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Regularity of Lipschitz Minima.

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The existence of Lipschitz minima was the fundamental step in Hilbert's proof of Dirichlet Principle (see [6] and [7]). Hilbert's Lemma in today's language would be written: «For any open, bounded set $\Omega \subset \mathbb{R}^n$ and any Lipschitz function $\gamma : \partial\Omega \to \mathbb{R}$ satisfying the B.S.C (Bounded Slope Condition), there exists a unique Lipschitz function $w \in \mathrm{Lip}(\mathbb{R}^n)$ satisfying (1) and $w_{|\partial\Omega} = \gamma$» (see [5]).

Its validity for a general F was remarked by A. Haar [8] in the 2-dimensional case. The results of Hilbert and Haar opened an interesting problem about the regularity of the minimizing functions according with the regularity of the function F. Hilbert conjectured (see the XIX Problem in [1]) that the Lipschitz minima are analytic if F is so.

E. Hopf [9] and C. B. Morrey [10] proved that the Hilbert conjecture was true for the C^1-minima.

The gap, from Lipschitz to C^1, was filled by the famous regularity result established by E. De Giorgi in 1957 [2].

What we do here, is to apply De Giorgi's method directly to the proof of Hilbert conjecture.

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NOTATION. Throughout this paper the symbol Ω means an open, bounded set in $\mathbb{R}^n$ with $n \geq 2$. By $\mathrm{Lip}(\Omega, \mathbb{R}^k)$ we denote the set of Lip-

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schitz functions $f: \Omega \to \mathbb{R}^k$ and by $\mathrm{Lip}_c(\Omega, \mathbb{R}^k)$ the set of functions in $\mathrm{Lip}(\Omega, \mathbb{R}^k)$ with compact support in Ω.

If $k = 1$ we write $\mathrm{Lip}(\Omega)$, $\mathrm{Lip}_c(\Omega)$ for $\mathrm{Lip}(\Omega, \mathbb{R})$ and $\mathrm{Lip}_c(\Omega, \mathbb{R})$.

In this paper we shall prove the following:

THEOREM 1. *If $F: \mathbb{R}^n \to \mathbb{R}$ is a strictly convex function of class C^2 and $w \in \mathrm{Lip}(\mathbb{R}^n)$ satisfies*

$$\int_\Omega F(Dw + D\varphi)\, dx \geq \int_\Omega F(Dw)\, dx\,, \tag{1}$$

for all $\varphi \in \mathrm{Lip}_c(\Omega)$, then w has Hölder-continuous first derivatives in Ω.

PROOF. We shall divide the proof in six steps.

STEP 1. We prove that the differential quotients of w satisfy the integro-differential equation (7), whose coefficients are defined by (7) and have the property (10).

The convexity assumption on F implies that (1) is equivalent to the following identity:

$$\int_\Omega DF(Dw)\, D\varphi\, dx = 0\,, \qquad \forall \varphi \in \mathrm{Lip}_c(\Omega)\,. \tag{2}$$

For any $e \in S^{n-1} = \{e \in \mathbb{R}^n \colon |e| = 1\}$ and $|t| < \mathrm{dist}(\mathrm{spt}\,\varphi, \partial\Omega)$, we get

$$\int_\Omega DF(Dw)\, D\varphi(x - te)\, dx = 0$$

and

$$\int_\Omega DF(Dw(x + te))\, D\varphi\, dx = 0\,. \tag{3}$$

If we subtract (2) from (3), we obtain

$$\int_\Omega D[F(Dw(x + te)) - F(Dw)]\, D\varphi\, dx = 0\,. \tag{4}$$

Since

$$DF(Dw(x + te)) - DF(Dw) = DF(Dw + s[Dw(x + te) - Dw])\,\Big|_{s=0}^{s=1},$$

we have

$$D_i[F(Dw(x+te)) - F(Dw)] = \tag{5}$$

$$= \int_0^1 \sum_{j=1}^n D_i D_j F(Dw + s[Dw(x+te) - Dw])\, D_j[w(x+te) - w]\, ds$$

$$= \sum_{j=1}^n a_{ij}(x)\, D_j[w(x+te) - w], \qquad \forall i = 1, \ldots, n$$

where

$$a_{ij}(x) = \int_0^1 D_i D_j F(Dw + s[Dw(x+te) - Dw])\, ds \tag{6}$$

are bounded, Lebesgue measurable functions for all $i, j = 1, \ldots n$.

Thanks to (5) the integral equation (4) becomes

$$\int_\Omega \sum_{i,j} a_{ij}(x)\, D_j[w(x+te) - w]\, D_i \varphi\, dx = 0 .$$

Dividing by $t \neq 0$, we obtain the following integral equation

$$\int_\Omega \sum_{i,j} a_{ij}(x)\, D_j u D_i \varphi\, dx = 0, \tag{7}$$

where

$$u(x) = u(x, t, e) = \frac{w(x+te) - w(x)}{t}$$

is a differential quotient of w.

Putting

$$p = Dw + s[Dw(x+te) - Dw],$$

with $s \in [0, 1]$ and denoting by K the Lipschitz constant of w, we have

$$|p| = |Dw + s[Dw(x+te) - Dw]| \leqslant (1-s)K + sK = K, \tag{8}$$

for all $s \in [0, 1]$ and for all x, t, e.

