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On a question of Deaconescu about automorphisms. - III

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1. - Introduction.

A theorem on embeddability of every countable set $\{G_\mu\}_{\mu \in I}$ of countable groups without involutions in a simple 2-generator group G in which every proper subgroup is either a cyclic group or contained in a subgroup conjugate to one of the embedding groups G_μ was proved in [2], and the generalizations of this theorem to the case of arbitrary sets $\{G_\mu\}_{\mu \in I}$ of groups without involutions were given in [3] and [4]. Recently an embedding scheme of a set of arbitrary groups (without mentioning the absence of involutions) into a simple group with a «well-described» lattice of subgroups and a given outer automorphism group was established in [5]. These constructions have given an opportunity to obtain minimal extensions of the subgroup lattices of the resulting groups G in comparison with the subgroup lattices of the embedding groups which has been used for construction of infinite groups with prescribed properties.

In this paper we concentrate on groups G satisfying the property that $\operatorname{Aut} H \cong N_G(H)/C_G(H)$ for all subgroups H of G. Such groups were referred to in [1] as MD-groups. At the Second International Conference on Algebra in Barnaul (Siberia, Russia, 1991) M. Deaconescu posed a problem on the existence of infinite MD-groups. It is easy to prove that the infinite dihedral group has the MD-property. J. C. Lennox and J. Wiegold showed in [1] that the only nontrivial finite MD-groups are Z_2 and S_3, and Theorem 2.1 of the same paper gave a complete classification of infinite metabelian MD-groups. This classifica-

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tion was extended in [8] to the case of radical groups. Recall that a group G is a *radical* group if the iterated series of Hirsch-Plotkin radicals reaches G. (Thus, for instance, locally nilpotent groups and hyperabelian groups are radical.) In fact, H. Smith and J. Wiegold proved in [8] that if G is an infinite radical MD-group, then G is metabelian and is thus an extension of a torsion-free locally cyclic subgroup A having finite type at every prime p by a cyclic group $\langle x \rangle$ of order 2, such that $xax^{-1} = a^{-1}$ for all $a \in A$. As a consequence, the authors of [8] obtained that every infinite locally soluble MD-group is, in fact, metabelian. Moreover, it was also proved in [8] that Z_2 and S_3 are the only periodic MD-groups which are nontrivial.

First of all we note that if there is a simple infinite MD-group L, then L is *complete*, that is, it has trivial centre and no outer automorphisms. Theorem A [5] does not provide us with examples of simple infinite MD-groups, since by assertion 13 of this theorem (with $H = \{1\}$), L contains infinite cyclic subgroups A with $N_L(A) = C_L(A)$. But J. Wiegold noted that if we obtain a modification of Theorem A [5] without such subgroups A, then it might work in our case.

Let $\{G_\mu\}_{\mu \in I}$ be an arbitrary set of nontrivial groups. We denote by Ω^1 the *free amalgam* of the groups G_μ, $\mu \in I$, that is, the set $\bigcup_{\mu \in I} G_\mu$ with $G_\mu \cap G_\nu = \{1\}$ whenever $\mu \neq \nu$. We say that the mapping $g\colon \Omega^1 \to G$ is an *embedding* of Ω^1 into G if it is injective and its restriction to every G_μ is a homomorphism

Let $\Omega = \Omega^1 \backslash \{1\} = \{a_j,\ j \in J\}$. Then a mapping $f\colon 2^\Omega \backslash \{\emptyset\} \to 2^\Omega$ is called *generating* on the set Ω if the following conditions hold:

1) if $C \subseteq G_\mu$ for some $\mu \in I$, then $f(C) = \langle C \rangle \backslash \{1\}$;

2) if $C \not\subseteq G_\mu$ for each $\mu \in I$ and $C = \{a, b\} \subseteq \Omega$, where a and b are involutions (such a subset C will be called *dihedral*), then $f(C) = C$;

3) if C is a finite non-dihedral subset of Ω and $C \not\subseteq G_\mu$ for each $\mu \in I$, then $f(C) = B$, where B is an arbitrary countable subset of Ω such that $C \subseteq B$ and if D is a finite subset of B, then $f(D) \subseteq B$;

4) if C is an infinite subset of Ω and $C \not\subseteq G_\mu$ for each $\mu \in I$, then $f(C) = \bigcup_{A \in T} f(A)$, where T is the set of all finite subsets of C.

For example, a generating mapping f on Ω can be defined in the following way: if $C \in 2^\Omega \backslash \{\emptyset\}$ and $C = \bigcup_{\mu \in I} C_\mu$, where $C_\mu = C \cap G_\mu$, $\mu \in I$, then $f(C) = \left(\bigcup_{\mu \in I} \langle C_\mu \rangle \right) \backslash \{1\}$ (we assume that $\langle C_\mu \rangle = \{1\}$ if $C_\mu = \emptyset$).

We denote by $G(1)$ the free product of the groups G_μ, $\mu \in I$. A group G having the presentation

$$G = \langle G(1) \| R = 1, R \in D \rangle \tag{1.1}$$

is called (*diagrammatically*) *aspherical* if every diagram on a sphere over (1.1) is either non-reduced or consists entirely of 0-cells. (All necessary information about diagrams can be found in [6].)

Let $G = \langle \Omega \rangle$, f an arbitrary generating mapping on Ω. We say that X is a *minimal* word (over the alphabet Ω) in G if it follows from $X = Y$ in G that $|X| \leq |Y|$, where $|Z|$ denotes the length of the word Z. Let W be the set of all non-empty words over the alphabet Ω written in the *normal form*, that is, every element X in W is written in the form $X_1 \dots X_k$, where each X_l, $1 \leq l \leq k$, is a nontrivial element of $G_{\mu(l)}$, $\mu(l) \in I$, and $\mu(l) \neq \mu(l+1)$ for $l = 1, \dots, k-1$. Then a mapping F: $2^W \setminus \{0\} \to 2^\Omega$ is defined in the following way: if $C \subseteq W$ and $C \neq \emptyset$, then let $V(C)$ be the set of all letters occuring in the expressions of words of C. Then we set $F(C) = f(V(C))$.

The main result of this paper is the following modification of Theorem A [5].

THEOREM A. *Let Ω^1 be the free amalgam of a family $\{G_\mu\}_{\mu \in I}$ of nontrivial groups, g_μ: $G_\mu \to H$ a set of arbitrary homomorphisms of the groups G_μ into a group H with kernels N_μ, $\mu \in I$, such that either H is a torsion-free group and a system of subgroups $\{g_\mu(G_\mu)\}_{\mu \in I}$ generates H or the group H is trivial, let $\{N_\mu\}_{\mu \in I_1}$, $I_1 \subseteq I$, be the set of nontrivial groups of the set $\{N_\mu\}_{\mu \in I}$, Ω_1^1 the free amalgam of the groups N_μ, $\mu \in I_1$. Also suppose that the set $\Omega = \Omega^1 \setminus \{1\}$ contains an involution, and let f be an arbitrary generating mapping on Ω with the property that if $C \nsubseteq G_\mu$ for each $\mu \in I$, then $f(C)$ contains an involution. If the set $\{N_\mu\}_{\mu \in I_1}$ contains either three groups or two groups of which one has order $\geqslant 3$, then the free amalgam Ω^1 of the groups G_μ can be embedded in an aspherical group $G = \langle \Omega \rangle$ with the following properties*:

1) *the free amalgam Ω_1^1 is embedded in a normal simple infinite subroup L of G such that $G/L \cong H$;*

2) *if $X \in L$ and is not conjugate in G to an element of any group G_μ, $\mu \in I$, then either X is an involution or X is of infinite order and is a product of two involutions in L;*

3) *if $XY = YX$ in G for some $X, Y \in G$, then either X and Y are in the same cyclic subgroup of G or X and Y lie in a subgroup conjugate to some group G_μ, $\mu \in I$;*

4) *every subgroup M of G is either a cyclic group or infinite dihedral, or $M \cap L = 1$ and the homomorphic image of M in $H \cong G/L$ has*

an element not conjugate to an element of any group $g_\mu(G_\mu)$, $\mu \in I$, *or if* M *is not cyclic or infinite dihedral, then* M *is conjugate in* G *to an extension* $G_{C,H'}$ *of a group* H' *by a normal subgroup* L_C (*that is,* $G_{C,H'}/L_C \cong H'$), *where* $H' \leqslant H$ *and* $L_C \leqslant L$, *and if every element of* L_C *is a minimal word in* G, *then* $C = F(L_C \setminus \{1\})$ *or* $C = \emptyset$ *in the case* $L_C = \{1\}$;

5) $L_C = R_C \cap L$, *where* $R_C = \langle C \rangle$ *for* $C \in 2^\Omega \setminus \{\emptyset\}$ *or* $R_C = \{1\}$ *in the case* $C = \emptyset$, *and if* $C \nsubseteq G_\mu$ *for each* $\mu \in I$, *then* $G_{C,H'} \leqslant R_C$, L_C *is a simple group,* $N_G(L_C) = R_C$ *and* $C_G(L_C) = \{1\}$;

6) *if* $C \nsubseteq G_\mu$ *for each* $\mu \in I$, *then* $\operatorname{Aut} L_C \cong R_C$ *and* $\operatorname{Out} L_C \cong R_C/L_C$ (*in particular,* $\operatorname{Aut} L \cong G$ *and* $\operatorname{Out} L \cong H$), *and if* $X \in R_C \setminus L_C$, *then the mapping* $g\colon L_C \to X^{-1}L_C X$ *is a regular automorphism of* L_C (*that is,* $g(a) = a$ *if and only if* $a = 1$) *if and only if there is no* $\mu \in I$ *such that* $X \in G_\mu \cap C$ *and* $[X, c] = 1$ *for some* $c \in G_\mu \cap C \cap \Omega_1$, *where* $\Omega_1 = \Omega_1^1 \setminus \{1\}$;

7) *if* $C \nsubseteq G_\mu$ *for each* $\mu \in I$, *then for each* $a \in C \cap \Omega_1$, *we have that* $L_C = \langle cbab^{-1}c^{-1},\ b, c \in C \rangle$ (*in particular,* $L = \langle cbab^{-1}c^{-1},\ b, c \in \Omega \rangle$, *where* a *is an arbitrary element of* Ω_1);

8) *if* X *is a minimal nontrivial word in the group* G, *then* $X \in R_C$ *if and only if* $F(\{X\}) \subseteq f(C)$;

9) *if* $\{G_\mu\}_{\mu \in J}$, $J \subseteq I$, *is a set of all groups having nontrivial intersections with a subgroup* R_C *of* G *and* $X \in Z^{-1}R_C Z$, *where* Z *is of minimal length among all words in* $R_C Z$ *and* $G_\mu Z$, $\mu \in J$, *then* $F(\{Z\}) \subseteq F(\{X\})$;

10) *if* $C \nsubseteq G_\mu$ *for each* $\mu \in I$ *and* M *is a subgroup of* G *in which every element is a minimal word in* G, *then* $\langle L_C, M \rangle \cap L = L_{C_1}$, *where* $C_1 = F(C \cup (M \setminus \{1\}))$;

11) *if* $N_\mu = 1$ *for some* $\mu \in I$ *and the homomorphism* $g_\nu\colon G_\nu \to H$ *is trivial for each* $\nu \in I \setminus \{\mu\}$, *then* G *is the semidirect product of* H *and* L;

12) *if a subgroup* M *of* G *is contained in some group* G_μ, $\mu \in I$, *then* $N_G(M) = N_{G_\mu}(M)$ *and* $C_G(M) = C_{G_\mu}(M)$;

13) *if a subgroup* M *of* G *is infinite dihedral and is not conjugate in* G *to a subgroup of any group* G_μ, $\mu \in I$, *then* $N_G(M)$ *is infinite dihedral and* $C_G(M) = \{1\}$;

14) *if an infinite cyclic subgroup* M *of* L *is not conjugate in* G *to a subsgroup of any group* G_μ, $\mu \in I$, *then* $N_G(M)/C_G(M) \cong Z_2$.

As an immediate consequence of Theorem A (with $H = \{1\}$), we have

THEOREM B. *Let* $\{G_\mu\}_{\mu \in I}$ *be an arbitrary set of nontrivial MD-groups containing either three groups or two groups of which one has order* $\geqslant 3$ *such that the free amalgam* Ω^1 *of the groups* G_μ, $\mu \in I$, *contains an involution, and also let f be an arbitrary generating mapping on* $\Omega = \Omega^1 \setminus \{1\}$ *with the property that if* $C \not\subseteq G_\mu$ *for each* $\mu \in I$, *then* $f(C)$ *contains an involution. Then the free amalgam* Ω^1 *can be embedded in a simple infinite MD-group* $L = \langle \Omega \rangle$ *such that every proper subgroup of L is either contained in an infinite dihedral subgroup of L, or conjugate in L to a subgroup* $R_C = \langle C \rangle$ *for some* $C \in 2^\Omega \setminus \{\emptyset\}$, *and* $a \in R_C \cap \Omega$ *if and only if* $a \in f(C)$.

REMARK. It is possible, but unknown to the author, that every nontrivial *MD*-group has an involution and thus the condition in Theorem B that the free amalgam Ω^1 of the groups G_μ, $\mu \in I$, contains an involution is redundant. It was noted by H. Smith that this conjecture is true for any nontrivial *MD*-group G whose cardinality does not exceed all the cardinals obtained from $\aleph_0$ by related exponentiation ω times, that is, does not exceed $\aleph_0, 2^{\aleph_0}, 2^{2^{\aleph_0}}, \ldots$. In fact, by the observation in [1], there are no $Z \times Z$ subgroups in such a group G. It follows from 1.2 [1] and Theorem 2 [8] that either G is finite and is therefore isomorphic to Z_2 or S_3, or G contains an element x of infinite order. In the second case, there exists $y \in G$ inverting x. Then y^2 centralizes x, but G has no $Z \times Z$ subgroups, so some even power of y is a power of x and is therefore both centralized and inverted by y, which completes the proof of the assertion.

