)$, which yield

$$\begin{aligned} f\left(\frac{\partial}{\partial s}g\right) - g\left(\frac{\partial}{\partial s}f\right) &= 2(\cos^2 2\theta + \sin^2 2\theta)\,\theta'(s) \\ &= 2\theta'(s). \end{aligned} \tag{41}$$

In this connection, we may denote $\phi = x + iy$, x and y being realvalued function of s and observe that the unit outward normal to the curve $|\phi|^2 \equiv$ constant in the (x, y)-plane is $\dfrac{\nabla K}{|\nabla K|}$, which, if a component $\tilde{\Gamma}(k)$ of $\Gamma(k)$ is a simple closed curve, coincides with $\left(\dfrac{\dot{v}}{\sqrt{\dot{u}^2+\dot{v}^2}}, \dfrac{-\dot{u}}{\sqrt{\dot{u}^2+\dot{v}^2}}\right)$ if a component of $E_1(k)$ (defined in (1.2)) situates at the left of $\tilde{\Gamma}(k)$, while is $\left(\dfrac{-\dot{v}}{\sqrt{\dot{u}^2+\dot{v}^2}}, \dfrac{\dot{u}}{\sqrt{\dot{u}^2+\dot{v}^2}}\right)$ if a component of $E_0(k)$ (defined in (1.1)) situates at the left of $\tilde{\Gamma}(k)$. Hence, by (3 9), (40) and (41), we obtain the following

THEOREM 6. *Suppose k is not a critical value and a component $\tilde{\Gamma}(k)$ of $\Gamma(k)$ is a simple closed curve. Then, if $\tilde{\Gamma}(k)$ is so described that a component of $E_0(k)$ situates at the left of $\tilde{\Gamma}(k)$, we have*

$$\int_{\tilde{\Gamma}(k)} \frac{\dot{u}\ddot{v} - \dot{v}\ddot{u}}{\dot{u}^2 + \dot{v}^2} ds = -\frac{1}{2} \int_{\tilde{\Gamma}(k)} \frac{\partial}{\partial s} \arg \phi ,$$

while, if $\tilde{\Gamma}(k)$ is so described that a component of $E_1(k)$ situates at the left of $\tilde{\Gamma}(k)$, we have

$$\int_{\tilde{\Gamma}(k)} \frac{\dot{u}\ddot{v} - \dot{v}\ddot{u}}{\dot{u}^2 + \dot{v}^2} = \frac{1}{2} \int_{\tilde{\Gamma}(k)} \frac{\partial}{\partial s} \arg \phi ,$$

where $\arg \phi$ *denotes the argument of* ϕ.

2.3 *We now proceed to calculate the sum of the last two terms in* (7) *in terms of* $|\phi|^2$, namely

$$\begin{aligned} &\left(\frac{\dot{v}}{\lambda^2}\frac{\partial}{\partial u} - \frac{\dot{u}}{\lambda^2}\frac{\partial}{\partial v}\right)\lambda^2 \\ (42) \qquad &= \left(\dot{v}\frac{\partial}{\partial u} - \dot{u}\frac{\partial}{\partial v}\right)\log \lambda^2 \\ &= i\left(-\dot{z}\frac{\partial}{\partial z} + \dot{\bar{z}}\frac{\partial}{\partial \bar{z}}\right)\log \lambda^2 . \end{aligned}$$

Assuming that $\phi \neq 0$ on $\Gamma(k)$, i.e. no umbilical point is on $\Gamma(k)$, we have, by (6) in Lemma 2 and the fact that H is constant,

$$\frac{(\lambda^2\phi)_{\bar{z}}}{\lambda^2\phi} = \frac{H_z}{\phi} \equiv 0\,,$$

and hence, there holds

$$\frac{(\lambda^2)_{\bar{z}}}{\lambda} = -\frac{\phi_{\bar{z}}}{\phi}\,,$$

and

$$\frac{(\lambda^2)_z}{\lambda^2} = -\frac{\bar{\phi}_z}{\bar{\phi}}\,.$$

Hence

$$-\dot{z}\frac{(\lambda^2)_z}{\lambda^2} + \dot{\bar{z}}\frac{(\lambda^2)_{\bar{z}}}{\lambda^2} = \dot{z}\frac{\bar{\phi}_z}{\bar{\phi}} - \dot{\bar{z}}\frac{\phi_{\bar{z}}}{\phi}\,, \tag{43}$$

in which

$$-\dot{z}\frac{\bar{\phi}_z}{\bar{\phi}} + \dot{\bar{z}}\frac{\phi_{\bar{z}}}{\phi} = \left(-\dot{z}\frac{(|\phi|^2)_z}{|\phi|^2} + \dot{\bar{z}}\frac{(|\phi|^2)_{\bar{z}}}{|\phi|^2}\right) - \left(-\dot{z}\frac{\phi_z}{\phi} + \dot{\bar{z}}\frac{(\bar{\phi})_{\bar{z}}}{\bar{\phi}}\right). \tag{44}$$