From (6) and (8), since F is of class C^2 and strictly convex, we obtain

(9) $$\mu_1 |\lambda|^2 \leq \sum_{i,j} a_{ij}(x)\, \lambda_i \lambda_j \leq \mu_2 |\lambda|^2, \qquad \forall \lambda \in \mathbb{R}^n,\ \forall x \in \mathbb{R}^n,$$

where

$$\mu_1 = \min\left\{\sum_{i,j} D_i D_j F(p)\, \lambda_i \lambda_j :\ |p| < K,\ |\lambda| = 1\right\} > 0,$$

$$\mu_2 = \max\left\{\sum_{i,j} D_i D_j F(p)\, \lambda_i \lambda_j :\ |p| < K,\ |\lambda| = 1\right\}.$$

REMARK 2. For any $\varepsilon > 0$, put $\Omega_{-\varepsilon} = \{x \in \Omega : \operatorname{dist}(x, (\mathbb{R}^n - \Omega)) > > \varepsilon\}$, we have that

$$u(x, t, e) = \frac{w(x + te) - w(x)}{t}, \qquad \forall x \in \Omega_{-\varepsilon},\ e \in S^{n-1},\ |t| < \varepsilon$$

is a family of bounded functions (where the bound is the Lipschitz constant K of w) that satisfies the integro-differential equation

(10) $$\int_{\Omega_{-\varepsilon}} \sum_{i,j} a_{ij}(x)\, D_j u D_i \varphi \, dx = 0, \qquad \forall \varphi \in \mathrm{Lip}_c(\Omega_{-\varepsilon}),$$

where $\{a_{ij}(x)\}$ are defined by (6) and satisfy (9).

For notational convenience the same letter Ω will be used to denote the set $\Omega_{-\varepsilon}$.

STEP 2. As consequence of (10) and (9) we obtain the so-called Caccioppoli inequalities (see [2]), i.e. for all $y \in \Omega$, $0 < \varrho_1 < \varrho_2 < \operatorname{dist}(y, \partial\Omega)$, $\gamma = \frac{\mu_2}{\mu_1}$ and $k \in \mathbb{R}$, we have

(11a) $$\frac{\gamma}{(\varrho_2 - \varrho_1)^2} \int_{A(k, \varrho_2)} (u(x) - k)^2 \, dx \geq \int_{A(k, \varrho_1)} |Du|^2 \, dx,$$

(11b) $$\frac{\gamma}{(\varrho_2 - \varrho_1)^2} \int_{B(\varrho_2) - A(k)} (u(x) - k)^2 \, dx \geq \int_{B(\varrho_1) - A(k)} |Du|^2 \, dx,$$

where

$$B(\varrho)=B_y(\varrho)=\{x\in\mathbb{R}^n:\ |x-y|<\varrho\},$$

$$A(k)=\{x\in\mathbb{R}^n:\ u(x)>k\},$$

$$A(k,\varrho)=A(k)\cap B(\varrho).$$

We restrict ourselves to the proof of (11a), since (11b) can be derived from the application of (11a) to the function $-u$.

Let be

$$\overline{\varphi}=\eta^2[(u-k)\vee 0],$$

where $\eta\in \mathrm{Lip}_c(\Omega)$. The function $\overline{\varphi}$ is in $\mathrm{Lip}_c(\Omega)$, moreover it vanishes in $\mathbb{R}^n-A(k)$. Hence substituting in the integral equation (7) the function $\overline{\varphi}$ in the place of φ, we obtain

$$\int_{A(k)}\sum_{i,j}a_{ij}(x)\,D_iuD_j\overline{\varphi}\,\mathrm{d}x=0\,,$$

that is,

$$\int_{A(k)}\sum_{i,j}a_{ij}(x)\,D_i(u-k)\,D_j[\eta^2(u-k)]\,\mathrm{d}x=0\,.$$

By a simple computation, we get

$$\text{(12)}\quad \int_{A(k)}\sum_{i,j}a_{ij}(x)\{D_i[\eta(u-k)]\,D_j[\eta(u-k)]-(u-k)^2D_i\eta D_j\eta-$$
$$(u-k)\,D_i\eta D_j[\eta(u-k)]+(u-k)\,D_i[\eta(u-k)]\,D_j\eta\}\,\mathrm{d}x=0\,.$$

Since the matrix $A=\{a_{ij}(x)\}$ is symmetric, we have

$$\int_{A(k)}\sum_{i,j}a_{ij}(x)\{D_i[\eta(u-k)]\,D_j[\eta(u-k)]-(u-k)^2D_i\eta D_j\eta\}\,\mathrm{d}x=0\,,$$

then by (9),

$$\text{(13)}\quad \mu_1\int_{A(k)}|D[\eta(u-k)]|^2\mathrm{d}x\leqslant\int_{A(k)}\sum_{i,j}a_{ij}(x)\{D_i[\eta(u-k)]\,D_j[\eta(u-k)]\}\,\mathrm{d}x$$
$$=\int_{A(k)}\sum_{i,j}(u-k)^2a_{ij}(x)\,D_i\eta D_j\eta\,\mathrm{d}x$$
$$\leqslant\mu_2\int_{A(k)}(u-k)^2\,|D\eta|^2\,\mathrm{d}x\,.$$