All infinite *MD*-groups constructed in [1] and [8] are metabelian and countable, and there were no examples of uncountable or simple infinite *MD*-groups. Now we have

COROLLARY 1. *For each infinite cardinal number* α, *there exists a simple MD-group L of cardinality* α.

PROOF. It is sufficient to take $\{G_\mu\}_{\mu \in I}$ to be a set of the groups of order 2 with $|I| = \alpha$ and L as a group in Theorem B for this set of groups and an arbitrary generating mapping f.

Another application of Theorem B is devoted to finitely generated infinite *MD*-groups. It follows from Corollary [8] and Corollary 2.4 [1] that the only infinite finitely generated locally soluble *MD*-group is the infinite dihedral group.

COROLLARY 2. *There exists a continuum of pairwise non-isomorphic 2-generator simple infinite MD-groups in which every maximal proper subgroup is infinite dihedral.*

PROOF. Let $G_\mu = \langle a_\mu \rangle$ be the group of order 2 for each $\mu \in \{1, 2, 3\}$. We define a generating mapping f on $\Omega = \{a_1, a_2, a_3\}$ in the only possible way: if $C \subseteq \Omega$ such that $C \not\subseteq G_\mu$ for each $\mu \in \{1, 2, 3\}$ and C is not dihedral (thus $C = \Omega$), then $f(C) = \Omega$. Then Theorem B applies to $\Omega^1 = \Omega \cup \{1\}$ and this mapping f and yields a simple infinite MD-group $G = \langle a_1, a_2, a_3 \rangle$ in which every maximal proper subgroup is infinite dihedral. It follows from assertions 2 and 3 of Theorem A that $G = \langle a_1 a_2, a_3 \rangle$. That there exists a continuum of pairwise non-isomorphic groups with necessary properties can be proved in a way as in the proof of Theorem 28.7 [6].

The proof of Theorem A will be heavily based on the results from [4]-[7]. Unless otherwise stated, all definitions and notation may be found in [6] and [7].

2. - Construction of the group G.

As in [6], we introduce the positive parameters

$$\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \iota,$$

where all the parameters are arranged according to «height», that is, each constant is chosen after its predecessor. Our proofs and some definitions are based on a system of inequalities involving these parameters. The values of the parameters can be chosen in such a way that all the inequalities hold. We then use the following notation:

$$\alpha' = 1/2 + \alpha, \quad \beta' = 1 - \beta, \quad \gamma' = 1 - \gamma, \quad h = \delta^{-1}, \quad d = \eta^{-1}, \quad n = \iota^{-1}.$$

We assume that h and n are integers. We also use the notation introduced in § 1.

We may assume that I is a well-ordered set. We also may assume that Ω^1 is a well-ordered set such that 1 is the maximal element of Ω^1 and if $a \in G_\mu \setminus \{1\}$ and $b \in G_\nu \setminus \{1\}$, where $\mu < \nu$, then $a < b$. On a set $\Omega_2 = \{ab \mid a \in \Omega^1, b \in \Omega \text{ and if } \{a, b\} \subseteq G_\mu \text{ for some } \mu \in I, \text{ then } a = 1\}$ we introduce an order in the following way: $ab \leq cd$ if and only if either $b < d$ or $b = d$ and $a \leq c$ (with respect to the ordering of Ω^1).

By the statement of Theorem A, there is a homomorphism of the free product $G(1)$ of the groups G_μ, $\mu \in I$, onto H such that its restriction to every group G_μ is equal to g_μ. Suppose that the kernel of this homomorphism is N.

Let $D_1 = \emptyset$, and suppose, by induction, that we have defined the set of relators $D_{i-1} \subseteq N$, $i \geq 2$, and set

$$G(i-1) = \langle G(1) \| R = 1, R \in D_{i-1} \rangle .$$

A word X (over the alphabet Ω) is called *free* in rank $i-1$ if X is not conjugate in rank $i-1$ to an element of Ω^1, that is, to an image in $G(i-1)$ of an element of one of the free factors G_μ. A non-empty word Y is said to be *simple* in rank $i-1$ if it is free in rank $i-1$, not conjugate in rank $i-1$ (that is, in $G(i-1)$) to a power of a shorter word and to a power of a period of rank $k<i$ and not conjugate in rank $i-1$ to the subword $S_{j+p-1,C}\,aS^{-1}_{j+p-1,C}A^l_p$ of a relator of the form (2.10) for a period A_p of rank $k<i$.

Now let P_i denote a maximal set of words of length i which are simple in rank $i-1$ with the property that $A, B \in P_i$ and $A \not\equiv B$ («$\equiv$» means letter-for-letter equality of words of the same length) implies that A is not conjugate in rank $i-1$ to B or B^{-1}. The words in P_i are called *periods* of rank i. A special role in construction of the group $G(i)$ will be played by the set P_i' of all periods of rank i which are not equal in rank $i-1$ to a product of two involutions (of $G(i-1)$). (For short, elements of P_i' will be called *non-dihedral periods of rank i.*) We may assume (see Lemma 4.18 below) that if $a, c \in \Omega_1$, $b, e, g \in \Omega$ and $d, f \in \Omega^1$ such that a is of infinite order (if such an a exists), $\{a, b\} \not\subseteq G_\mu$, $\{c, e\} \not\subseteq G_\nu$ and $\{c, g\} \not\subseteq G_\tau$ for each $\mu, \nu, \tau \in I$, $fg, de \in \Omega_2$, $fg \neq de$ and $fge^{-1}d^{-1} \neq c$ in the case $c^2 = 1$, then the words

$$A_0 = a[a, b]^k a[a, b]^{2k} a[a, b]^{3k}, \qquad A_m = aA_0^m,$$

$$B_0 = (cfg)^{-1}[[c, de]^k, [c, fg]](cfg), \qquad B_m = (cfg)^{-1}[c, de]^k (cfg) B_0^m$$

are non-dihedral periods of some ranks for each $k > n^7$ and m, $|m| > k^3 n^7$.

For each period $A \in P_i' \cap N$, we fix a maximal subset Y_A such that

1) if $T \in Y_A$, then $1 \leqslant |T| < d|A|$;

2) each double coset of the pair $\langle A\rangle$, $\langle A\rangle$ of subgroups of $G(i)$ contains at most one word in Y_A and this word is of minimal length among the words representing this double coset;

3) if $T \in Y_A$, then $T \in N$ and $F(\{T\}) \subseteq F(\{A\})$.

We may assume (see Lemma 4.18 below) that if a power F^t of a period F of some rank is conjugate to a word BC^m for some $m > n$, where C is a non-dihedral period of rank i not equal to A_0 or B_0, $|B| < \iota^2(m|C|)^{1/3}$ and $BCB^{-1} \neq C^{\pm 1}$ in $G(i)$, then $t = 1$.

For each period $A \in P_i' \cap N$, we introduce the ordering of the set of natural numbers (or a finite segment of it) on the set Y_A such that the first element of the set Y_A belongs to Ω_1 (it follows from the statement of Theorem A that $Y_A \cap \Omega_1 \neq \emptyset$) and if $A = A_m$ or $A = B_m$, where $m = 0$

or $|m| > k^3 n^7$, then the first element of the set Y_A is a or $\min(c, h)$ (with respect to the ordering of Ω^1), respectively, where $h = d$ if $d \in \in \Omega_1$, otherwise $h = 1$. We denote this order by $\leqslant_A$.

For each period $A \in P_i' \cap N$, $i \geqslant 7$, we now construct some relations. If $A = A_m$, $|m| > k^3 n^7$, for some $a \in \Omega_1$ and $b \in \Omega$ such that $\{a, b\} \not\subseteq G_\mu$ for each $\mu \in I$ and a is of infinite order, then for each l, $n \leqslant l \leqslant 3n$, we introduce a relation

$$a^{-1} A^n a A^{n+l} a A^{n+6n+l} \dots a A^{n+6n(h-2)+l} = 1\,. \tag{2.1}$$

If $A = B_m$, $|m| > k^3 n^7$, for some $c \in \Omega_1$, $e, g \in \Omega$ and $d, f \in \Omega^1$ such that $\{c, e\} \not\subseteq G_\mu$, $\{c, g\} \not\subseteq G_\nu$ for each $\mu, \nu \in I$, $fg, de \in \Omega_2$, $fg \neq de$ and $fge^{-1}d^{-1} \neq c$ in the case $c^2 = 1$, then for each l, $3n < l \leqslant 5n$, and $T = = (cfg)^{-1}[c, de]^k(cfg)$, $k > n^7$, we consider a relation

$$T^{-1} A^n T A^{n+l} T A^{n+6n+l} \dots T A^{n+6n(h-2)+l} = 1\,, \tag{2.2}$$

and if $b_1 = \min(c, e)$, $b_2 = \min(de, fg)$ (with respect to the ordering of Ω_2) and $T_j = (cfg)^{-1}[c, de]^j(cfg)$, $j = 1, 2$, then we set

$$(cfg)^{-1} b_j c b_j^{-1} (cfg) A^n T_j A^{6n} T_j A^{12n} \dots T_j A^{6n(h-1)} = 1 \tag{2.3}$$

for each j, $1 \leqslant j \leqslant 2$. Let $T \in Y_A$ and $T \neq a$ in the case $A = A_m$, $|m| > > k^3 n^7$. If a is the minimal element of the set Y_A and $T \neq a$, then we introduce a relation

$$a A^n T A^{3n} T A^{9n} \dots T A^{3n+6n(h-2)} = 1\,, \tag{2.4}$$

and if $T = a$, then it follows from the definition of the set P_i that there exists $b \in F(\{A\})$ such that $\{a, b\} \not\subseteq G_\mu$ for each $\mu \in I$, and we consider a relation

$$bab^{-1} A^n T A^{3n} T A^{9n} \dots T A^{3n+6n(h-2)} = 1\,. \tag{2.5}$$

If a is the first element of the set Y_A, $T \in Y_A \backslash \{a\}$ and $T \neq \neq (cfg)^{-1}[c, de]^k(cfg)$, $k > n^7$, in the case $A = B_m$, $|m| > k^3 n^7$, then we introduce a relation

$$a A^n T A^{5n} T A^{11n} \dots T A^{5n+6n(h-2)} = 1\,, \tag{2.6}$$

and if $T = a$, then, as above, we set

$$bab^{-1} A^n T A^{5n} T A^{11n} \dots T A^{5n+6n(h-2)} = 1 \tag{2.7}$$

for some $b \in F(\{A\})$ such that $\{a, b\} \not\subseteq G_\mu$ for each $\mu \in I$. And if $T \in Y_A$, then let T_1 be the minimal element of the set $Y_A \backslash \{a^{\pm 1}\}$ such that

$T <_A T_1$ (if such an element T_1 exists). Then we consider a relation

(2.8) $$T_1 A^n TA^{7n} TA^{13n} \dots TA^{n+6n(h-1)} = 1 .$$

The relations (2.1)-(2.8) are taken from the definition of the group G in Theorem A [5]. We can ensure assertions 2 and 14 of Theorem A by imposing relations of the form $SAS^{-1} = A^{-1}$, where A is a non-dihedral period and the conjugating word S is an involution and contains long (compared to A) l-aperiodic subwords for small values of l.

Let C be a finite non-dihedral subset of Ω such that $C = C^{-1}$, where $C^{-1} = \{a^{-1}, a \in C\}$, and $C \not\subseteq G_\mu$ for each $\mu \in I$. Lemma 4.1 gives a sequence $Q_{1,C}, Q_{2,C}, \dots$ of non-empty 7-aperiodic reduced words over the alphabet Ω with $|Q_{j+1,C}| = |Q_{j,C}| + 2$ and $V(\{Q_{j,C}\}) = C$ for all $j = 1, 2, \dots$, where $V(\{Z\})$ denotes the set of all letters occuring in the expression of the word Z over the alphabet Ω. Let a, b be arbitrary fixed elements of C such that $\{a, b\} \not\subseteq G_\mu$ for each $\mu \in I$. For $j \geq 2$, we define auxiliary words $S_{j,C}$ by the formula

$$S_{j,C} \equiv Q_{r(j),C}(ab)^{100} Q_{r(j)+1,C}(ab)^{100} \dots Q_{r(j)+n^2,C}(ab)^{100},$$

where $r(j)$ is chosen to be a sufficiently large number such that if we set $r(1) = 0$, then $r(j) > r(j-1) + n^2$ and $|Q_{r(j),C}| > n^4 j^2$.

Let $A \in P_i' \cap N$, where $i \geq 2$. It follows from the definition of the set P_i' that $C = V(\{A\}) \cup V(\{A\})^{-1}$ is a finite symmetric (that is, $C = C^{-1}$) non-dihedral subset of Ω which is not contained in any group G_μ, $\mu \in I$. Also suppose that k is the minimal positive integer such that $S_{k,C}$ has not been involved in the definition of the group $G(i-1)$, and set $j = \max(i, k)$. It is obvious that the family $\{A_1 = A, A_2, \dots, A_t\}$, $t \geq 1$, of all elements of $P_i' \cap N$ with $V(\{A_p\}) \cup V(\{A_p\})^{-1} = C$, $1 \leq p \leq t$, is finite. By the statement of Theorem A, $f(C)$ contains an involution $a \in \Omega_1$. Now for each p, $1 \leq p \leq t$, we introduce a relation

(2.9) $$(S_{j+p-1,C}\, a S_{j+p-1,C}^{-1}) A_p (S_{j+p-1,C}\, a S_{j+p-1,C}^{-1})^{-1} A_p = 1 .$$

It is convenient to consider (2.9) with its consequences

(2.10) $$(S_{j+p-1,C}\, a S_{j+p-1,C}^{-1}) A_p^l (S_{j+p-1,C}\, a S_{j+p-1,C}^{-1})^{-1} A_p^l = 1$$

for each p, $1 \leq p \leq t$, and $l \neq 0$.