But

$$\begin{aligned} &-\dot{z}\frac{\bar{\phi}_z}{\bar{\phi}} + \dot{\bar{z}}\frac{\phi_{\bar{z}}}{\phi} \\ &= -\frac{1}{2}\left(\left(\dot{z}\frac{\partial}{\partial z} - \dot{\bar{z}}\frac{\partial}{\partial \bar{z}}\right)\log\bar{\phi}\right) - \frac{1}{2}\left(\left(\dot{z}\frac{\partial}{\partial z} + \dot{\bar{z}}\frac{\partial}{\partial \bar{z}}\right)\log\bar{\phi}\right) \\ &\quad -\frac{1}{2}\left(\left(\dot{z}\frac{\partial}{\partial z} - \dot{\bar{z}}\frac{\partial}{\partial \bar{z}}\right)\log\phi\right) + \frac{1}{2}\left(\left(\dot{z}\frac{\partial}{\partial z} + \dot{\bar{z}}\frac{\partial}{\partial \bar{z}}\right)\log\phi\right), \end{aligned}$$

and

$$\begin{aligned} &-\dot{z}\frac{\phi_z}{\phi} + \dot{\bar{z}}\frac{(\bar{\phi})_{\bar{z}}}{\bar{\phi}} \\ &= -\frac{1}{2}\left(\left(\dot{z}\frac{\partial}{\partial z} - \dot{\bar{z}}\frac{\partial}{\partial \bar{z}}\right)\log\phi\right) - \frac{1}{2}\left(\left(\dot{z}\frac{\partial}{\partial z} + \dot{\bar{z}}\frac{\partial}{\partial \bar{z}}\right)\log\phi\right) \\ &\quad -\frac{1}{2}\left(\left(\dot{z}\frac{\partial}{\partial z} - \dot{\bar{z}}\frac{\partial}{\partial \bar{z}}\right)\log\bar{\phi}\right) + \frac{1}{2}\left(\left(\dot{z}\frac{\partial}{\partial z} + \dot{\bar{z}}\frac{\partial}{\partial \bar{z}}\right)\log\bar{\phi}\right). \end{aligned}$$

Hence

$$
\begin{aligned}
&-\dot{z}\,\frac{\overline{\phi}_z}{\overline{\phi}}+\dot{\bar{z}}\,\frac{\phi_{\bar{z}}}{\phi}\\
&=-\dot{z}\,\frac{\phi_z}{\phi}+\dot{\bar{z}}\,\frac{(\overline{\phi})_{\bar{z}}}{\overline{\phi}}-\left(\left(\dot{z}\,\frac{\partial}{\partial z}+\dot{\bar{z}}\,\frac{\partial}{\partial \bar{z}}\right)\log\frac{\overline{\phi}}{\phi}\right)\\
&=-\dot{z}\,\frac{\phi_z}{\phi}+\dot{\bar{z}}\,\frac{(\overline{\phi})_{\bar{z}}}{\overline{\phi}}-\frac{\partial}{\partial s}\log\frac{\overline{\phi}}{\phi}\,.
\end{aligned}
$$

Thus, (44) yields

$$
-\dot{z}\,\frac{\overline{\phi}_z}{\overline{\phi}}+\dot{\bar{z}}\,\frac{\phi_{\bar{z}}}{\phi}=-\frac{1}{2}\left(-\dot{z}\,\frac{(|\phi|^2)_z}{|\phi|^2}+\dot{\bar{z}}\,\frac{(|\phi|^2)_{\bar{z}}}{|\phi|^2}\right)-\frac{1}{2}\left(\frac{\partial}{\partial s}\log\frac{\overline{\phi}}{\phi}\right),
$$

and then (43) yields

$$
-\dot{z}\,\frac{(\lambda^2)_z}{\lambda^2}+\dot{\bar{z}}\,\frac{(\lambda^2)_{\bar{z}}}{\lambda^2}=-\frac{1}{2}\left(-\dot{z}\,\frac{(|\phi|^2)_z}{|\phi|^2}+\dot{\bar{z}}\,\frac{(|\phi|^2)_{\bar{z}}}{|\phi|^2}\right)+\frac{1}{2}\left(\frac{\partial}{\partial s}\log\frac{\overline{\phi}}{\phi}\right).
$$

Inserting this into (42), we obtain, along $\Gamma(k)$,

$$
\begin{aligned}
&\left(\frac{\dot{v}}{\lambda^2}\frac{\partial}{\partial u}-\frac{\dot{u}}{\lambda^2}\frac{\partial}{\partial v}\right)\lambda^2\\
&=-\frac{1}{2}\left(\left(\dot{v}\,\frac{\partial}{\partial u}-\dot{u}\,\frac{\partial}{\partial v}\right)\log|\phi|^2\right)+\frac{i}{2}\left(\frac{\partial}{\partial s}\log\frac{\overline{\phi}}{\phi}\right)\\
&=-\frac{1}{2}\left(\left(\dot{v}\,\frac{\partial}{\partial u}-\dot{u}\,\frac{\partial}{\partial v}\right)\log|\phi|^2\right)+\frac{\partial}{\partial s}\arg\phi\,.
\end{aligned}
$$

In virtue of this identity, (14) and Theorem 6, we thus complete the proof of Main Theorem 1.

