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Some New Partially Symmetric Designs and their Resolution.

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ABSTRACT - In this note we study a resolution (a generalisation of a parallelism) in the (new) partially symmetric designs of the type $\mathcal{S} = \mathcal{D} \backslash \mathcal{D}'$ where $\mathcal{D}'$ is a tight (Baer) subdesign in the symmetric $2-(v, k, \lambda)$ design $\mathcal{D}$ (with $\lambda > 1$).

1. - Introduction.

Throughout this note let $\mathcal{D}$ be a symmetric $2-(v, k, \lambda)$ design (with $\lambda > 1$) and let $\mathcal{D}'$ be a symmetric $2-(v', k', \lambda')$ subdesign of $\mathcal{D}$. By Jungnickel [4] $\mathcal{D}'$ is a *tight* subdesign of $\mathcal{D}$ iff each block of $\mathcal{D} \backslash \mathcal{D}'$ meets $\mathcal{D}'$ in a constant number x of points. Furthermore if $\lambda = \lambda'$ (and then $x = 1$) we say $\mathcal{D}'$ is Baer subdesign of $\mathcal{D}$.

By Hughes [2] a square 1-design $\mathcal{S}$ is a *partial symmetric design* (a PSD) if there exist integers $\lambda_1, \lambda_2 \geq 0$ such that two points are on λ_1 or λ_2 common blocks; two blocks of $\mathcal{S}$ contains λ_1 or λ_2 common points and all such that $\mathcal{S}$ is connected. We say then $\mathcal{S}$ is PSD for $(v_1, k_1, \lambda_1, \lambda_2)$ (where v_1 is the number of points (blocks) in $\mathcal{S}$ and k_1 block (point)-size of $\mathcal{S}$).

The concept of a divisibility and resolution (a generalisation of the parallelism) we take as in [3] (pp. 206, 154) and [1] (pp. 45, 39).

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2. – Results.

2.1 Proposition. *Let $\mathcal{D}'$ be a Baer subdesign of $\mathcal{D}$. Then $\mathcal{S} = \mathcal{D}\backslash\mathcal{D}'$ is a* PSD *for* $(v_1 = v - v', k_1 = k - 1, \lambda_1 = \lambda - 1, \lambda_2 = \lambda)$ *and the relation* $\|$ (*for arbitrary blocks b and c in* $\mathcal{S}$)

$$b \| c \text{ iff } b \text{ and } c \text{ lie on the same point in } \mathcal{D}'$$

is an equivalence relation and, in this sense, $\mathcal{S}$ is divisible.

Proof. It is clear that $\mathcal{S}$ is a square 1-design with $v_1 = v - v'$ points (blocks) and with point (block)-size $k - 1$. Each point in $\mathcal{D}\backslash\mathcal{D}'$ is exactly on one block of $\mathcal{D}'$ and thus two points in $\mathcal{D}\backslash\mathcal{D}'$ lie exactly on $\lambda - 1$ or λ blocks of $\mathcal{D}\backslash\mathcal{D}'$. Furher any two blocks of $\mathcal{D}\backslash\mathcal{D}'$ lie exactly on 0 or 1 points in $\mathcal{D}'$ and their integers in $\mathcal{D}\backslash\mathcal{D}'$ are $\lambda - 1$ or λ.

The partition of the set of all blocks in $\mathcal{S}$ onto subsets of blocks passing through some point from $\mathcal{D}'$ is disjoint and therefore the $\|$ is an equivalence relation. ■

2.2 Remark. By 2.1, $\mathcal{S} = \mathcal{D}\backslash\mathcal{D}'$ (where $\mathcal{D}'$ is a Bear subdesign of $\mathcal{D}$) is a PSD with a divisibility. But, in general, $\mathcal{S}$ cannot have a resolution. For instance $\mathcal{S} = \mathcal{D}\backslash\mathcal{D}'$, where $\mathcal{D}$ is a symmetric $2 - (16, 6, 2)$ design with a symmetric $2 - (4, 3, 2)$ subdesign $\mathcal{D}'$.