Let $A \in P_i$. If $A \in P_i' \cap N$, then we denote by $r(A)$ the length of the subword $S_{j+p-1,C}\, a S_{j+p-1,C}^{-1}$ in the relation (2.9) for $A_p \equiv A$, otherwise we set $r(A) = n$.

The left-hand sides of the relations (2.1)-(2.10) form the set U_i of *relators* of rank i. All relators of the form (2.9) and (2.10) are called *relators of the second type* while the others are called *relators of the first* *type*. For each $i \geq 2$, we set $D_i = D_{i-1} \cup U_i$, and the group $G(i)$ is defined by its presentation:

$$G(i) = \langle G(1) \| R = 1,\ R \in D_i \rangle. \tag{2.11}$$

Finally, we define

$$G = \langle G(1) \| R = 1,\ R \in D = \bigcup_{i \geq 1} D_i \rangle.$$

3. - Cells and maps.

By a *diagram of rank* i, where $i \geq 2$, we mean a diagram over the presentation (2.11). *Cells of the first type of rank* i correspond to the relators (2.1)-(2.8) in U_i and *cells of the second type of rank* i to the relators (2.10) in U_i. Contours of cells, in the diagrams under consideration, split into sections according to (2.1)-(2.10). Those sections of a cell Π of the first type of rank i with labels $(A^{n+s})^{\pm 1}$ are called *long sections of the first type of rank* i while the others are called *short sections* of Π. If the label of a section p of a cell Π of the second type of rank i is equal to $S_{j+p-1,C}\, a S_{j+p-a,C}^{-1}$, then we say that this section is a *special section of rank* i, and if q corresponds in (2.10) to a subword $A_p^{\pm l}$, then we say that q is a *nonspecial section of rank* i. A nonspecial section q of rank i is called *long* if $|q| > n^2 i$. Further, if, with the preceding notation, $|p| \geq |q|$ ($|p| < |q|$), then p and q (q and p) are called a *long section of the second type of rank* i and a *short section* of Π, respectively.

The definitions of the type of a diagram Δ over $G(i)$ or over G, the compatibility between cells of the first type (that is, the notion of a j-pair) and the A-compatibility of sections of the contour are the same as in § 41 [6] and § 13 [6]. Also we will use the definition of self-compatible cells in Δ from [6, p. 462], with the words «long sections» replaced by «special sections».

We now consider cells of the second type Π and Π' with contours $s_1 t_1 s_2 t_2$ and $s_1' t_1' s_2' t_2'$ such that

$$\begin{cases} \phi(s_1) \equiv \phi(s_2) \equiv \phi(s_1') \equiv \phi(s_2') \equiv S_{j+p-1,C}\, a S_{j+p-1,C}^{-1}, \\ \phi(t_1) \equiv \phi(t_2) \equiv A_p^l, \qquad \phi(t_1') \equiv \phi(t_2') \equiv A_p^k. \end{cases} \tag{3.1}$$

(We should remember that $S_{j+p-1,C}\, a S_{j+p-a,C}^{-1} = (S_{j+p-1,C}\, a S_{j+p-a,C}^{-1})^{-1}$ in $G(1)$, since a is an involution.) We define the notion of compatibility between a special section of rank i and a section of a contour of a diagram and between s_{i_1} and s'_{i_2}, where $i_1, i_2 \in \{1, 2\}$, in the same way as in [6, p. 461].

If Π and Π' are distinct cells of the second type with some sections compatible in Δ, then we say that (Π, Π') is a *cancellable pair* (or an *i-pair*) in the following sense. Cutting along a compatible path, making a 0-refinement and excising a subdiagram Γ with two cells Π and Π' from Δ, we make a hole in Δ such that the label of its boundary is equal in rank 1 to a word B. Consider the following cases.

1) If special sections, say s_1 and s'_1, of the cells Π and Π' are compatible in Δ, them by (3.1), B is equal to $(S_{j+p-1,C}aS^{-1}_{j+p-1,C})\cdot \cdot A_p^{k+l}(S_{j+p-1,C}aS^{-1}_{j+p-1,C})A_p^{k+l}$ in $G(1)$, which corresponds to (2.10) when $k+l \neq 0$. Hence, in pasting one cell of the second type (or a 0-cell) in place of two cells Π and Π', we decrease the type of Δ.

2) If the cells Π and Π' have nonspecial sections, say t_1 and t'_1, compatible in Δ, then by (3.1), $lk < 0$ and B is equal in $G(1)$ to $SA_p^{l_1}SA_p^{l}SA_p^{l_2}SA_p^{k}$, where $S \equiv S_{j+p-1,C}aS^{-1}_{j+p-1,C}$ and $l+k = l_1 + l_2$. Now, in pasting at most two cells of the second type (corresponding to the relations (2.10) for the powers $A_p^{l_1}$ and $A_p^{l_2}$ when $l_j \neq 0, j = 1, 2$) without nonspecial sections compatible in Δ in place of Π and Π', the type of Δ does not increase.

In this way, we can interpret equations and relations of conjugacy using reduced diagrams (over $G(i)$ and over G), that is, diagrams without j-pairs.

A map Δ is called a *N-map* if the following conditions hold.

N1. For cells of the first type and their sections, Δ satisfies conditions B1, B2, B4 and B6-B10 in the definition of a B-map (see [6, p. 225]).

N2. The length of any long section of the first type of rank j is less than $6hnj$.

N3. Every subpath of length $\leqslant \max(j, 2)$ of any nonspecial section of rank j is geodesic in Δ.

N4. Let Γ be a contiguity submap of p to q with $(p, \Gamma, q) \geqslant \varepsilon$, where p is a special or long nonspecial section of a cell of the second type, q is a long section of the first type of rank j or a nonspecial section of rank j. Then $|\Gamma \wedge q| < (1+\gamma)j$. If p is a special or long nonspecial section of rank k, q is a section of a cell of rank j and $(p, \Gamma, q) \geqslant \varepsilon$, then $k < j$.

N5. The Γ-contiguity degree of a special or long nonspecial section of a cell of the second type or any long section of the first type to a special section of a cell of the second type is less than ε for any submap Γ.

N6. If Γ is a contiguity submap of a long section q_1 of the first type of rank j to a nonspecial section q_2 of rank k with $j \neq k$ and $(q_1, \Gamma, q_2) \geqslant \varepsilon$, then $|\Gamma \wedge q_2| < (1+\gamma)k$.

N7. If q is a long section of the first type of rank j or a nonspecial section of rank j and the Γ-contiguity degree of a cell π to q is greater than 1/3, then $|\Gamma \wedge q| < (1+\gamma)j$.

N8. If q is a special section of a cell of the second type of rank j, then $|q| > n^6 j^2$.

N9. Long sections of a cell of the second type have equal lengths.

As in [6], we also introduce the notion of a smooth section of a contour. A subpath q of a contour of a N-map Δ is declared a *smooth section of the first type of rank* i if it has the following properties.

F1. For cells of the first type, conditions S2 and S4 in the definition of a smooth section in a B-map (see [6, p. 226]) are true for q.

F2. For a contiguity submap Γ of p to q, where p is a special or long nonspecial section of a cell of the second type of rank j, it follows from $(p, \Gamma, q) \geqslant \varepsilon$ that $j < i$ and $|\Gamma \wedge q| < (1+\gamma)i$.

F3. If the Γ-contiguity degree of a cell π to q is greater than 1/3, then $|\Gamma \wedge q| < (1+\gamma)i$ and $\Gamma \wedge q = t_1 t_2 t_3$, where $|t_1|$, $|t_3| < \iota|t_2|$ and every subpath of length $\leqslant \max(i, 2)$ in t_2 is geodesic in Δ.

A section q of $\partial\Delta$ is called a *smooth section of the second type* (*of rank* i) if the contiguity degree of a special or long nonspecial section of a cell of the second type or any long section of the first type to it is less than ε.

In the definition of a contiguity submap Γ of a cell Π to a section q of another cell or to a section q in $\partial\Delta$ (respectively, to a cell Π') (see [6, pp. 220, 221]), we assume that $q_1, \dots, q_u$ (respectively, $s_0, \dots, s_l$) are all special and long nonspecial sections of the contour of Π (respectively, of the contour of Π') if Π (respectively, Π') is a cell of the second type.

The definitions of a N-map and its smooth sections enable us to apply to them the results in §§ 20-24 [6] (with B-maps replaced by N-maps). But we need to insert some amendments in their statements. The assertions 2 and 3 of Lemma 20.3 [6] have now the following form: 2) if a subpath p of a smooth section of the first type of rank j (respectively, of a smooth section of the second type) of a contour of a N-map is a section of a contour of a submap Γ, then p is a smooth section of the first type of rank j (respectively, a smooth section of the second type) in $\partial\Gamma$; 3) if a subpath p of a long section of the first type of rank j or of a nonspecial section of rank j of a cell Π (respectively, of a special section of a cell Π of the second type of rank j) is a section of a contour of a submap Γ in a N-map and Π does not occur in Γ, then p is a smooth section of the first type of rank j (respectively, a smooth section of the second type of rank j) in $\partial\Gamma$. In the statement of Lemma 20.4 [6] we require that q is a long section of a cell Π of the first type and add the phrase «and if q is a long section of a cell Π of the second type, then $2|q| < |\partial\Pi| \leqslant 4|q|$». We claim that assertion 1) of Lemma 21.1 [6] is also true for a nonspecial section q_1' of a cell of the second type, and this lemma has now the additional assertion: 3) $|p_1| = |p_2| = 0$ if q_1' is a special section of a cell of the second type or a smooth section of the second type. In the statements of Lemmas 21.2 [6], 21.4 [6], 21.10-21.16 [6], 23.7-23.11 [6], the words «a long section of a cell» should be replaced by «a special or long nonspecial section of a cell of the second type or a long section of the first type». The conclusion of Lemma 21.16 [6] is true if q_2 is a short section of a cell of the first type. The conclusions of Lemmas 22.3 [6], 22.4 [6] and 23.17 [6] should be corrected as in Lemmas 40.19-40.21 [6], respectively.

In the definitions of a C-map (see [6, pp. 244, 245]) and of a D-map (see [6, p. 257]) we demand that every section of the first kind (respectively, every long section) is a smooth section of the first type of rank j and consider the following additional cases.

By a *C-map* we also understand a circular or annular N-map Δ whose contours have the form $p_1 t_1 t_2 s_1 t_3 \dots t_{l+1} s_l t_{l+2} t_{l+3} p_2 q$, $1 \leqslant l \leqslant 4$, (in the case of a circular map) or $s_1 t_1 \dots s_l t_l t_{l+1}$, $l = 1, 2$ or $l = 4$, and q (in the annular case), where $s_1, \dots, s_l$ are called *long sections of the first kind*, $t_1, \dots, t_{l+3}, p_1, p_2$ *short sections*, and q a *long section of the second kind*; all sections are assumed reduced and, for some j, the following conditions hold.

C1′. Every long section of the first kind is a smooth section of rank j and $|s_1|, \dots, |s_l| > n^2 j$.

C2′. The length of one of the long sections of the first kind is greater than $n^4 j$.

C3′. The long section q of the second kind is either smooth or geodesic.

C4′. Every short section is either a smooth section of rank j or geodesic.

C5′. The length of any short section is not greater than $n^2 j$.

C6′. The same as C7 (see [6, p. 245]).

By a *D-map* we also mean a N-map $\varDelta$ on a sphere with one, two or three holes and with contours q_1, q_2 and q_3 (q_1 and q_2 or only q_1), where for each $r \in \{1, 2, 3\}$, either $|q_r| = 1$, or $q_r = s_0 t_0 \dots s_l t_l$, $l \leqslant 3$, $q_r = q_r' q_r'' q_r'''$, $s_0, \dots, s_l$ and q_r', q_r'' are called *long sections*, $t_0, \dots, t_l$ and q_r''' are called *short sections*, and for some j, the following conditions are satisfied.

D1′. The length of one of the contours is greater than 1.

D2′. Every long section is a smooth section of rank j.

D3′. The length of one of the long sections $s_0, \dots, s_l$ is greater than $\zeta n j$ and $|q_r'| > n^6 j^2$.

D4′. The short sections $t_0, \dots, t_l$ and q_r''' are geodesic and the length of every short section t_s, $0 \leqslant s \leqslant l$, (respectively, of q_r''') is less than $\max(dj, \iota |s_0|^{1/3})$ (respectively, than dj).

The changes in the definition of a C-map imply the corresponding corrections in the statements of Lemmas 23.1 [6] and 23.2 [6]. Lemmas 23.18-23.20 [6] and 24.3-24.5 [6] are stated only for C-maps satisfying conditions C1-C7 (see [6, p. 245]) and for D-maps with conditions D1-D6 (see [6, p. 257]), respectively.

Now we fix these alterations.

LEMMA 3.1. *With the alterations mentioned above, all the results in* §§ 20-24 [6] *are true for N-maps.*

PROOF. All the proofs of the listed results work in the case of N-maps if we use the definition of a N-map and make the following changes.

1) In the proof of Lemma 21.6 [6], it follows from N5 and the definition of a smooth section of the second type that q is not a special section of a cell of the second type or a smooth section of the second type. If q is not geodesic in $\partial\varDelta$ or it is not a short section of a cell of the first type, then by N7, N1, N3 and F3, we have that $q_2 = t_1 t_2 t_3$, where $|t_1|$, $|t_3| < \iota |t_2|$, and $t_2 = t' t''$ such that t' is a geodesic path and $|t''| < \gamma |t'|$. Hence

$$|t'| \leqslant |t_1^{-1} s (t'' t_3)^{-1}| < |s| + (\gamma + 4\iota)|t'|$$

and

$$|q_2| < (1 + \gamma + 4\iota)|t'| < (1 + \gamma + 4\iota)(1 - \gamma - 4\iota)^{-1} |s| < (1 + 3\gamma)|s| .$$

2) As in [6, p. 232], we introduce the weight function on the edges of a N-map $\varDelta$. If q is a long section of any ordinary or special cell of the

first type or q is a special or long nonspecial section of any ordinary or special cell of the second type, then for any edge e in q we set $\nu(e) = |q|^{-1/3}$. The weights of all other edges in Δ are assumed to be zero. The weight of a path, a cell, or a submap is defined as in [6, p. 232].