Making for η the following choice

$$\eta(x) = 1\,, \qquad \forall x \in B(\varrho_1)\,,$$

$$\eta(x) = 0\,, \qquad \forall x \in \Omega - B(\varrho_2)\,,$$

$$\eta(x) = \frac{\varrho_2 - |x - y|}{\varrho_2 - \varrho_1}\,, \qquad \forall x \in B(\varrho_2) - B(\varrho_1)\,,$$

we get

$$\mu_1 \int_{A(k,\,\varrho_1)} |D(u-k)|^2 \mathrm{d}x \leqslant \frac{\mu_2}{(\varrho_2 - \varrho_1)^2} \int_{A(k,\,\varrho_2)} (u-k)^2 \mathrm{d}x\,,$$

that is (11a).

STEP 3. Here we recall the definition of «perimeter of a set» following [3] and prove two inequalities which will be useful later.

Let E be a Lebesgue measurable set and A an open set in $\mathbb{R}^n$, we define the *perimeter of E in A* as

$$P(E, A) = \sup\left\{ \int_E \operatorname{div} g \, \mathrm{d}x : g \in \mathrm{Lip}_c(A, \mathbb{R}^n),\ |g(x)| \leqslant 1 \right\}.$$

If $A = \mathbb{R}^n$, we shall write $P(E)$ for $P(E, \mathbb{R}^n)$.

It is useful now to remember the global and local isoperimetric inequalities (see [3], [4], [5]).

For any Lebesgue measurable $E \subset \mathbb{R}^n$

$$\text{(14}a\text{)} \qquad \min\{|E|, |\mathbb{R}^n - E|\}^{\frac{n-1}{n}} \leqslant (n\omega_n^{1/n})^{-1} P(E)\,,$$

and (see [2])

$$\text{(14}b\text{)} \qquad \min\{|E \cap B(\varrho)|, |(\mathbb{R}^n - E) \cap B(\varrho)|\}^{\frac{n-1}{n}} \leqslant \beta_1(n)\, P(E, B(\varrho))\,,$$

with

$$\beta_1(n) = [(1 - 2^{-1/n})\, n\omega_n^{1/n}]^{-1},$$

where ω_n is the Lebesgue measure of the unit ball of $\mathbb{R}^n$.

The following Lemma establishes a connection between the gradient and the perimeter of level sets of Lipschitz functions.

LEMMA 3. *For all $f \in \mathrm{Lip}(\mathbb{R}^n)$ and for all $\varrho, t \in \mathbb{R}$ with $\varrho > 0$ we have*

$$\liminf_{\varepsilon \to 0^+} \varepsilon^{-1} \int_{A(t,\varrho) - A(t+\varepsilon,\varrho)} |Df(x)|\,\mathrm{d}x \geqslant P(A(t), B(\varrho)).$$

PROOF. Given $\varepsilon > 0$, let us denote with $f_\varepsilon(x)$ the function defined by

$$f_\varepsilon(x) = \varepsilon^{-1}\{[t \vee f(x) \wedge (t+\varepsilon)] - t\}.$$

Let us observe that

(15) $\quad |Df_\varepsilon(x)| = \varepsilon^{-1}|Df(x)|, \quad$ for almost all $x \in [A(t,\varrho) - A(t+\varepsilon,\varrho)]$,

and

(16) $\quad |Df_\varepsilon(x)| = 0, \quad$ for almost all $x \in B(\varrho) \setminus [A(t,\varrho) - A(t+\varepsilon,\varrho)]$.

Hence we have

$$\int_{B(\varrho)} |Df_\varepsilon(x)|\,\mathrm{d}x = \varepsilon^{-1} \int_{A(t,\varrho) - A(t+\varepsilon,\varrho)} |Df(x)|\,\mathrm{d}x. \tag{17}$$

The functions $f_\varepsilon(x) \in \mathrm{Lip}(\mathbb{R}^n)$ for all $\varepsilon > 0$. Moreover it is obvious that

$$0 \leqslant f_\varepsilon(x) \leqslant 1, \qquad \forall x \in \mathbb{R}^n$$

and

$$\lim_{\varepsilon \to 0^+} f_\varepsilon(x) = \chi_{A(t)}(x), \qquad \forall x \in \mathbb{R}^n,$$

where $\chi_{A(t)}$ is the characteristic function of $A(t)$.