But we have

2.3 Proposition. *Let $\mathcal{D} = PG_2(3, q)$ (q a prime power). Then $\mathcal{D}$ has Baer $2 - (q + 1, q + 1, q + 1)$ subdesign $\mathcal{D}'$ and the $\mathcal{S} = \mathcal{D}\backslash\mathcal{D}'$ has a strong resolution.*

Proof. By [4] $\mathcal{D}'$ exists. Any class of blocks are all blocks in $\mathcal{S} = \mathcal{D}\backslash\mathcal{D}'$ through any point in $\mathcal{D}'$. Two different classes are disjoint.

Each of these classes have exactly $m = q^2 + q + 1 - (q - 1) = q^2$ blocks and any two blocks in the same class have exactly $\lambda - 1$ points in common. Two blocks in the different classes have exactly λ points in common. Finally, each point in $\mathcal{D}\backslash\mathcal{D}'$ lies exactly on $\lambda - 1$ blocks of any one class. Thus, $\mathcal{S}$ have a strong resolution. ■

In general, let $\mathcal{D} = PG_{2d}(2d + 1, q)$ ($d \geqslant 2$) be the design of points and hyperplanes of the $(2d + 1)$-dimensional projective space over $GF(q)$. Then, by [4], $\mathcal{D}$ has a tight (c, c, c)-subdesign $\mathcal{D}'$ with $c = q^d + \ldots + q + 1$.

In general, we cannot say anything of the relation $\|$ (as in 2.1 and 2.3) in $\mathcal{S} = \mathcal{D}\backslash\mathcal{D}'$. Namely, the partition of $\mathcal{S}$, corresponding to $\|$, is not disjoint. Further, we cannot say anything of a inner (outer) constant (for $\|$). But we have

2.4 PROPOSITION. *Let* $\mathcal{D} = PG_{2d}(2d+1, q)$ $(d \geq 2)$ *and let* $\mathcal{D}'$ *be a tight* (c, c, c)*-subdesign. Then* $\mathcal{S} = \mathcal{D}\backslash\mathcal{D}'$ *is a* PSD *having a strong resolution.*

PROOF. The parameters of $\mathcal{D}$ and $\mathcal{D}'$ are $v = q^{2d+1} + \ldots + q + 1$, $k = q^{2d} + \ldots + q + 1$, $\lambda = q^{2d-1} + \ldots + q + 1$ and $c = q^{d} + \ldots + q + 1$. It is not difficult to check that any point in $\mathcal{S} = \mathcal{D}\backslash\mathcal{D}'$ is exactly on x blocks of $\mathcal{D}'$ where $x = q^{d-1} + \ldots + q + 1$. Thus $\mathcal{S}$ is a square $1 - (v - c, k - x, k - x)$ design. Further we have:

$$\text{from } 1 + (x(x-1))/\overline{\lambda}_1 = c \quad \text{we get } \overline{\lambda}_1 = q^{d-2} + \ldots + q + 1;$$

$$k - c = \ldots = q^{d+1} \cdot (q^{d-1} + \ldots + q + 1) = q^{d+1} \cdot x$$

and

$$\lambda - c = \ldots = q^{d+1}(q^{d-2} + \ldots + q + 1) = q^{d+1}\overline{\lambda}_1 .$$

Here there is an automorphism $y \mapsto y + u$ exchanging the points resp. the blocks (pointwise) in $\mathcal{D}'$ (Singer cycle in $\mathcal{D}'$, generated additively with u in Z_v). Thus we conclude that the blocks in $\mathcal{D}'$ $(\cong PG_{d-1}(d, q))$ have x sets, each of these sets has exactly q^{d+1} points (in $\mathcal{D}\backslash\mathcal{D}'$) and any two blocks in $\mathcal{D}'$ have exactly $\overline{\lambda}_1$ sets in common. We are calling these sets «points». So any two points of $\mathcal{D}\backslash\mathcal{D}'$ are in one or in two «points». Therefore through two points in $\mathcal{D}\backslash\mathcal{D}'$ pass, according with this, x or $\overline{\lambda}_1$ blocks from $\mathcal{D}'$. Thus, on two points of $\mathcal{S}$ lie $\lambda_1 = \lambda - x$ or $\lambda_2 = \lambda - \overline{\lambda}_1$ common blocks. By [4] (2.1), we get this for the intersections of the blocks in $\mathcal{S}$. Thus, $\mathcal{S}$ is a PSD for