3) In the proof of Lemma 21.8 [6], we note that any long section of a cell of the second type is either special or long nonspecial. It follows from N5 that p is not a special section of a cell of the second type. Hence by N1, N4 and N6, if p is not a short section of a cell of the first type, then $|t_2| < (1+\gamma)j$.

4) In the proof of Lemma 21.13 [6], it follows from N4 that Π_1 is a cell of the first type. If Π_2 is a cell of the second type, then by N2, N8 and the definition of a long nonspecial section of a cell of the second type, we have that

$$\nu(q_1) \leqslant |q_1|^{2/3} < (6h\iota|q_2|)^{2/3} = (6h\iota)^{2/3}\nu(q_2) < (6h\iota)^{2/3}\nu(\Pi_2).$$

5) In the proof of Lemma 21.17 [6], we also consider a distinguished contiguity submap Γ of a short section q_1' of a cell Π of the second type of rank j with $|q_1'| \leqslant n^2 j$ to q_2', where q_2' is a special or long nonspecial section of a cell of the second type or a long section of the first type. Set $\partial(q_1', \Gamma, q_2') = p_1 q_1 p_2 q_2$, and let E_1 denote the sum of all the $\nu(q_2)$ as Γ runs through all such submaps in Δ. By Lemma 21.1 [6] for Γ, we have that

$$|q_2| < (\beta')^{-1}(4h\varepsilon^{-1}+1)|q_1|.$$

By summing over all Γ for short section q_1' of a fixed cell Π of the second type of rank j, where $|q_1'| \leqslant n^2 j$, and using N8 and N9, we deduce the following estimate for the corresponding part E_Π of the total sum E_1:

$$E_\Pi \leqslant 4(\beta')^{-1}(4h\varepsilon^{-1}+1)n^2 j < \eta(|s_1|^{1/2} + |s_2|^{1/2}),$$

where s_1 and s_2 are special sections of Π. By the definition of the weight function, we have that $E_\Pi < \eta\nu(\Pi)$, whence $E_1 \leqslant \eta\nu(\Delta)$.

6) The weights of the edges belonging to the contours of cells in a C-map Δ is now defined as in 2).

7) In the proof of Lemma 23.9 [6] we note that, by F2 and the definition of a smooth section of the second type, Π is a cell of the first type.

8) In the proof of Lemma 23.12 [6], one more possibility arises when p is a short section of a cell Π of the second type of rank k and $|p| \leqslant n^2 k$. It follows from Lemma 21.1 [6] and Theorem 22.4 [6] that

$|q_2| < (\beta')^{-1}(4h\varepsilon^{-1}+1)n^2k < 5h\varepsilon^{-1}n^2k$. Hence

$$\nu(q_2) \leqslant |q_2|^{2/3} < (5h\varepsilon^{-1}n^{-4}k^{-1})^{2/3}(n^6k^2)^{2/3} < \iota^2\nu(\Pi).$$

9) In the proof of Lemma 23.13 [6], if a C-map Δ satisfies conditions C1′-C5′, then as in 8), we have that $\nu(q_2) < (5h\varepsilon^{-1}n^2k)^{2/3}$, and by C2′ and C5′,

$$F/\nu(\Delta) < 9(5h\varepsilon^{-1}n^2k)^{2/3}/(n^4k)^{2/3} < \iota.$$

10) In the proof of Lemma 23.16 [6], if Π is a cell of the second type, then it follows from N4 that the sum of the lengths of the contiguity arcs of the form $\Gamma_k' \wedge s_i$ is less than $4\varepsilon\sum |s_i|$.

11) In the proof of Lemma 24.2 [6] for D-maps with conditions D1′-D4′ we should use the argument in the proof of Lemma 5 [7].

4. – Main lemmas.

We start this section with

LEMMA 4.1. *Suppose that C is a finite non-dihedral subset of Ω such that $C = C^{-1}$ and $C \not\subseteq G_\mu$ for each $\mu \in I$. Then there is a sequence $Q_{1,C}, Q_{2,C}, \ldots$ of non-empty 7-aperiodic reduced words over the alphabet Ω with $|Q_{j+1,C}| = |Q_{j,C}| + 2$ and $V(\{Q_{j,C}\}) = C$ for all $j = 1, 2, \ldots$.*

PROOF. Let W be a 6-aperiodic word in a 2-letter alphabet $\{x, y\}$. By the statement of the lemma, there exists $\mu \in I$ such that $C \cap G_\mu = \{a_1, \ldots, a_t\} \neq \emptyset$, $C \backslash G_\mu = \{b_1, \ldots, b_s\} \neq \emptyset$ and $t + s > 2$. Then putting $x \to \prod_{j=1}^{t}(a_j b_1 \ldots a_j b_s)$, $y \to a_1 b_s$ yields a word Q which is at least 7-aperiodic relative to Ω. It is clear that $V(\{Q\}) = C$. Thus our assertion follows from Theorem 4.6 [6].

LEMMA 4.2. *Put $S \equiv S_{j,C}aS_{j,C}^{-1} \equiv U_1VU_2$, where $|V| \geqslant \eta\iota|S|$, $a \in \Omega$ and C is a finite non-dihedral subset of Ω such that $C = C^{-1}$ and $C \not\subseteq G_\mu$ for each $\mu \in I$. Then* 1) *if $V^{\pm 1}$ is a subword in $S' \equiv S_{l,C_1}bS_{l,C_1}^{-1}$, then $j = l$ and $C = C_1$;* 2) *and* 3) *are as assertions* 3) *and* 4) *in the statement of Lemma* 41.1 [6]; 4) *S is a 200-aperiodic word.*

PROOF. We can repeat the proof of Lemma 41.1 [6]. The last assertion of the lemma follows from the definition of the words $S_{j,C}$ (see § 2).

Lemmas 4.3 to 4.21 are verified simultaneously by induction on the rank which, thanks to Lemmas 4.19-4.21, enables us to use the results on N-maps in the previous section.

Lemmas 4.3-4.6 are stated and proved in exactly the same way as Lemmas 41.4 [6], 41.5 [6], 26.1 [6] and 26.2 [6], respectively, if in the statements of Lemmas 26.1 [6] and 26.2 [6] we replace the words «of the second type» by «of the first type» and «of the form (2) in § 25» by «of the form (2.1)-(2.8)». The statements of Lemmas 4.7-4.12 are the same as for Lemmas 25.5-25.10 [6] with the corrections from [7], respectively, but in Lemmas 4.7 and 4.12 we also must add the condition $|t| \leqslant |A|$ for the coordinating path t. Lemmas 4.7 to 4.15 are proved for a fixed $i \geqslant 0$ by simultaneous induction on the sum L of the lengths of the periods. In the proof of Lemma 4.7 we also require one addendum to the proof of Lemma 25.5 [6], the notation of which will be used below.

In order to apply Lemma 22.1 [6] in part 3) of the proof of Lemma 25.5 [6], we must show that there are no self-compatible cells in Δ'.

Suppose that Δ' has a self-compatible cell Π. First of all we may assume that such a cell Π is unique in Δ', since otherwise let Π_1 and Π_2 be self-compatible cells in Δ' corresponding to the relators (2.10) for $B_1^{t_1}$ and $B_2^{t_2}$, where B_r is a non-dihedral period of rank $i_r \leqslant i$ and t_r is a nonzero integer, $r = 1, 2$. Hence $XB_1^{mt_1}X^{-1} = B_2^{-mt_2}$ in some rank $k \leqslant i$ for a word X and an integer m such that $2|X| < \iota m$ and $m \geqslant \max(r(B_1), r(B_2))$. Then by Lemma 4.15 (which can be applied, since it follows from Lemma 23.17 [6] for Δ and the definition of relators of the second type that $L' = |B_1| + |B_2| < L = 2|A|$), $B_1 \equiv B_2$ and $X = SB_1^l$ in $G(i)$, where either $S = 1$ in $G(i)$ or S is the subword $S_{j+p-1,C}\,aS_{j+p-1,C}^{-1}$ of the relator (2.9) for B_1. It is obvious that $|t_1| = |t_2|$. The diagram Δ' splits into three annular diagrams Δ_1, Δ_2 and Δ_3, where Δ_2 consists of the cells Π_1 and Π_2 and the annular subdiagram Δ_4 of Δ' with contours corresponding to nonspecial sections of Π_1 and Π_2 such that Δ_4 does not contain these cells. Now excising the subdiagram Δ_2 (in the case when $t_1 = -t_2$) or Δ_4 together with Π_2 (if $t_1 = t_2$) from Δ' and pasting together the corresponding contours of Δ_1 or of Π_1 and Δ_3, we obtain (after cancellation of j-pairs) a reduced annular diagram Δ_1' with $\tau(\Delta_1') < \tau(\Delta')$ whose contours are the same as the contours of Δ'. Thus we may assume that Δ' contains the only self-compatible cell.

The diagram Δ' splits into three annular diagrams Δ_1, Δ_2 and Δ_3, where Δ_2 is the cell Π, and the contours u and v of Δ_2 have labels of the form $B^{\pm k}$. By Lemma 4.20 and the reducibility of Δ', the contours u and v of the cell Π are smooth sections in Δ'. We can join a vertex o_1 of the path p_1 to a vertex o_2 on q' by a path $w_1 w_2 w_3$, where w_2 is a special section of Π and by Lemma 22.1 [6], $|w_1| < (1/2 + \gamma)(|p_1| + |u|)$ and

$|w_3| < (1/2 + \gamma)(|v| + |p_2|) + \gamma|q'|$. By Theorem 22.4 [6] and the definition of relations (2.9) and (2.10), we have that $|v| = |u| < (\beta')^{-1}|p_1|$, and applying the estimates for $|q'|$, $|p_1|$ and $|p_2|$ (see part 3) in the proof of Lemma 25.5 [6]), we deduce that

$$|w_1| < (1/2 + \gamma)(1 + (\beta')^{-1})\alpha|A| < \delta|q_2| \tag{4.1}$$

and

$$|w_3| < \gamma((\beta')^{-1} - 1)|q_2| + (\gamma + (1/2 + \gamma)(1 + (\beta')^{-1})\alpha)|A| < 2\beta\gamma|q_2|. \tag{4.2}$$

By Lemma 23.17 [6], the perimeter of every cell in Δ is less than $\gamma^{-1}|A|$. We note that in the course of cancelling cells Π_1 and Π_2 of the second type in Δ', they may be replaced by a cell π with longer perimeter, but the special sections of π are of the same length as those of Π_1 and Π_2. Thus

$$|w_2| < \gamma^{-1}|A|. \tag{4.3}$$

The labels of the paths $w_1^{-1}q_2$ and $w_2 w_3$ are related in $G(i)$, by Lemma 11.4 [6], by an equation of the form $\phi(w_1^{-1}q_2) = \phi(u)^l\phi(w_2)\phi(w_3)$, where l is an integer, that is, we obtain a reduced circular diagram Δ_4' of rank i with contour $s_1 t_1 s_2 t_2$, where $\phi(s_1) = \phi(w_1)$, $\phi(t) = \phi(u)^l$, $\phi(s_2) = \phi(w_2)\phi(w_3)$ and $\phi(t_2) = \phi(q_2^{-1})$. By (4.1)-(4.3), we have that

$$|s_1| < \delta|t_2| \tag{4.4}$$

and

$$|s_2| < \gamma^{-1}|A| + 2\beta\gamma|t_2| < 3\beta\gamma|t_2|. \tag{4.5}$$

Excising cells of the second type with long nonspecial sections compatible with t from Δ_4', we obtain a diagram Δ_4 with contour $s_1 t_1 s_2 t_2$ in which, by Lemma 4.20, t_1 and t_2 are smooth sections of ranks $|B|$ and $|A|$, respectively. It follows from Theorem 22.4 [6] and (4.4), (4.5) that

$$|t_1| > \beta'|t_2| - |s_1| - |s_2| > (1 - 2\beta)|t_2| \tag{4.6}$$

and

$$|t_1| < (\beta')^{-1}(|t_2| + |s_1| + |s_2|) < (\beta')^{-1}(1 + \delta + 3\beta\gamma)|t_2| < (1 + \alpha)|t_2|. \tag{4.7}$$

By (4.3) and the definition of relations (2.9) and (2.10), we also obtain that

$$|B| < \iota^6 |w_2| < \gamma^{-1} \iota^6 |A| < \iota^5 |A|. \tag{4.8}$$

By Lemma 22.5 [6], we can excise from Δ_4 a subdiagram Δ_5 with contour $s_1' t_1' s_2' t_2'$, where t_1' and t_2' are subpaths in t_1 and t_2, respectively, such that $|s_1'|$, $|s_2'| < \alpha |B|$ and $|t_j'| > |t_j| - M$, $j = 1, 2$, where $M = \gamma^{-1}(|s_1| + |s_2| + |B|)$. Now we deduce from (4.4)-(4.8) that

$$|t_2'| > (1 - \gamma^{-1}(\delta + 3\beta\gamma + \iota^5)) |t_2| > 2h|A|/3 \tag{4.9}$$

and

$$|t_1'| > (1 - 2\beta - \gamma^{-1}(\delta + 3\beta\gamma + \iota^5)) |t_2| > 2h|A|/3 > n^5 |B|. \tag{4.10}$$

Now we note that if Δ_4' has a cell of the second type with a long nonspecial section compatible with t, then $|t_1| > 2r(B)$, and it follows from (4.7) and (4.10) that

$$|t_1'| > (1 - \alpha)(1 + \alpha)^{-1} |t_1| > r(B). \tag{4.11}$$

Finally, by applying Lemmas 4.14 and 4.13 to Δ_5 (it is possible since $L' = |B| + |A| < L = 2|A|$, by (4.8)), we deduce from (4.9)-(4.11) that $A \equiv B$. This contradicts the simplicity of A in rank i and the inequality $|B| < |A|$.