Let $g \in \mathrm{Lip}_c(B(\varrho), \mathbb{R}^n)$ be such that $|g(x)| \leqslant 1$. Then

$$\int_{A(t)} \operatorname{div} g\,\mathrm{d}x = \int \chi_{A(t)} \operatorname{div} g\,\mathrm{d}x = \lim_{\varepsilon \to 0^+} \int f_\varepsilon(x) \operatorname{div} g\,\mathrm{d}x.$$

Since

$$\int f_\varepsilon(x) \operatorname{div} g\,\mathrm{d}x = -\int g Df_\varepsilon(x)\,\mathrm{d}x \leqslant \int_{B(\varrho)} |Df_\varepsilon(x)|\,\mathrm{d}x,$$

we get

$$\int_{A(t)} \operatorname{div} g \, \mathrm{d}x \leq \liminf_{\varepsilon \to 0^+} \int_{B(\varrho)} |Df_\varepsilon| \, \mathrm{d}x .$$

On taking the supremum over all such g and recalling (17), we have

$$P(A(t), B(\varrho)) \leq \liminf_{\varepsilon \to 0^+} \int_{B(\varrho)} |Df_\varepsilon(x)| \, \mathrm{d}x =$$

$$= \liminf_{\varepsilon \to 0^+} \varepsilon^{-1} \int_{A(t, \varrho) - A(t+\varepsilon, \varrho)} |Df(x)| \, \mathrm{d}x . \qquad \blacksquare$$

Lemma 3 leads us to two important inequalities (see [2]).

FIRST INEQUALITY. *For all* $f \in \mathrm{Lip}(\mathbb{R}^n)$ *and for all* $k, \lambda, \varrho \in \mathbb{R}$ *with*

$$k < \lambda , \qquad \varrho > 0 ,$$

we have

$$\beta_1 \int_{A(k, \varrho) - A(\lambda, \varrho)} |Df(x)| \, \mathrm{d}x \geq (\lambda - k)[\tau(k, \lambda, \varrho)]^{\frac{n-1}{n}} , \tag{18}$$

where $\tau(k, \lambda, \varrho) = \min \{ |A(\lambda, \varrho)| , |B(\varrho) - A(k, \varrho)| \}$ *and* β_1 *is the constant in the isoperimetric inequality* (14).

PROOF. The isoperimetric inequality (14*b*) implies

$$\beta_1 P(A(t), B(\varrho)) \geq \min \{ |A(t, \varrho)| , |B(\varrho) - A(t, \varrho)| \}^{\frac{n-1}{n}} ,$$

then it follows from Lemma 3 that

$$\beta_1 \liminf_{\varepsilon \to 0^+} \varepsilon^{-1} \int_{A(t, \varrho) - A(t+\varepsilon, \varrho)} |Df(x)| \, \mathrm{d}x \geq \tag{19}$$

$$\geq \min \{ |A(t, \varrho)| , |B(\varrho) - A(t, \varrho)| \}^{\frac{n-1}{n}} .$$

If

$$2 |A(t, \varrho)| \leq |B(\varrho)| , \qquad \forall t \geq k , \tag{20}$$

the inequality (19) becomes

$$(21)\qquad \beta_1 \liminf_{\varepsilon\to 0^+} \varepsilon^{-1} \int_{A(t,\varrho)-A(t+\varepsilon,\varrho)} |Df(x)|\,dx \geq |A(t,\varrho)|^{\frac{n-1}{n}}, \qquad \forall t \geq k .$$

The function $F(t) = \int_{A(t,\varrho)} |Df(x)|\,dx$ is a non-increasing function of t and then

$$\int_{A(k,\varrho)-A(\lambda,\varrho)} |Df(x)|\,dx \geq -\int_k^\lambda \frac{dF(t)}{dt}\,dt =$$

$$= \int_k^\lambda \left[\liminf_{\varepsilon\to 0^+} \varepsilon^{-1} \int_{A(t,\varrho)-A(t+\varepsilon,\varrho)} |Df(x)|\,dx\right] dt ,$$

for all $k, \lambda \in \mathbb{R}$ with $k < \lambda$.

Using this inequality and integrating (21) with respect to t from k to λ $(k<\lambda)$, we get

$$\beta_1 \int_{A(k,\varrho)-A(\lambda,\varrho)} |Df(x)|\,dx \geq \int_k^\lambda |A(t,\varrho)|^{\frac{n-1}{n}}\,dt .$$

Since $|A(t,\varrho)|^{\frac{n-1}{n}}$ is a non-increasing function of t, we have

$$(22)\qquad \beta_1 \int_{A(k,\varrho)-A(\lambda,\varrho)} |Df(x)|\,dx \geq (\lambda-k)|A(\lambda,\varrho)|^{\frac{n-1}{n}} = (\lambda-k)[\tau(k,\lambda,\varrho)]^{\frac{n-1}{n}} .$$

If

$$(23)\qquad 2\,|A(t,\varrho)| \geq |B(\varrho)|, \qquad \forall t \leq \lambda ,$$

considering the non-decreasing function $|B(\varrho)-A(t,\varrho)|^{\frac{n-1}{n}}$ and proceeding as before, we obtain

$$(24)\qquad \beta_1 \int_{A(k,\varrho)-A(\lambda,\varrho)} |Df(x)|\,dx \geq (\lambda-k)\,|B(\varrho)-A(k,\varrho)|^{\frac{n-1}{n}} =$$

$$= (\lambda-k)[\tau(k,\lambda,\varrho)]^{\frac{n-1}{n}} .$$

If neither inequality (20) nor inequality (23) are satisfied then there must exist $\bar{t} \in [k,\lambda]$ such that

$$2\,|A(t,\varrho)| \leq B(\varrho), \qquad \forall t \geq \bar{t}$$

and

$$2\,|A(t,\varrho)| > B(\varrho), \qquad \forall t < \bar{t}.$$

Applying (22) over the interval $[\bar{t}, \lambda]$ and (24) over $[k, \bar{t}]$, we obtain again (18). ∎

SECOND INEQUALITY. *For all $f \in \mathrm{Lip}(\mathbb{R}^n)$ and for all $k, \varrho \in \mathbb{R}$ with*

$$\varrho > 0, \qquad 0 < 2\,|A(k,\varrho)| \leq |B(\varrho)|, \tag{25}$$

we have

$$\int_{A(k,\varrho)} (f(x)-k)^2\,\mathrm{d}x \leq \beta_2\,|A(k,\varrho)|^{2/n} \int_{A(k,\varrho)} |Df(x)|^2\,\mathrm{d}x, \tag{26}$$

where $\beta_2 = 4\beta_1^2$.