3. Proof of Theorem 5.

Assume that Theorem 5 is false. That is, assuming k and M satisfy the same conditions as in Main Theorem 1, a component $\tilde{\Gamma}_1(k)$ of $\Gamma(k)$ encloses a simply-connected component $\tilde{E}_1(k)$ of $E_1(k)$, adjacent to which situates a component $\tilde{E}_0(k)$ of $E_0(k)$ enclosed by $\tilde{\Gamma}_1(k)$ and *another component* $\tilde{\Gamma}_2(k)$ of $\Gamma(k)$, $\tilde{\Gamma}_2(k)$ being a simple closed curve. (cf. Figure 5).

Then, by Theorem 2, Main Theorem 1, and the location of $\tilde{E}_1(k)$, we have

(45)
$$\int_{\tilde{\Gamma}_1(k)} K_g\, ds = -\frac{1}{4} \int_{\tilde{\Gamma}_1(k)} (\nabla \log|\phi|^2)\cdot n_1\, ds + 2\pi\,,$$

where n_1 denotes the unit outward normal of $\tilde{\Gamma}_1(k)$ with respect to $\tilde{E}_1(k)$. Likewise, by Main Theorem 1 and the location of $\tilde{E}_0(k)$, we have

(46)
$$\int_{\tilde{\Gamma}_2(k)} K_g\, ds = -\frac{1}{4} \int_{\tilde{\Gamma}_2(k)} (\nabla \log|\phi|^2)\cdot n_2\, ds$$

where n_2 denotes the unit outward normal of $\tilde{\Gamma}_2(k)$ with respect to $\tilde{E}_0(k)$. However, by Green's Theorem and the partial differential equation (5),

$$\int_{\tilde{E}_0(k)} K dA = -\frac{1}{4} \int_{\tilde{\Gamma}_1(k)} (\nabla \log|\phi|^2)\cdot n_1\, ds + \frac{1}{4} \int_{\tilde{\Gamma}_2(k)} (\nabla \log|\phi|^2)\cdot n_2\, ds\,.$$

(Delete small neighborhoods of umbilical points inside $\tilde{E}_0(k)$, if exists, first and then perform a limiting procedure to obtain this.) Thus, by (4.5) and (4.6)

$$\int_{\tilde{E}_0(k)} K dA = -\int_{\partial \tilde{E}_0(k)} K_g\, ds - 2\pi\,,$$

contradicting (3.1) and the fact that $\tilde{E}_0(k)$ is diffeomorphic to a planar annulus.

REFERENCES

[1] H. S. BRASCAMP - E. LIEB, *On extention of the Brunn-Minkowski and Prekoja-Leindler theorems*, J. Funct. Anal., **11** (1976), pp. 366-389.

[2] L. A. CAFFARELLI - A. FRIEDMAN, *Convexity of solutions of semilinear elliptic equations*, Duke. Math. J., **52** (1985), pp. 31-456.

[3] J. T. CHEN - W. H. HUANG, *Convexity of capillary surfaces in outer space*, Invent. Math., **67** (1982), pp. 253-259.

[4] R. FINN, *Existence criteria for capillary free surfaces without gravity*, Indiana Univ. Math. J., **32** (1982), pp. 439-460.

[5] R. FINN, *Comparison Principles in Capillarity*, Calculus of Variations, Lecture Notes in Mathematics, vol. 1357, Springer-Verlag, 1988.

[6] H. Hopf, *Lectures on Differential Geometry in the Large*, Lecture Notes in Mathematics, vol. 1000, Springer-Verlag, 1983.
[7] W. H. Huang, *Superhamonicity of curvatures for surfaces of constant mean curvature*, Pacific J. of Math., **152**, no. 2 (1992), pp. 291-318.
[8] W. H. Huang - Chun-Chi Lin, *Negatively Curved Sets in Surfaces of Constant Mean Curvature in* $\mathbb{R}^3$, Arch. Rat. Mech. Anal., **141**, no. 2 (1998), pp. 105-116.
[9] N. J. Korevarr, *Convex solutions to nonlinear elliptic and parabolic boundary value problems*, Indiana Univ. Math. J., **32** (1983), pp. 603-614.
[10] N. J. Korevarr - J. L. Lewis, *Convex solutions to nonlinear elliptic equations having constant rank Hessians*, Arch. Rat. Mech. Anal., **32** (1987), pp. 19-32.
[11] H. Wente, *Counterexample to a conjecture of H. Hopf*, Pacific J. Math., **121** (1986), pp. 193-243.

Manoscritto pervenuto in redazione il 15 aprile 1998.