Now we should explain the possibility of use of Lemmas 4.7-4.12 for the study of contiguity submaps in the diagrams under consideration and complete their proofs, since Lemma 4.20 gives a complicated form for smooth sections of the first type. For this purpose we introduce E-diagrams.

By an *E-diagram* we understand a reduced circular diagram Δ with contour $p_1 q_1 p_2 q_2$, where $q_r = s_{0r} t_{1r} s_{1r} \dots t_{k(r)r} s_{k(r)r} t_{k(r)+1r} s_{k(r)+1r}$, $r = 1, 2$, such that

E1. Δ is a N-map;

E2. the label of t_{l1} is an A_1-periodic word, where $A_1^{\pm 1}$ is either a period of rank i or a simple word in rank i (possibly $|t_{l1}| = 0$);

E3. the label of t_{l2} is an A_2-periodic word, where $A_2^{\pm 1}$ is a period of rank $k \leq i$ (possibly $|t_{l2}| = 0$);

E4. q_1 and q_2 are smooth sections of the first type in Δ;

E5. $|q_2| > \zeta n k$;

E6. $r(\Delta) < k$;

E7. if $|s_{l1}| > 0$ for some $l \in \{0, 1, \ldots, k(1) + 1\}$, then $A_1^{\pm 1} \in P_i' \cap \cap N$ and $\phi(s_{l1})$ is a subword of S_1, where S_1 is the subword $S_{j+p-1,C}aS_{j+p-1,C}^{-1}$ of the relator (2.9) for the period $A_1^{\pm 1}$;

E8. if $|s_{l2}| > 0$ for some $l \in \{0, 1, \ldots, k(2) + 1\}$, then $A_2^{\pm 1} \in P_k' \cap \cap N$ and $\phi(s_{l2})$ is a subword of S_2, where S_2 is the subword $S_{j+p-1,C}aS_{j+p-1,C}^{-1}$ of the relator (2.9) for the period $A_2^{\pm 1}$;

E9. $|s_{lr}| = |S_r|$ for each $l \in \{1, \ldots, k(r)\}$ $(r = 1, 2)$;

E10. there is no a contiguity submap of a section of q_r to a distinct section of q_r, $r = 1, 2$;

E11. for each indexes $l \in \{1, \ldots, k(2) - 2\}$ and $s \in \{1, \ldots, k(1) + + 1\}$, there are no contiguity submaps $\Gamma_p, p = 0, 1, 2$, of s_{l+p2} to a section t_{s1} such that $(s_{l+p2}, \Gamma_p, t_{s1}) > 1 - \alpha$;

E12. $|p_r| < dk$ for each $r = 1, 2$.

LEMMA 4.13. *In an E-diagram Δ, we have that* 1) *if* $k(1) > 0$, *then* $k(2) > 0$, $A_1 \equiv A_2^{\pm 1}$ *and the exist sections* s_{l1} *and* s_{t2} *compatible in* Δ, *where* $l \in \{1, \ldots, k(1)\}$ *and* $t \in \{1, \ldots, k(2)\}$; 2) *if* $k(1) = 0$ *and* $k = i$, *then* $k(2) = 0$, *and if, in addition,* $|t_{12}| > \eta nk$, *then* $A_1 \equiv A_2^{\pm 1}$, *and it follows from* $A_1 \equiv A_2^{-1}$ *(respectively, from* $A_1 \equiv A_2$*) that* t_{11} *and* t_{12} *are* A_1*-compatible in* Δ *(respectively,* A_1*-anticompatible in* Δ *and* A_1 *is a product of two involutions in rank* $|A_1| - 1$); 3) *if* $k = i$, $A_2^{\pm 1} \in P_k' \cap N$ *and* $|q_2| > \iota r(A_2^{\pm 1})$, *then there exist sections of* q_1 *and* q_2 *compatible in* Δ; 4) *if* $k < i$ *and* $q_2 = t_{12}$, *then* $|q_1| < 2i$ *and* $|s_{01}|$, $|s_{11}| < \eta |t_{11}|$; 5) *if* $k < i$ *and* $q_1 = t_{11}$, *then* $|q_1| < 1000i$; 6) *if* $k < i$, $A_2^{\pm 1} \in P_k' \cap N$, $|q_2| > > \alpha r(A_2^{\pm 1})$ *and* $q_1 \neq t_{11}$, *then* $|q_1| < \iota r(A_1^{\pm 1})$ *and* $|s_{01}|$, $|s_{11}| < \iota |t_{11}|$.

PROOF. We proceed by induction on $\tau(\Delta)$. All sections s_{lr} and t_{lr}, $r = 1, 2$, are called *long*, and the definitions of the distinguished contiguity submaps and of the weight function in Δ are exactly the same as those in the case of a D-map. Now we can obtain the assertion of Lemma 24.1 [6] for Δ, but we require the following addendum to the proof of this result. The important role in the proof of Lemma 24.1 [6] is played by the fact that the number of long sections in $\partial\Delta$ is bounded by a constant $(2(h + 2))$. Suppose that Δ contains a D-cell π with a section p such that there exist contiguity submaps of p to 10 distinct long sections of $\partial\Delta$. Then we obtain a subdiagram Γ of Δ with contour $l_1 f_1 l_2 f_2$, where f_1 and f_2 are subpaths of q_r for some $r \in \{1, 2\}$ and p, respectively, f_1 contains a subpath t of some section s_{lr} such that $|t| > |S_r|/3$ and $|l_1|$, $|l_2| < 2h\varepsilon^{-1}|p|$, by Lemma 21.1 [6]. If π is a cell of the first type, then by E6 and the definition of relators of the first type, we have that $|p| < 6hnk$, and we arrive at a contradiction to Theorem 22.4 [6] and the inequality $|S_r| > n^6 k^2$. If π is a cell of the second type, then, again by Lemma 21.1 [6] and E6, $|l_1|$, $|l_2| < dk$, and hence Γ is an E-diagram with $\tau(\Gamma) < \tau(\Delta)$. Now the case when $|f_2| \leqslant \alpha |S_r|$ is impossible, by Theorem 22.4 [6], and we arrive at a contradiction to the induction hypothesis and the inequality $r(\pi) < k$ (by E6). Thus if p is a section of a cell in Δ, then there are contiguity submaps of p to at most 9 distict long sections of $\partial\Delta$, but that is all we need for the proof of the assertion of Lemma 24.1 [6] for Δ.

Suppose that long sections t_{lr} and s_{lr} contain $\eta_{lr}|t_{lr}|$ and $\theta_{lr}|s_{lr}|$ outer edges. Then by Lemma 24.1 [6], we have that $L_1' + L_2' > (1 - \delta^{1/2})(L_1 + L_2)$, where $L_1' = \sum \eta_{lr}|t_{lr}|^{2/3}$, $L_2' = \sum \theta_{lr}|s_{lr}|^{2/3}$, $L_1 = \sum |t_{lr}|^{2/3}$ and $L_2 = \sum |s_{lr}|^{2/3}$.

Consider the case when $k(1) > 0$. We distinguish the subset J of the index set such that $|t_{lr}| \leqslant n^3|A_r|$ for each $(l, r) \in J$. It is obvious, by the definition of long sections, Theorem 22.4 [6] and E9, that

$$L_3' = \sum_{(l,\, r) \in J} \eta_{lr}|t_{lr}|^{2/3} < \iota L_2 ,$$

hence

$$L_4' + L_2' > (1 - \delta^{1/2} - \iota)(L_4 + L_2), \tag{4.12}$$

where $L_4' = L_1' - L_3'$ and $L_4 = \sum_{(l,\, r) \notin J} |t_{lr}|^{2/3}$. Then there exists an index (l, r) such that either $\theta_{lr} > 1 - \alpha$ or $\eta_{lr} > 1 - \alpha$ and $(l, r) \notin J$. Consider these cases.

a) In the first case, by E10, there is a contiguity submap Γ of s_{lr} to q_{3-r} such that $(s_{lr}, \Gamma, q_{3-r}) > 1 - \alpha$. It follows from Corollary 22.1 [6], Lemma 4.21 and the definition of a smooth section of the second type that $r(\Gamma) = 0$. Suppose that $|s_{lr}| \geqslant \iota|S_r| > 400i$. Then by Lemma 4.2 and the inequality $|s_{lr}| \geqslant \iota|S_r| > 400i \geqslant 400k$, we have that there exists a section s_{t3-r} for some index t such that s_{lr} and s_{t3-r} are compatible in Δ. By Lemma 4.2, we also have that $A_1 \equiv A_2^{\pm 1}$. If $l = 0$ or $l = k(r) + 1$, then it is obvious that $t = 0$ or $t = k(3 - r) + 1$, respectively, and we can proceed by induction on the number of long sections in $\partial\Delta$.

b) Suppose that $\theta_{l1} > 1 - \alpha$ and $|s_{l1}| < \iota|S_1|$. Then there exists a contiguity submap Γ of s_{l1} to q_2 such that $r(\Gamma) = 0$. It follows from Lemma 4.2 and (4.12) that we can neglect the value of $|s_{l1}|^{2/3}$ in our considerations.

c) Suppose that $\theta_{l2} > 1 - \alpha$ and $|S_2| \leqslant 400ni$. We show, by induction on $k(2)$, that there exists an index (l, r) such that either $\theta_{lr} > 1 - \alpha, r = 1$ and $|s_{lr}| \geqslant \iota|S_1|$ or $\eta_{lr} > 1 - \alpha$ and $(l, r) \notin J$. By (4.12) and *b*), this assertion is obvious when there are no indexes $l_1, l_2, l_3 \in \in \{1, \ldots, k(2)\}$ such that $\theta_{l_1 2}, \theta_{l_2 2}, \theta_{l_3 2} > 1 - \alpha$ (in particular when $k(2) < 3$). But if such indexes exist, then there are contiguity submaps Γ_t, $t = 1, 2, 3$, of $s_{l_t 2}$ to q_1 such that $(s_{l_t 2}, \Gamma_t, q_1) > 1 - \alpha$ and $r(\Gamma_t) = 0$, and by E11 and Lemma 4.2, two of these contiguity submaps define a contiguity submap Γ of q_2 to q_1 which is an E-diagram with $k(1) > 0$ and a smaller number $k(2)$, and by induction hypothesis, our assertion is true for Γ and therefore for Δ.

d) Now we consider the second case when there is a contiguity submap Γ of t_{lr} with $|t_{lr}| > n^3 |A_r|$ to q_{3-r} such that $(t_{lr}, \Gamma, q_{3-r}) > 1 - \alpha$. If $r = 2$, then by Lemma 21.1 [6] and the previous considerations, we can apply Lemma 23.14 [6] to Γ (with $\Gamma \wedge t_{l2}$ is the long section of the second kind) and obtain, using Lemma 4.2 and Theorem 22.4 [6], that $\Gamma \wedge q_1$ contains a subpath t of a section t_{s1} for some index s with $|t| > |\Gamma \wedge q_1|/2$. Then by Lemma 4.12 and the definition of the sets of periods in G, either $A_1 \equiv A_2^{\pm 1}$ and it follows from $A_1 \equiv A_2^{-1}$ (respectively, from $A_1 \equiv A_2$) that t_{s1} and t_{l2} are A_1-compatible in Δ (respectively, A_1-anticompatible in Δ and A_1 is a product of two involutions in rank $|A_1| - 1$), or $|\Gamma \wedge q_1| < 2i, k < i$ and we can neglect in (4.12) the weight of the edges in $\partial\Gamma$. But in the first case, it follows from $k(1) > 0$ that $A_1 \equiv A_2^{-1}$ and Δ consists of three subdiagrams Δ_1, Γ and Δ_2, and proceeding by induction on the number of long sections in $\partial\Delta$ and using Lemma 4.2, we obtain the assertion of the lemma.

Let $r = 1$. If $k = i$, then we can repeat the previous considerations in the case when $r = 2$. Suppose that $k < i$. Using Lemma 21.1 [6], we again can apply Lemma 23.14 [6] and (4.12) to the contiguity submap Γ, where $\Gamma \wedge t_{l1}$ is a long section of the second kind. Now we note that, by Lemmas 4.12 and 4.2, if Γ' is a contiguity submap of a section s of q_2 to t_{l1} such that $|\Gamma' \wedge s| > \zeta^{-1}|A_2|$, then $|\Gamma' \wedge t_{l1}| < 200i$. Using this fact, Lemma 21.1 [6], Theorem 22.4 [6], (4.12) and E11, we arrive at a contradiction to the inequality $|t_{l1}| > n^3 i$.

Thus the assertion of the lemma is proved in the case $k(1) > 0$. If $k(1) = 0$, then $0 \leqslant k(2) \leqslant 2$ (by Lemma 4.2 and E11), and the remaining assertions of the lemma follow easily from assertion 1 of the lemma, Lemmas 23.15 [6], 4.12 and 4.2 and Theorem 22.4 [6].