PROOF. Lemma 3 and the isoperimetric inequality (14*b*) imply

$$\begin{aligned}\beta_1 \liminf_{\varepsilon\to 0^+} \varepsilon^{-1} \int_{A(t,\varrho)-A(t+\varepsilon,\varrho)} |Df(x)|\,\mathrm{d}x \geq \\ \geq \min\{|B(\varrho)-A(t,\varrho)|,\ |A(t,\varrho)|\}^{\frac{n-1}{n}}.\end{aligned} \tag{27}$$

Since

$$|A(t,\varrho)| \leq |A(k,\varrho)|, \qquad \forall t \geq k,$$

we have by (25) and (27)

$$\beta_1 \liminf_{\varepsilon\to 0^+} \varepsilon^{-1} \int_{A(t,\varrho)-A(t+\varepsilon,\varrho)} |Df(x)|\,\mathrm{d}x \geq |A(t,\varrho)|^{\frac{n-1}{n}}, \qquad \forall t \geq k,$$

so that

$$\beta_1 \liminf_{\varepsilon\to 0^+} \varepsilon^{-1} \int_{A(t,\varrho)-A(t+\varepsilon,\varrho)} |Df(x)|\,\mathrm{d}x \geq |A(k,\varrho)|^{-1/n}\,|A(t,\varrho)|, \qquad \forall t \geq k.$$

If we integrate this last inequality with respect to t over the interval $(k, +\infty)$, we get

$$\beta_1\,|A(k,\varrho)|^{1/n} \int_{A(k,\varrho)} |Df(x)|\,\mathrm{d}x \geq \int_k^{+\infty} |A(t,\varrho)|\,\mathrm{d}t,$$

that is,

$$\beta_1 \, |A(k,\varrho)|^{1/n} \int_{A(k,\varrho)} |D(f(x)-k)| \, \delta x \geqslant \int_{A(k,\varrho)} (f(x)-k)\, dx . \tag{28}$$

Being

$$A(k,\varrho) = \{x \in B(\varrho) : f(x) > k\} = \{x \in B(\varrho) : (f(x)-k) > 0\}$$

$$= \{x \in B(\varrho) : [(f(x)-k) \vee 0]^2 > 0\},$$

we can apply (28) to the function $[(f(x)-k) \vee 0]^2$, obtaining

$$\beta_1 \, |A(k,\varrho)|^{1/n} \int_{A(k,\varrho)} |D(f(x)-k)^2| \, \mathrm{d}x \geqslant \int_{A(k,\varrho)} (f(x)-k)^2 \, \mathrm{d}x .$$

Since

$$[D(f(x)-k)]^2 = 2(f(x)-k)\, Df(x),$$

we have

$$\beta_1 \, |A(k,\varrho)|^{1/n} \int_{A(k,\varrho)} 2(f(x)-k)\, |Df(x)| \, \mathrm{d}x \geqslant \int_{A(k,\varrho)} (f(x)-k)^2 \, \mathrm{d}x .$$

Now, using Schwarz-Hölder inequality, we get

$$\int_{A(k,\varrho)} (f(x)-k)^2 \, \mathrm{d}x \leqslant$$

$$\leqslant 2\beta_1 \, |A(k,\varrho)|^{1/n} \left[\int_{A(k,\varrho)} |Df(x)|^2 \, \mathrm{d}x \right]^{1/2} \left[\int_{A(k,\varrho)} (f(x)-k)^2 \, \mathrm{d}x \right]^{1/2},$$

then

$$\left[\int_{A(k,\varrho)} (f(x)-k)^2 \, \mathrm{d}x \right]^{1/2} \leqslant 2\beta_1 \, |A(k,\varrho)|^{1/n} \left[\int_{A(k,\varrho)} |Df(x)|^2 \, \mathrm{d}x \right]^{1/2}.$$

By taking the square, we can conclude

$$\int_{A(k,\varrho)} (f(x)-k)^2 \, \mathrm{d}x \leqslant \beta_2 \, |A(k,\varrho)|^{2/n} \int_{A(k,\varrho)} |Df(x)|^2 \, \mathrm{d}x ,$$

where $\beta_2 = 4\beta_1^2$. ■

STEP 4. What we are going to prove now is an estimate for the oscillation of the differential quotients u, directly derived from (11*a*) and (11*b*).