$$(v - c, k - x, \lambda_1 = \lambda - x, \lambda_2 = \lambda - \overline{\lambda}_1) =$$

$$= (q^{2d+1} + \ldots + q^{d+1}, q^{2d} + \ldots + q^{d}, q^{2d-1} + \ldots + q^{d}, q^{2d-1} + \ldots + q^{d-1}).$$

Any resolution-class in $\mathcal{S}$ is formed from all blocks in $\mathcal{S}$ passing through any block in $2 - (c, x, \overline{\lambda}_1)$ design $\mathcal{D}'$. One has exactly

$$\frac{k-c}{x} = \frac{q^{d+1}x}{x} = q^{d+1}$$

blocks in each resolution-class and exactly

$$\frac{v-c}{q^{d+1}} = \frac{q^{2d+1} + \ldots + q + 1 - (q^{d} + \ldots + q + 1)}{q^{d+1}} = \ldots = c$$

(disjoint!) classes. There are exactly $(\lambda - x)/x = \dots = q^d$ blocks from each resolution-class passing through any point in $\mathcal{S} = \mathcal{D} \setminus \mathcal{D}'$.

Finally, our resolution is strong with inner and outer constant $\lambda - x$ and $\lambda - \bar{\lambda}_1$ respectively. ■

AN ILLUSTRATION. $\mathcal{D} = PG_4(5, 2)$ (with a tight subdesign for $(7, 7, 7)$). The initial block in $\mathcal{D}$ (in the form of a difference set) is

$$1_0 = 0, 1, 2, 3, 4, 6, 7, 8, 9, 12, 13, 14, 16, 18, 19, 24, 26, 27, 28,$$
$$32, 33, 35, 36, 38, 41, 45, 48, 49, 52, 54, 56.$$

(All blocks 1_i are formed by (mod. 63) addition of the $1, 2, \dots, 62$ respectively.) The points in $\mathcal{D}'$ are 0, 9, 18, 27, 36, 45, 54.

The resolution-classes are:

$$(0) \wedge (9) = (0) \wedge (45) = (9) \wedge (45) =$$
$$= \{1_7, 1_{31}, 1_{37}, 1_{39}, 1_{44}, 1_{56}, 1_{59}, 1_{60}\},$$

$$(0) \wedge (18) = (0) \wedge (27) = (18) \wedge (27) =$$
$$= \{1_{11}, 1_{14}, 1_{15}, 1_{25}, 1_{49}, 1_{55}, 1_{57}, 1_{62}\},$$

$$(0) \wedge (36) = (0) \wedge (54) = (36) \wedge (54) =$$
$$= \{1_{22}, 1_{28}, 1_{30}, 1_{35}, 1_{47}, 1_{50}, 1_{51}, 1_{61}\},$$

$$(9) \wedge (18) = (9) \wedge (54) = (18) \wedge (54) =$$
$$= \{1_2, 1_5, 1_6, 1_{16}, 1_{40}, 1_{46}, 1_{48}, 1_{53}\},$$

$$(9) \wedge (27) = (9) \wedge (36) = (27) \wedge (36) =$$
$$= \{1_1, 1_3, 1_8, 1_{20}, 1_{23}, 1_{24}, 1_{34}, 1_{58}\},$$

$$(18) \wedge (36) = (18) \wedge (45) = (36) \wedge (45) =$$
$$= \{1_4, 1_{10}, 1_{12}, 1_{17}, 1_{29}, 1_{32}, 1_{33}, 1_{43}\},$$

$$(27) \wedge (45) = (27) \wedge (54) = (45) \wedge (54) =$$
$$= \{1_{13}, 1_{19}, 1_{21}, 1_{26}, 1_{38}, 1_{41}, 1_{42}, 1_{52}\},$$

This is the 4-resolution with the constants 12 and 14.

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