LEMMA 4.14. *Let Δ be a reduced diagram which is a N-map, t_1 and t_2 subpaths of some contours of Δ such that the label of t_r is an A_r-periodic word, where $A_r^{\pm 1}$ is a period of rank i_r, $i_2 \leqslant i_1 \leqslant i$, and also let the diagram Δ' and the sections t_1' and t_2' of some contours of Δ' be obtained from Δ, t_1 and t_2, respectively, by excising all cells of the second type having long nonspecial sections compatible with t_1 or t_2. If Γ is a* *circular subdiagram of Δ' with contour $p_1 q_1 p_2 q_2$, where q_r is a subpath of t_r, $|p_r| < di_2$, $r = 1, 2$, and $|q_2| > \zeta n i_2$, then Γ is an E-diagram (with the standard partition of the contour) such that $k(1) = 0$.*

PROOF. We use the notation in the definition of an E-diagram. Conditions E1-E3, E5, E7-E9 and E12 follow from the statement of the lemma, condition E4 is implied by Lemma 4.20 and the statement of the lemma and condition E6 follows from Lemma 23.17 [6]. We also derive E10 from the reducibility of Δ and the argument in the proof of condition H7 in Lemma 4.19. The nonfulfilment of E11 implies the existence of two special sections s_{l2} and s_{l+12} of a cell π of the second type such that there are the maximal contiguity submaps Γ_0 and Γ_1 of s_{l2} and s_{l+12} to a section t_{s1} of q_2 with $r(\Gamma_p) = 0$ and $(s_{l+p2}, \Gamma_p, t_{s1}) > 1 - \alpha$, $p = 0, 1$. If $|s_{l2}|/2 < |t_{l+12}|$, then by Lemma 23.15 [6], we obtain a contiguity submap Γ_2 of t_{l+12} to t_{s1} with $(t_{l+12}, \Gamma_2, t_{s1}) > 1 - \alpha$. In any case, using Lemma 23.15 [6], we obtain that there exists a contiguity submap Γ' of π to t_{s1} in Δ such that $(\pi, \Gamma', t_{s1}) > 1/2$ and Γ' can be decomposed into Γ_t, $t = 0, 1, 2$, and at most two subdiagrams whose perimeters are small compared to the perimeters of Γ_0 and Γ_1, where Γ_2 is the maximal contiguity submap of t_{l+12} to t_{s1}. By the definition of the set P_{i_1}, $|\Gamma' \wedge \wedge t_{s1}| > i_1$, and moreover, by the arguments in the proof of Lemma 4.13, $\Gamma_2 \wedge t_{s1}$ does not contain a subpath t such that the label of t is a subword of S_2 and $|t| > \iota |S_2|$. Repeating the argument in the proof of Theorem 26.2 [6] and using the definition of the relators of the second type, we have that A_1 is conjugate in rank $< i_1$ to the subword $S_2 A_2^l$ of the relator of the second type corresponding to π, which contradicts the definition of P_{i_1}.

Thus Γ is an E-diagram, and if $k(1) > 0$, then by Lemma 4.13, $k(2) > 0$ and there exist sections s_{l1} and s_{t2} compatible in Δ, which contradicts the reducibility of Δ.

LEMMA 4.15. *Suppose that $Z_1 A_1^{m_1} Z_2 = A_2^{m_2}$ in $G(k)$ and $m = \min(|m_1|, |m_2|)$, where A_r is a period of rank i_r, $r = 1, 2$, $i_2 \leqslant i_1 \leqslant i$ and $k \geqslant 1$. If $|Z_1| + |Z_2| < \iota m$ and $m \geqslant \max(r(A_1), r(A_2))$, then $A_1 \equiv A_2$ and either $Z_r \in \langle A_1 \rangle$ in $G(s)$ or Z_r is equal in $G(s)$ to $SA_1^{l_r}$, $r = 1, 2$, where $s = \max(k, i_1)$ and either A_1 is a non-dihedral period from N and S is the subword $S_{j+p-1} a S_{j+p-1}^{-1}$ of the relator (2.9) for A_1, or A_1 is a product of two involutions X_1 and X_2 in rank $i_1 - 1$ and $S = X_1$.*

PROOF. Let Δ be a reduced circular diagram of rank k for the considered equality with contour $z_1 t_1 z_2 t_2$, where $\phi(z_r) \equiv Z_r$, $r = 1, 2$, and $\phi(t_1) \equiv A_1^{m_1}$, $\phi(t_2) \equiv A_2^{-m_2}$. It is a N-map, by Lemma 4.19. Excising cells of the second type with long nonspecial sections compatible with t_1 or t_2

from Δ, we obtain a diagram Δ' with contour $z_1 t_1' z_2 t_2'$, where $|t_r'| \geqslant |t_r|$, $r = 1, 2$. It follows from Lemma 22.5 [6] that Δ' contains a subdiagram Γ with contour $p_1 q_1 p_2 q_2$, where q_1 and q_2 are subpaths of t_1' and t_2', such that $|p_r| < \alpha i_2$ and

$$|q_r| > (1 - 2\gamma^{-1}\iota) m i_r > r(A_r) i_r / 2,$$

$r = 1, 2$. By Lemma 4.14, Γ is an E-diagram with $k(1) = 0$, and by Lemma 4.13, $A_1 \equiv A_2$ and there are subpaths of q_1 and q_2 which are A_1-compatible in Γ (respectively, A_1-anticompatible in Γ and $A_1 = X_1 X_2$ in rank $i_1 - 1$ for some involutions X_1 and X_2). Hence $Z_r \in \langle A_1, S \rangle$ (respectively, $Z_r \in \langle A_1 \rangle X_1 \langle A_1 \rangle$) in $G(s)$, $r = 1, 2$, and the assertion of the lemma follows from the fact that, by (2.9), $SA_1 = A_1^{-1} S$ (respectively, $X_1 A_1 = A_1^{-1} X_1$) in $G(i_1)$.

LEMMA 4.16. *Every word X is conjugate in rank $i \geqslant 0$ to A, where A is either a power of a period of rank $j \leqslant i$ or a power of a word which is simple in rank i, or an element of Ω^1, or the subword $S_{j+p-1,C}\, a S_{j+p-1,C}^{-1} A_p^l$ of a relator of the form* (2.10) *for a period A_p of rank $j \leqslant i$, and also in some diagram for the conjugacy of X and A^ε, where $|\varepsilon| = 1$ and the case when $\varepsilon = -1$ is possible only if A is a power of a non-dihedral period, no cells are self-compatible.*

PROOF. The first assertion of the lemma can be proved as Lemma 34.2 [6], taking into consideration the definition of a simple word in rank i.

Suppose that a diagram Δ for the conjugacy of X and A has a self-compatible cell Π with the label of a nonspecial section equal to E^l. By the argument in the proof of Lemma 4.7, we may assume that such a cell Π is unique in Δ. Then there exists a subdiagram Δ_1 of Δ for the conjugacy of E^l and A with $|\Delta_1(2)| < |\Delta(2)|$. It follows from the definition of a simple word, Lemma 4.20 and Theorem 22.4 [6] that A is a power of a period of rank $j \leqslant i$, and it follows from Lemma 4.15 (as it was shown in the proof of Lemma 4.7) that $E^l = A^{\pm 1}$ in rank i. Now a subdiagram Δ_2 of Δ for the conjugacy of X and E^{-l} is required.

LEMMA 4.17. *Let A be a period of rank $j \leqslant i$ and $B_1 = A^m T A^{2m} T A^{3m} T$, $B_2 = [A^m, T]$, $B_3 = TA^m$, where a word T and an integer m are chosen in such a way that $TAT^{-1} \neq A^{\pm 1}$ in $G(i)$, $|T| < \iota^2 (|m|\,|A|)^{1/3}$ and $|m| > n$. Then B_s, $s \in \{1, 2, 3\}$, is not conjugate in rank i to the subword $S \equiv S_{j+p-1,C}\, a S_{j+p-1,C}^{-1} A_p^l$ of a relator of the form* (2.10) *for a period A_p of rank $k \leqslant i$.*

PROOF. Assuming the contrary, we obtain a reduced annular diagram Δ' whose contours have the form $p' = t_1 s_1' t_2 \dots t_3 s_3' t_4$ and $q = q_1 q_2$, where $\phi(q_1) \equiv S_{j+p-1,C} a S_{j+p-1,C}^{-1}$, $\phi(q_2) \equiv A_p^{-l}$, $\phi(s_j') \equiv A^{\pm mj}$ or $|s_j'| = 0$, $j \in \{1, 2, 3\}$, and $\phi(t_r) \equiv T^{\pm 1}$ or $|t_r| = 0$, $r \in \{1, 2, 3, 4\}$ (it depends on the word B_s, $1 \leqslant s \leqslant 3$). We may assume, by Lemma 4.16, that Δ' has no self-compatible cells. Hence Δ' is a N-map, by Lemma 4.19. Excising cells of the second type with long nonspecial sections compatible with s_j', $j = 1, 2, 3$, from Δ', we obtain a diagram Δ from Δ' in which p' and all sections s_j' are replaced by p and s_j, $j = 1, 2, 3$, and by Lemma 4.20, the sections s_j are smooth in Δ. Moreover, $|s_j| \geqslant |s_j'|$, $j = 1, 2, 3$, and pasting some cells of the second type corresponding to relators (2.10) for some powers of A_p to the contour q, we may assume, by Lemmas 4.20 and 4.21, that the sections q_1 and q_2 are also smooth in Δ. Thus Δ is a D-map with long sections s_j, $1 \leqslant j \leqslant 3$, and q_1, q_2. It follows from Lemma 24.2 [6] that in Δ there is a system of regular pairwise disjoint contiguity submaps of long sections of the contours to long sections of the contours such that the sum of the lengths of the contiguity arcs of these submaps is greater than $\beta'(|s_1| + |s_2| + |s_3| + |q_1| + |q_2|)$.

If Γ is a contiguity submap of a section s_j to s_j in which at least one of the contiguity arcs has the length greater than $\zeta |s_j|$, then in the case $\phi(p') \equiv B_s$, $s \in \{1, 3\}$, we obtain, by Lemmas 4.14, 4.13, 24.2 [6] and 3 [7], that $TAT^{-1} = A^{\pm 1}$ in rank i, which contradicts the choice of T. If $s = 2$, then again using Lemmas 4.14 and 4.13, we have that A^k is conjugate in rank i to an involution for some integer k, and we arrive at a contradiction to Lemma 4.20 and Theorem 22.4 [6].

The existence of a contiguity submap Γ of s_{j_1} to a distinct long section s_{j_2}, $j_1, j_2 \in \{1, 2, 3\}$, in which $|\Gamma \wedge s_{j_1}| > \zeta |s_{j_1}|$ or $|\Gamma \wedge s_{j_2}| > \zeta |s_{j_2}|$ is impossible, since if $\phi(p') \equiv B_1$, then using Lemmas 4.14 and 4.13, we can repeat the argument in the proof of Lemma 3.5 [5], and in the case when $\phi(p') \equiv B_2$, it follows from Lemmas 4.14 and 4.13 that $TAT^{-1} = A^{\pm 1}$ in rank i, which contradicts the choice of T.

If Γ is a contiguity submap of q_1 to q_2, then, as in the proof of condition N7 in Lemma 4.19, $|\partial\Gamma| < 400|A_p|$.

Such small values do not affect the resulting estimates, thus we may assume that in Δ there are no contiguity submaps of s_{j_1} to s_{j_2}, $j_1, j_2 \in \{1, 2, 3\}$, and of q_1 to q_2. It follows from Theorem 22.4 [6], Lemmas 24.2 [6], 21.1 [6] and 4.2 and the definition of the words B_s, $1 \leqslant s \leqslant 3$, that for each $j \in \{1, 2, 3\}$, then is a contiguity submap of s_j to q such that the length of the contiguity arc of this submap to s_j is greater than $|s_j|/2$. Now if Γ is a contiguity submap of s_j to q_1, then by Corollary 22.1 [6], Lemma 21.7 [6] and N5, we have that $r(\Gamma) = 0$, and it follows from Lemma 4.2 that $|\Gamma \wedge s_j| < 200j$. Hence by Lemmas 4.14, 4.13 and

4.2, Theorem 22.4 [6] and Lemma 21.1 [6], we have that either there are subpaths of the sections s_j which are A-compatible with q_2 (of course, if $|s_j| > 0$) or $|s_j| < 10\iota|S|$, $j = 1, 2, 3$. In the first case, if $s = 1$ or $s = 2$, then $|s_2| > 0$ and $TAT^{-1} = A^{\pm 1}$, which contradicts the choice of T. If $s = 3$, then, by Lemmas 4.14 and 4.13, we again arrive at a contradiction to the choice of T, since $A \equiv A_p$. In the second case, it follows from Lemma 24.2 [6] and Theorem 22.4 [6] that there exist $r \in \{1, 2\}$ and a contiguity submap Γ of q_r to p such that $|\Gamma \wedge q_r| > |S|/2$, and using Lemma 21.2 [6], we arrive at a contradiction to Theorem 22.4 [6], which completes the proof of the lemma.

LEMMA 4.18. *The choice of the set of periods of the group G is correct.*

PROOF. Using Lemmas 4.13-4.17 and 4.19-4.21, we can repeat the proof of Lemma 3.1 [5]. But it might be suspected that a reduced annular diagram Δ for the conjugacy of $B_1 C_1^{m_1}$ and $(B_2 C_2^{m_2})^{-1}$ has a self-compatible cell Π, where C_j is a non-dihedral period of rank i_j not equal to A_0 and B_0, $|B_j| < \iota^2(m_j|C_j|)^{1/3}$, $m_j > n$, $B_j C_j B_j^{-1} \neq C_j^{\pm 1}$ in $G(i_j)$, $j = 1, 2$, and $B_1 C_1^{m_1}$, $(B_2 C_2^{m_2})^{-1}$ are conjugate to a period F. By Theorem 22.4 [6], Lemma 23.16 [6] and the proof of Lemma 27.3 [6], these elements are conjugate to F in rank $< |F|$. Hence we may assume that $r(\Delta) < |F|$. But it follows from the proof of Lemma 4.16 that Π is a cell of rank $|F|$, and we arrive at a contradiction to our assumption about $r(\Delta)$.

The following three lemmas will be proved simultaneously by induction on the type of the diagram.