LEMMA 4. *For all* $\sigma \in (0,1)$ *we define*

$$\theta = \theta(\sigma) = \min\left\{\frac{(1-\sigma)^n \omega_n}{2}, \left[\frac{\sigma^2 2^{-n-2}}{1+\gamma+\beta_2}\right]^n\right\}, \tag{29}$$

and

$$c^2 = \varrho^{-n}\theta^{-1}\int_{A(k,\varrho)} (u(x)-k)^2\,\mathrm{d}x\,, \tag{30}$$

where β_2 *is the constant in* (26) *and* γ *is the constant in Caccioppoli inequalities. Then if* $k \in \mathbb{R}$ *and*

$$B(\varrho) \subset \Omega\,, \qquad |A(k,\varrho)| \leqslant \varrho^n \theta\,,$$

we have

$$\sup_{B(\varrho-\sigma\varrho)} |u| \leqslant k + \sigma c\,. \tag{31}$$

PROOF. Putting

$$\varrho_h = \varrho - \sigma\varrho + 2^{-h}\sigma\varrho\,, \qquad k_h = k + \sigma c - 2^{-h}\sigma c\,,$$

for all integer $h \geqslant 0$, we have

$$\varrho = \varrho_0 > \varrho_1 > \ldots > \varrho_h > \ldots > (1-\sigma)\,\varrho \tag{32}$$

and

$$k = k_0 < k_1 < \ldots < k_h < \ldots < k + \sigma c\,. \tag{33}$$

The definition of θ implies

$$\theta \leqslant \frac{1}{2}\omega_n(1-\sigma)^n,$$

then

$$|A(k_h, \varrho_{h+1})| \leqslant \varrho^n\theta \leqslant \frac{1}{2}\omega_n\varrho_{h+1}^n, \qquad \forall h \geqslant 0\,,$$

that is the condition for using inequality (26),

$$\int_{A(k_h,\varrho_{h+1})} (u(x)-k_h)^2 dx \leqslant \beta_2 |A(k_h,\varrho_{h+1})|^{2/n} \int_{A(k_h,\varrho_{h+1})} |Du|^2 dx, \quad \forall h \geqslant 0.$$

Writing (11a) for $\varrho_1 = \varrho_{h+1} < \varrho_2 = \varrho_h$, $k = k_h$, we have

$$\gamma 4^{h+1} \int_{A(k_h,\varrho_h)} (u(x)-k_h)^2 dx \geqslant \sigma^2 \varrho^2 \int_{A(k_h,\varrho_{h+1})} |Du|^2 dx, \qquad h \geqslant 0.$$

Combining these two last inequalities, we obtain

$$(34) \quad \int_{A(k_h,\varrho_{h+1})} (u(x)-k_h)^2 dx \leqslant \beta_2 |A(k_h,\varrho_{h+1})|^{2/n} \gamma 4^{h+1} \sigma^{-2} \varrho^{-2} \cdot \int_{A(k_h,\varrho_h)} (u(x)-k_h)^2 dx.$$

Noting that

$$\int_{A(k_h,\varrho_{h+1})} (u(x)-k_h)^2 dx \geqslant \int_{A(k_{h+1},\varrho_{h+1})} (u(x)-k_h)^2 dx \geqslant \geqslant \int_{A(k_{h+1},\varrho_{h+1})} (u(x)-k_{h+1})^2 dx$$

and

$$(u(x)-k_h)^2 \geqslant (k_{h+1}-k_h)^2 = \sigma^2 c^2 4^{-(h+1)}, \qquad \forall x \in A(k_{h+1},\varrho_{h+1}),$$

from (34) we obtain the following two inequalities

$$\sigma^2 c^2 4^{-(h+1)} |A(k_{h+1},\varrho_{h+1})| \leqslant \frac{\beta_2 |A(k_h,\varrho_h)|^{2/n} \gamma 4^{h+1}}{\sigma^2 \varrho^2} \int_{A(k_h,\varrho_h)} (u(x)-k_h)^2 dx,$$

$$\int_{A(k_{h+1},\varrho_{h+1})} (u(x)-k_{h+1})^2 dx \leqslant \frac{\beta_2 |A(k_h,\varrho_h)|^{2/n} \gamma 4^{h+1}}{\sigma^2 \varrho^2} \int_{A(k_h,\varrho_h)} (u(x)-k_h)^2 dx.$$

Thanks to these and the definitions of θ and c, it is easy to prove, by induction on h, that

(35) $$|A(k_h, \varrho_h)| < \varrho^n \theta 2^{-2nh}, \qquad \forall h \geq 0,$$

(36) $$\int_{A(k_h, \varrho_h)} (u(x) - k_h)^2 \, dx \leq \varrho^n \theta c^2 2^{-2nh}, \ \forall h \geq 0.$$

Then (35) implies

(37) $$|A(k + \sigma c, \varrho - \sigma\varrho)| = 0,$$

that is

(38) $$\sup_{B(\varrho - \sigma\varrho)} u \leq k + \sigma c.$$

Considering $-u$ and proceeding as before, we obtain (31). ■

STEP 5. We can study now the behaviour of the oscillation of u on $B(\varrho)$ as $\varrho \to 0$.

LEMMA 5. *There exists a number* $\eta = \eta(n, \gamma) > 0$ *such that, for all* $\varrho \in \mathbb{R}$ *with*

$$0 < 4\varrho < \operatorname{dist}(y, (\mathbb{R}^n - \Omega)),$$

we have

$$\operatorname{osc}(u, B(\varrho)) \leq (1 - \eta) \operatorname{osc}(u, B(4\varrho)),$$

where

$$\operatorname{osc}(u, B(\varrho)) = \sup_{B(\varrho)} u - \inf_{B(\varrho)} u.$$

PROOF. Let us put

$$\omega = \operatorname{osc}(u, B(4\varrho)) = \sup_{B(4\varrho)} u - \inf_{B(4\varrho)} u = \mu_1 - \mu_2,$$

and

$$\bar{\mu} = \frac{\mu_1 + \mu_2}{2}.$$

At least one of the following two inequalities must be true:

$$2\,|A(\bar{\mu}, 2\varrho)| < |B(2\varrho)|\,, \tag{39a}$$

$$2\,|B(2\varrho) - A(\bar{\mu}, 2\varrho)| < |B(2\varrho)|\,. \tag{39b}$$

Assume that (39*a*) is true. If (39*b*) is satisfied we would use the same arguments for the function $-u$.
For all $\lambda \leqslant \frac{\omega}{4}$, we put

$$D(\lambda) = A(\mu_1 - 2\lambda, 2\varrho) - A(\mu_1 - \lambda, 2\varrho)\,.$$

From the Schwarz inequality we have

$$\left(\int_{D(\lambda)} |Du|\,\mathrm{d}x\right)^2 \leqslant |D(\lambda)| \int_{D(\lambda)} |Du|^2\,\mathrm{d}x\,. \tag{40}$$

Noting that

$$|A(\mu_1 - \lambda, 2\varrho)| \leqslant |A(\bar{\mu}, 2\varrho)| \leqslant |B(2\varrho) - A(\mu_1 - 2\lambda, 2\varrho)|\,,$$

and recalling (18), we obtain

$$\left(\int_{D(\lambda)} |Du|\,\mathrm{d}x\right)^2 \geqslant \lambda^2 \beta_1^{-2}\,|A(\mu_1 - \lambda, 2\varrho)|^{\frac{2n-2}{n}}\,. \tag{41}$$

Since $D(\lambda) \subset A(\mu_1 - 2\lambda, 2\varrho)$, we have

$$\int_{D(\lambda)} |Du|^2\,\mathrm{d}x \leqslant \int_{A(\mu_1 - 2\lambda, 2\varrho)} |Du|^2\,\mathrm{d}x\,,$$

then applying (11*a*) with $\varrho_1 = 2\varrho < \varrho_2 = 4\varrho$ and $k = \mu_1 - 2\lambda$, we get

$$\int_{D(\lambda)} |Du|^2\,\mathrm{d}x \leqslant \gamma 4^{-1} \varrho^{-2} \int_{A(\mu_1 - 2\lambda, 4\varrho)} (u - \mu_1 + 2\lambda)^2\,\mathrm{d}x \leqslant \gamma \varrho^{-2} \lambda^2 |B(4\varrho)|\,. \tag{42}$$

Combining this last inequality with (40) and (41), we obtain

$$|D(\lambda)| \geqslant \frac{\varrho^2}{\beta_1^2 \gamma}\,|A(\mu_1 - \lambda, 2\varrho)|^{\frac{2n-2}{n}}\,|B(4\varrho)|^{-1}, \qquad \forall \lambda \leqslant \frac{\omega}{4}\,. \tag{43}$$

Now, let $h = h(n, \gamma) \in \mathbb{N}$ be the first integer such that

$$h \geqslant \frac{\gamma \beta_1^2\,|B(4\varrho)|\,|B(2\varrho)|}{\varrho^{2n}} \left[\theta\left(\frac{1}{2}\right)\right]^{\frac{2-2n}{n}}, \tag{44}$$

and put

$$\eta = \eta(n, \gamma) = 2^{-(h+2)}. \tag{45}$$

For all $\lambda = 2^m \eta\omega$ with $m = 1, \ldots, h$ we have $\lambda \leq \frac{\omega}{4}$, then from (43) it follows

$$|D(2^m \eta\omega)| \geq \frac{\varrho^2}{\beta_1^2 \gamma} |A(\mu_1 - 2^m \eta\omega, 2\varrho)|^{\frac{2n-2}{n}} |B(4\varrho)|^{-1}$$

$$\geq \frac{\varrho^2}{\beta_1^2 \gamma} |A(\mu_1 - 2\eta\omega, 2\varrho|^{\frac{2n-2}{n}} |B(4\varrho)|^{-1},$$

for all $m = 1, \ldots h$.