LEMMA 4.19. *Any reduced diagram Δ of rank $i + 1$ on a disk, annulus or sphere with three holes, and without self-compatible cells in the last two cases, is a N-map.*

PROOF. Properties N2, N8 and N9 follow from the choice of relators.

The verifications of N1 and N7 for a cell π of the first type are the same as those for B1-B4 and B6-B10 in Lemmas 26.3 [6] and 26.4 [6].

In checking N3, let q be a subpath of a nonspecial section of a cell Π of the second type of rank j in Δ with $|q| \leqslant \max(j, 2)$, p a geodesic subpath in Δ homotopic to q. The case where Π does not occur in a submap Γ with contour $p^{-1}q$ is treated like condition B1 in Lemma 26.4 [6]. When Π is contained in Γ, we replace q by the path q_1 complementing q in $\partial\Pi$

and arrive at a submap Γ_1 with contour $p^{-1}q_1$ not containing Π (that is, a N-map by induction). We show that such a submap Γ_1 is impossible. To prove this by contradiction, we consider a putative counter-example with a minimal number of D-cells. It is obvious that Γ_1 is a C-map satisfying conditions C1′-C5′ in which the long section of the second kind is of the trivial length, since q_1 contains special sections of Π. We arrive at a contradiction to Lemma 23.15 [6] and condition N4.

We can verify N6 in the same way as B2 in the proof of Lemma 26.3 [6].

In checking N5 we may assume, by the reducibility of Δ and Lemmas 4.20, 4.21 and 4.2, that the contiguity arcs $\Gamma \wedge p$ and $\Gamma \wedge q$ are smooth in $\partial\Gamma$. Arguing by induction, we may also assume that Γ is a 0-contiguity submap. Hence it follows from Corollary 22.1 [6], Lemma 21.7 [6] and Lemma 4.2 that p and q are special sections of cells of the second type such that p and q are compatible in Δ, since $(p, \Gamma, q) \geqslant \varepsilon$. In the case when p and q are sections of distinct cells of the second type, these cells form a cancellable pair in Δ, contrary to the fact that Δ is reduced. If p and q are sections of the same cell of rank $j \leqslant i+1$, then a period A_p of rank j is of finite order in $G(i+1)$, and a diagram Δ' for the equation $A_p^k = 1$ is of smaller type than Δ. Now using Lemma 4.20, we arrive at a contradiction to Theorem 22.4 [6].

Verifying N4, we note that Γ is a N-map, since $\tau(\Gamma) < \tau(\Delta)$, and it follows from Lemmas 4.20 and 4.21 applied to Γ, condition N2 and the reducibility of Γ that q_1 and q_2 are smooth sections in $\partial\Gamma = p_1 q_1 p_2 q_2$. If p is a special section of a cell of the second type, then it follows from Corollary 22.1 [6], Lemma 21.7 [6] and N5 that $r(\Gamma) = 0$, and the desired inequality now follows from part 3) of Lemma 4.2. If p is a long nonspecial section of a cell of the second type of rank k, then applying Lemma 21.1 [6] to Γ, we obtain that $|p_1|, |p_2| < 2h\varepsilon^{-1}c$, where $c = \min(k, j)$. Then by Lemma 4.12 and the definition of the relations (2.9) and (2.10), either $|q_2| < (1+\gamma)j$ or q_1 and q_2 are compatible in Δ. But the second case is impossible, since if q_2 is a long section of the first type, then we arrive at a contradiction to N2 and the definition of a long nonspecial section, and the assumption that q_2 is a nonspecial section contradicts the reducibility of Δ.

Let Γ be a contiguity submap of p to q with $(p, \Gamma, q) \geqslant \varepsilon$, where p is a special or long nonspecial section of rank k, q is a section of a cell of rank j. It follows from N5 for Γ that q is not a special section of a cell of the second type. Again we may assume that q_1 and q_2 are either smooth or gedesic sections in $\partial\Gamma = p_1 q_1 p_2 q_2$, and by the first part of N4, it remains to consider the case when q_2 is a short section of a cell of the first type. Then by Theorem 22.4 [6], Lemma 21.1 [6] and the definitions of special and long nonspecial sections of rank k and of the relations in G, we have that

$$(\beta' \varepsilon n^2 - 4h\varepsilon^{-1})k < |q_2| < dj,$$

therefore $k < j$.

It remains to prove N7 when π is a cell of the second type of rank k in Δ. By induction, Γ is a N-map with $\partial\Gamma = p_1 q_1 p_2 q_2$. It follows from Lemmas 21.1 [6], 4.20 and 4.21 for Γ, the reducibility of Δ and the definition of the relators of the second type that Γ is a C-map with the long section q_2 of the second kind.

First we consider the case when Γ has the only long section of the first kind. Then there is a decomposition $q_1 = t_1 s_1 t_2$, where $|t_1|$, $|t_2| < 2n^2 k$ and s_1 is a subpath of a section of π with $|s_1| > n^6 k^2/2$. Let Γ_1 be the maximal contiguity submap of s_1 to q_2 (the existence of such a submap Γ_1 follows from Lemma 23.15 [6]). It is possible, by Lemma 23.15 [6] and the maximality of Γ_1, to decompose Γ into three submaps Γ_1, Γ_2 and Γ_3 with $\partial\Gamma_2 = p_1 t_1 l_1 p_3 f_1$ and $\partial\Gamma_3 = p_4 l_2 t_2 p_2 f_2$, where l_1, l_2 and f_1, f_2 are subpaths of q_1 and q_2, respectively, $|l_1|$, $|l_2| \leqslant n^4 k$, and by Lemma 21.1 [6], $|p_1|, \ldots, |p_4| < 2h\varepsilon^{-1}k$. By Lemma 23.15 [6] and Theorem 22.4 [6], we have that

$$|\Gamma_1 \wedge q_2| \geqslant \beta' |\Gamma_1 \wedge s_1| - |p_3| - |p_4| > \beta' n^6 k^2/4 - 4h\varepsilon^{-1}k > dn^5 k, \tag{4.13}$$

and

$$|f_v| \leqslant (\beta')^{-1}(|p_v| + |t_v| + |l_v| + |p_{2+v}|) < \\ < (\beta')^{-1}(4h\varepsilon^{-1} + 2n^2 + n^4)k < 2n^4 k, \tag{4.14}$$

$v = 1, 2$.

We note that, by the proof of N4, we could have a stronger form of N4 if we replaced $|\Gamma \wedge q| < (1+\gamma)j$ by $|\Gamma \wedge q| < (1 + 3\gamma/4)j$. Hence

$$|\Gamma_1 \wedge q_2| < (1 + 3\gamma/4)j, \tag{4.15}$$

and it follows from (4.13)-(4.15) that

$$|q_2| = |f_1| + |f_2| + |\Gamma_1 \wedge q_2| < \\ < (1+\iota)|\Gamma_1 \wedge q_2| < (1+\iota)(1 + 3\gamma/4)j < (1+\gamma)j,$$

as required.

Passing to the case when Γ has at least two long sections of the first kind, we consider a possible contiguity submap Λ of s_1 to s_2 in Δ, where

s_1 and s_2 are distinct sections of π, such that π is not contained in Λ. Of course, we may assume that Λ is the maximal contiguity submap of s_1 to s_2. It follows from Corollary 22.1 [6], Lemma 21.7 [6], Lemmas 4.20, 4.21 and condition N5 that if s_j is a special section of π for some $j \in \{1, 2\}$, then $r(\Lambda) = 0$. The diagram Δ has no self-compatible cells, hence if s_1 and s_2 are special sections of π, then $|\partial\Lambda| < 2\varepsilon|s_1|$ and there exists a submap Λ_1 of Δ with contour $l_1 + tl_2$, where t is a nonspecial section of π and l_j is a subpath of s_j, $j = 1, 2$. The case when $|l_1| = |l_2| = 0$ is impossible, as it was shown in the proof of condition N5. By Corollary 22.1 [6], Lemma 21.7 [6], condition N5, the reducibility of Δ and Lemma 4.20 for Λ_1, we have that $r(\Lambda_1) = 0$, and the maximality of Λ and Lemma 4.2 lead to the inequality $|l_1|, |l_2| < 200k$.

If s_1 and s_2 are nonspecial sections of π, then these sections are not compatible in Δ, since otherwise (using the notation in the definition of the relations (2.9) and (2.10)) it follows from Corollary 22.1 [6] and conditions N4-N6 that $S_{j+p-1,C}\,a S_{j+p-1,C}^{-1} A_p^s = 1$ in rank i for some integer s. If $s = 0$, then $a = 1$ in $G(i)$, which contradicts Lemma 23.16 [6], otherwise A_p is of finite order in $G(i)$, and we arrive at a contradiction to Lemma 4.20 and Theorem 22.4 [6]. Therefore, there is a submap Λ_1 of Δ with contour $l_1 s l_2 p$ where s is a special section of π, l_j is a subpath of s_j, $j = 1, 2$, and $|p| < 2h\varepsilon^{-1}k$ (by Lemma 21.1 [6]). We may assume that Λ_1 is a C-map, and using Lemma 23.15 [6] and the maximality of Λ, we arrive at a contradiction to N4.

Finally, if s_1 is a special section and s_2 is a nonspecial section of π, then $r(\Lambda) = 0$, and by Lemma 4.2, we have that $|\Lambda \wedge s_1| = |\Lambda \wedge s_2| < < 200k$.

Such small values do not affect the resulting estimates, so we may assume that there are no contiguity submaps of s_1 to s_2 not containing π, where s_1 and s_2 are sections of the cell π. Therefore, Γ can be decomposed into several submaps of the following two kinds: submaps of the first kind contain long subpaths of long sections of the first kind in Γ and submaps of the second kind do not contain such subpaths and so their perimeters are small compared to the perimeter of one of the submaps of the first kind, by the definition of the relations (2.9) and (2.10) and the condition $(\pi, \Gamma, q) > 1/3$. Now if we assume that $|q_2| \geqslant (1 + \gamma)j$, then repeating the arguments in the proof of Lemma 26.2 [6] and using the definition of the relators of the second type, we obtain that a period B of rank j is conjugate in rank i to the subword $S_{t,C}\,a S_{t,C}^{-1} A_p^l$ of the relator of the second type corresponding to π. Hence $B^2 = 1$ in $G(i)$, which contradicts Lemma 4.20 and Theorem 22.4 [6].

The proof of the lemma is complete.

Lemma 4.20. *Let q be a section of a contour of a reduced diagram Δ of rank $i+1$ which is a N-map such that the label of q is an A-periodic word, where $A^{\pm 1}$ is either a simple word in rank $i+1$ or a period of rank $k \leqslant i+1$, and in the latter case, let the diagram Δ' and the section q' of some contour of Δ' be obtained from Δ and q, respectively, by excising all cells of the second type having long nonspecial sections compatible with q. Then q (in the first case) and q' (in the second case) are smooth sections of the first type of rank $|A|$ in $\partial\Delta$ and $\partial\Delta'$, respectively.*

Proof. In the case when A is a simple word in rank $i+1$, the verification of conditions F1-F3 is the same as that of N1, N4, N3 and N7, respectively, in Lemma 4.19.

Consider the second case. The verification of S4 for a cell π of the first type is done in the same way as that of condition B4 in Lemma 4.19, by Lemma 4.2 and the argument in the proof of Lemma 4.13. In the course of verifying condition S2 for a cell π of the first type of rank k, we arrive at a contiguity submap Γ of a long section p of π to q' with $(p, \Gamma, q') \geqslant \varepsilon$. By Lemmas 21.1 [6], 4.14 and Theorem 22.4 [6], Γ is an E-diagram. If $k > i+1$, then $q_1 = t_{11}$ in Γ, and we arrive at a contradiction to Lemma 4.13. Hence $k < i+1$ and $k(2) = |s_{02}| = |s_{12}| = 0$ in Γ, and by Lemmas 4.13 and 4.2, we can repeat the argument in the first part of the proof of condition N7 in Lemma 4.19.

Condition F2 can be verified using Lemmas 4.13 and 4.2 and the previous considerations.

Let the Γ-contiguity degree of a cell π of rank k to q' is greater than 1/3. Then Γ is an E-diagram, by Lemmas 21.1 [6] and 4.14, and it follows from Lemma 4.13 and the reducibility of Δ that $k < i+1$ and $k(1) = 0$. Hence, again by Lemma 4.13, $\Gamma \wedge q' = s_{01} t_{11} s_{11}$ with $|s_{01}|, |s_{11}| < \iota |t_{11}|$, and the assertion about t_{11} can be proved in the same way as N3 in Lemma 4.19. Finally, the proof of the inequality $|t_{11}| < (1+\gamma)(1+2\iota)^{-1}(i+1)$ is the same as that of a stronger version of N7 in Lemma 4.19.

Lemma 4.21. *If the label of a section q of a contour of a N-map Δ of rank $i+1$ is visually equal to a subword of one of the words $S_{j+p-1,C}\, a S^{-1}_{j+p-1,C}$ (in the definition of the relations (2.9) and (2.10)) and Δ has no sections of cells of the second type compatible with q, then q is a smooth section of the second type in $\partial\Delta$.*

Proof. This is similar to the verification of condition N5 in Lemma 4.19.

5. – Proof of Theorem A.

We may obtain all the assertions of Theorem A if we repeat the proof of Theorem A [5] with the following remarks.

1) In order to prove assertion 2 of Theorem A, we note that, by Lemma 4.16 and the maximality of the sets of periods P_i, $i \geqslant 1$, if $X \in L$ and X is not conjugate in G to an element of any group G_μ, $\mu \in I$, then either $X^2 = 1$, since X is conjugate to the subword $S_{j+p-1, C} a S_{j+p-1, C}^{-1} A_p^l$ of a relator of the form (2.10), or X is conjugate in G to a power of a period A of some rank and therefore is of infinite order (by Theorem 22.4 [6] and Lemma 4.20), and in the latter case, we have that $A \in N$, since the group $H \cong G/L$ is either torsion-free or trivial, hence X is a product of two involutions (by the definition of the relations (2.9) and (2.10)).