To simplify the notation, let us put

$$k = \mu_1 - 2\eta\omega .$$

Being the sets $\{D(2^m \eta\omega)\}_{1 \leq m \leq h}$ disjoint and contained in $B(2\varrho)$, we have

$$|B(2\varrho)| \geq \sum_{m=1}^{h} |D(2^m \eta\omega)| \geq h \frac{\varrho^2}{\beta_1^2 \gamma} |A(k, 2\varrho)|^{\frac{2n-2}{n}} |B(4\varrho)|^{-1},$$

then, recalling (44), we obtain

$$|B(2\varrho)| \geq |B(2\varrho)| \varrho^{-2n+2} \left[\theta\left(\frac{1}{2}\right)\right]^{\frac{2-2n}{n}} |A(k, 2\varrho)|^{\frac{2n-2}{n}},$$

from which

$$|A(k, 2\varrho)| \leq \theta\left(\frac{1}{2}\right) \varrho^n < \theta\left(\frac{1}{2}\right) (2\varrho)^n.$$

Being

$$(u - k)^2 \leq (\mu_1 - k)^2 = (2\eta\omega)^2, \qquad \forall x \in A(k, 2\varrho),$$

we have

$$\int_{A(k, 2\varrho)} (u(x) - k)^2 \, dx \leq (2\eta\omega)^2 |A(k, 2\varrho)| < (2\eta\omega)^2 \theta\left(\frac{1}{2}\right) \varrho^n. \tag{46}$$

It follows then from Lemma 4

$$\sup_{B(2\varrho-\sigma\varrho)} |u| \leqslant k + \sigma c\,,$$

where from (46),

$$c^2 = (2\varrho)^{-n}\left[\theta\left(\frac{1}{2}\right)\right]^{-1}\left[\int_{A(k,\,2\varrho)} (u(x)-k)^2\,\mathrm{d}x\right] < (2\eta\omega)^2.$$

Since

$$k + \sigma c \leqslant \mu_1 - \eta\omega \quad \text{and} \quad 2\varrho - \sigma\varrho = \frac{3}{2}\varrho > \varrho\,,$$

we have

$$\sup_{B(\varrho)} |u| \leqslant \mu_1 - \eta\omega\,,$$

then

$$\begin{aligned} \operatorname{osc}(u,\ B(\varrho)) &\leqslant \sup_{B(\varrho)} u - \inf_{B(\varrho)} u \\ &\leqslant \sup_{B(\varrho)} u - \inf_{B(4\varrho)} u \\ &\leqslant \mu_1 - \eta\omega - \mu_2 = (1-\eta)\operatorname{osc}(u,\ B(4\varrho)). \qquad \blacksquare \end{aligned}$$

STEP 6. We are finally able to prove the following:

LEMMA 6. *There exists a number* $a = a(n, \gamma) \in (0, 1)$ *such that, for all* $d \in \mathbb{R}$ *with*

$$0 < 2d < \operatorname{dist}(y, (\mathbb{R}^n - \Omega)),$$

we have

$$(47) \qquad |u(x_1) - u(x_2)| \leqslant 2K\left(\frac{4\,|x_1 - x_2|}{d}\right)^a, \qquad \forall x_1, x_2 \in B_y\left(\frac{d}{2}\right),$$

where K *is the Lipschitz constant of* w.

PROOF. Let x_1, $x_2 \in B_y\left(\frac{d}{2}\right)$ and consider the ball $B(\varrho) = B_{x_1}(\varrho)$ of radius $\varrho = |x_2 - x_1|$ centered in x_1. Let m be an integer such that

$$4^{-m-1} d < \varrho \leqslant 4^{-m} d, \tag{48}$$

then Lemma 5 implies

$$\operatorname{osc}(u, B_{x_1}(\varrho)) \leqslant \operatorname{osc}(u, B_{x_1}(4^{-m} d)) \leqslant (1-\eta)^m \operatorname{osc}(u, B_{x_1}(d)) \leqslant (1-\eta)^m 2K. \tag{49}$$

From (48) we get

$$-m - 1 < \log_4 \varrho - \log_4 d,$$

that is

$$m > -1 - \log_4 \frac{\varrho}{d}.$$

Hence, recalling (49), we have

$$\operatorname{osc}(u, B_{x_1}(\varrho)) \leqslant (1-\eta)^{-1-\log_4 \frac{\varrho}{d}} 2K \leqslant \left(\frac{4\varrho}{d}\right)^{-\log_4(1-\eta)} 2K.$$

Put $a = -\log_4(1-\eta)$, where η is the constant defined in Lemma 5 from (45), we obtain

$$\operatorname{osc}(u, B_{x_1}(\varrho)) \leqslant 2K \left(\frac{4\varrho}{d}\right)^a,$$

from which it follows (47). ■

We can conclude that the differential quotients

$$u(x) = u(x, t, e) = \frac{w(x + te) - w(x)}{t}$$

are Hölder-continuous in every ball $B_y\left(\frac{d}{2}\right) \subset \Omega$ with $0 < 2d < \operatorname{dist}(y, (\mathbb{R}^n - \Omega))$, for all $e \in S^{n-1}$ and $t \in \mathbb{R} - \{0\}$ sufficiently small. Moreover the Hölder constant and exponent are independent of t and e. Therefore from (47), letting $t \to 0$, we obtain

$$\lim_{t \to 0} |u(x_1, t, e) - u(x_2, t, e)| = |D_e u(x_1) - D_e u(x_2)| \leqslant 2K \left(\frac{4|x_1 - x_2|}{d}\right)^a,$$

for all $x_1, x_2 \in B_y\left(\frac{d}{2}\right)$ and for all $e \in S^{n-1}$.

This completes the proof of Theorem 1.

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