2) Suppose that $XY = YX$ in G for some $X, Y \in G$ and consider the following cases.

a) If $X \in G_\mu$ for some $\mu \in I$, then by Lemmas 4.16 and 4.19, we can repeat the proof of Lemma 34.10 [6] and obtain that $Y \in G_\mu$.

b) In the case when $X = A^k$, where A is a period of some rank, let m be an integer such that $mk > r(A)$ and $2|Y| < \iota mk$. Then it follows from Lemma 4.15 that $Y \in \langle A \rangle$.

c) If X is the subword $S_{j+p-1, C} a S_{j+p-1, C}^{-1} A_p^l$ of a relator of the form (2.10), then, pasting some cells of the second type corresponding to relators (2.10) for some powers of A_p to the contours of a reduced annular diagram Δ for the conjugacy of X and X, we may assume, by Lemmas 4.16 and 4.19-4.21, that Δ is a D-map with long sections $q_1, \ldots, q_4$, where $\phi(q_1) \equiv \phi(q_3) \equiv S_{j+p-1, C} a S_{j+p-1, C}^{-1}$ and $\phi(q_2) \equiv A_p^{l_1}$, $\phi(q_4) \equiv A_p^{l_2}$ for some integers l_1 and l_2. It follows from Lemmas 24.2 [6] and 4.2 and the definitions of smooth sections that the sections q_1 and q_3 are compatible in Δ, hence $X = Y$ in G.

Thus assertion 3) of Theorem A is proved.

3) The assertion in Theorem A about regular automorphisms of a group L_C follows from assertion 3) of Theorem A and the condition that the group H is either torsion-free or trivial.

4) All the results in [5] about the mapping F are not affected, since by the definition of the relations (2.9) and (2.10), we have that $F(\{S_{j+p-1, C} a S_{j+p-1, C}^{-1}\}) = F(\{A_p\})$.

5) In the definitions (from [4] and [5]) of I-diagrams and of H-maps we further require that they have no self-compatible cells and

every section of the first kind (in the definition of H-maps in [5]) is a smooth section of the first type of rank j.

6) In the statement of Lemma 3.2 [5] we replace the condition $|k| > 100\zeta^{-1}$ by $|k| > n$ and $|W| < \iota^2(|k||C|)^{1/3}$. In the proof of this lemma, it follows from Lemmas 4.16 and 4.17 and the maximality of the sets of periods P_i, $i \geqslant 1$, that the word $[C^k, W]$ is conjugate in G to a word V, where either $|V| = 1$ or V is a power of a period A. Repeating the argument in the proof of Lemma 4.16, we may assume that the diagram Δ_0 for this conjugacy on a sphere with three holes has no self-compatible cells. Now, taking into consideration Lemmas 4.20, 4.13 and 4.14, we can repeat the proof of Lemma 3.2 [5] without essential changes.

7) In the statements of Lemmas 3.5 [5] and 3.6 [5] we further require that $|T| < \iota^2(|m||A|)^{1/3}$.

8) In the proof of Lemma 3.5 [5], it is necessary to show that $X \equiv \equiv A^m TA^{2m} TA^{3m} T$ is conjugate in G to a power C^t of a non-dihedral period C, that is, C is not a product of two involutions in rank $|C| - 1$. Assuming the contrary, we have, by Lemma 4.16, that $C^t = = X_1 W_1 X_1^{-1} X_2 W_2 X_2^{-1}$ in rank $|C| - 1$, where W_k, $k = 1, 2$, is either an involution in Ω or the subword $S_k \equiv S_{k, j+p-1, C} a_k S_{k, j+p-1, C}^{-1} A_{k, p}^{l_k}$ of a relator of the form (2.10) for a non-dihedral period $A_{k, p}$ of rank $< |C|$. Using the statement of Lemma 3.5 [5], we obtain a reduced diagram Δ' on a sphere with three holes with contours q_1', q_2 and q_3, where $\phi(q_1') \equiv \equiv X$, $\phi(q_2) \equiv W_1$, $\phi(q_3) \equiv W_2$. Let the diagram Δ and its contour q_1 be obtained from Δ' and q_1', respectively, by excising all cells of the second type having long nonspecial sections compatible with sections s_r', $1 \leqslant \leqslant r \leqslant 3$, of q_1', where $\phi(s_r') \equiv A^{mr}$. Pasting some cells of the second type corresponding to relators (2.10) for some powers of the period $A_{k, p}$ to the contour q_{k+1}, $k = 1, 2$, we may assume, by Lemmas 4.16 and 4.19-4.21, that Δ is a D-map in which we choose long sections in each contour q_s, $1 \leqslant s \leqslant 3$, in the obvious way. If there exists a contiguity submap Γ of p_1 to p_2 in which the length of the contiguity arc $\Gamma \wedge p_r$ is greater than $\varepsilon|p_r|$ (or $\zeta^{-1}|A_{r, p}|$), $r = 1, 2$, where p_1 and p_2 are sections of the contours q_2 and q_3, respectively (such contiguity submaps are called *long*), then by Lemmas 4.12 and 4.2, p_1 and p_2 are compatible and $W_1 = = W_2 A_{1, p}^s$ for some integer s. Thus, pasting a cell with the label $(W_1 A_{1, p}^l)^2$ for some integer l, we obtain that C^t is conjugate in G to a power of $A_{1, p}$, which contradicts Lemma 4.15. Thus there are no long contiguity submaps of a section of q_2 to a section of q_3. Now using Lemmas 24.2 [6], 4.13, 4.14 and 4.2 and repeating the arguments in the proofs of Lemmas 6 [7] and 4.17, we have that there is no a contiguity submap Γ of s_{j_1} to s_{j_2},

where $j_1, j_2 \in \{1, 2, 3\}$ such that either $|\Gamma \wedge s_{j_1}| > \zeta |s_{j_1}|$ or $|\Gamma \wedge s_{j_2}| > > \zeta |s_{j_2}|$ (such contiguity submaps are called *long*), and there exists contiguity submaps Γ_1 and Γ_3 of long sections, say s_1 and s_3, of q_1 to q_k for some $k \in \{2, 3\}$ such that $|\Gamma_t \wedge s_t| > |s_t|/10$, $t = 1, 3$. By Lemma 21.1 [6], Theorem 22.4 [6] and the argument in the proof of Lemma 4.14, we have that $|q_k| > 1$ and Γ_t is an E-diagram for each $t \in \{1, 3\}$. If $|\Gamma_t \wedge q_k| > \iota |S_k|$, then by Lemmas 4.14 and 4.13 and the definition of the relators of the second type, some subpaths of the sections s_t, $t = = 1, 3$, and q_k are compatible in Δ, hence, using again Lemmas 24.2 [6], 4.14, 4.13 and 4.2, we obtain that $TAT^{-1} = A^{\pm 1}$ in G, which contradicts the choice of T. If $|\Gamma_t \wedge q_k| \leq \iota |S_k|$ for some $t \in \{1, 3\}$, then by Lemma 4.13, it is also true for Γ_{4-t}. Therefore, Δ can be decomposed into subdiagrams Γ_1, Γ_3 and Δ_1, Δ_2, where for some $r \in \{1, 2\}$, the diagram Δ_r is circular while Δ_{3-r} is an annular diagram, the contour (or one of the contours) of Δ_1 contains t_3 and the contour (or one of the contours) of Δ_2 contains the path $t_1 s_2 t_2$.

Consider the diagram Δ_2. As it was noted above, there are no long contiguity submaps of s_{j_1} to s_{j_2} for each $j_1, j_2 \in \{1, 2, 3\}$ and of sections of q_2 to sections of q_3. Moreover, by Lemma 4.13, there is no a long contiguity submap of a section of q_k to s_j, $j = 1, 2, 3$, since $|A| < |A_{k,p}|$. If q_{5-k} is one of the contours of Δ_2 and there is a long contiguity submap of s_2 to q_{5-k}, then we obtain a circular subdiagram Δ_3 of Δ_2 with the same condition for long contiguity submaps between the sections of the contour as that for Δ_2. In any case, we arrive at a contradiction to Lemma 24.2 [6] which is true for Δ_2 (or Δ_3), by the argument in the proof of Lemma 4.13.

9) In the proof of Lemma 3.5 [5], it might be suspected that one more possibility could arise: $T_1 C^l T_1^{-1} = C^{-l}$, that is, $A^{2m} T A^{2m} T A^{3m} T^2 A^{2m} T A^{3m} T = 1$. Then repeating the argument in the proof of Lemma 3.3 [5] and using Lemmas 4.20, 4.14, 4.13 and 3 [7], we obtain that $TAT^{-1} = A^{\pm 1}$, which contradicts the choice of T.

10) In Lemma 3.6 [5] the fact that C is a non-dihedral period can be proved in the same way as in 8).

11) In the proof of Lemma 3.7 [5], we choose integers p, t and m according to the statements of Lemmas 3.2 [5], 3.5 [5] and 3.6 [5].

12) The following remark will be useful for proving the assertions of Theorem A. We note that if a subgroup M of G is not cyclic and is not conjugate in G to a subgroup of any group G_μ, $\mu \in I$, then M contains an element conjugate in G to a power of a period of some rank. In fact, assuming the contrary, we obtain, by Lemmas 4.16 and 4.19, a N-map Δ on a sphere with three holes with contours q_1, q_2 and q_3, where

the label of q_k is either an element of Ω or the subword $W_k \equiv \equiv S_{k,j+p-1,C} a_k S_{k,j+p-1,C}^{-1} A_{k,p}^{l_k}$ of a relator of the form (2.10), $1 \leqslant k \leqslant 3$. It was shown in the proof of Theorem 35.1 [6] that the case when $|q_k| = = 1$, $k = 1, 2, 3$, is impossible. We may assume that $|q_1| > 1$. Pasting some cells of the second type corresponding to relators (2.10) for some powers of $A_{k,p}$ to the contour q_k, $k = 1, 2, 3$, we may assume, by Lemmas 4.19-4.21, that Δ is a D-map with the natural choice of long sections. Then by Lemmas 24.2 [6], 4.14, 4.13 and 4.2, either a power of $A_{1,p}$ is conjugate in G to $\phi(q_k) \in \Omega$ for some $k \in \{2, 3\}$, which is impossible by Lemma 4.20 and Theorem 22.4 [6], or we may assume that $|q_2| > 1$ and there are sections p_1 and p_2 of q_1 and q_2, respectively, which are compatible in Δ. Therefore $\phi(q_3)$ is conjugate in G to a power of $A_{1,p}$, which contradicts our assumption.

13) Assertion 12 of Theorem A follows immediately from 2) and Lemma 34.10 [6].

14) If a subgroup M of G is infinite dihedral and is not conjugate in G to a subgroup of any G_μ, $\mu \in I$, then, as it was shown in 12), we may assume that M contains a power A^t of a period A. By 2), $C_G(A^t) = \langle A \rangle$, hence $C_G(M) = \{1\}$. The assertion about $N_G(M)$ can be proved in the same way as in Theorem A [5].

15) If an infinite cyclic subgroup M of L is not conjugate in G to a subgroup of any group G_μ, $\mu \in I$, then we may assume, by Lemma 4.16, the maximality of the sets P_i, $i \geqslant 1$, and the choice of the group H, that $M = \langle A^t \rangle$, where $A \in L$ is a period, $|t| \geqslant 1$. By 2), $C_G(M) = \langle A \rangle$, and it follows from assertion 2 of Theorem A that A is a product of two involutions X_1 and X_2 in L. By the proof of assertion 13 of Theorem A, we have that $N_G(M) = \langle X_1, X_2 \rangle$, which completes the proof of assertion 14 of Theorem A.

16) Let $A \in P_i \cap N$ for some $i \geqslant 1$ and T a word such that $T \notin \langle A \rangle$, the coset $T\langle A \rangle$ does not contain an involution and the double coset $\langle A \rangle T \langle A \rangle$ does not contain an element XTX^{-1}, where X is an involution with the property that $XAX^{-1} = A^{-1}$, and also assume that if $A \in P_i'$, then $ST\langle A \rangle$ does not have an involution, where S is the subword $S_{j+p-1,C}\, a S_{j+p-1,C}^{-1}$ of the relator (2.9) for $A_p \equiv A$. Let m be an integer with $|T| < \iota^2(|m|\,|A|)^{1/3}$ and $|m| > n$. By Lemma 3.2 [5], $V \equiv [A^m, T]$ is conjugate to a power of a period B. We show that, in our case, B is a non-dihedral period. Assuming the contrary and using Lemmas 4.16 and 4.19, we obtain a N-map Δ' on a sphere with three holes with contours q_1', q_2 and q_3, where $\phi(q_1') \equiv V$ and $\phi(q_k)$ is either an involution in Ω or the subword $W_k \equiv S_{k,j+p-1,C} a_k S_{k,j+p-1,C}^{-1} A_{k,p}^{l_k}$ of a relator of the form (2.10), $k = 2, 3$. In exactly the same way as in 8), we obtain a dia-

gram Δ and its contour q_1 from Δ' and q_1', respectively, such that Δ is a D-map if we choose long sections in the contours in the obvious way. Pasting a cell of the second type with the label $(W_2 A_{2,p}^s)^2$ for some nonzero integer s, we also may assume that there are no sections p_2 and p_3 of q_2 and q_3, respectively, such that they are compatible in Δ. Moreover, there is no a contiguity submap Γ of s_{j_1} to s_{j_2}, $j_1, j_2 \in \{1, 2\}$, such that either $|\Gamma \wedge s_{j_1}| > \zeta |s_{j_1}|$ or $|\Gamma \wedge s_{j_2}| > \zeta |s_{j_2}|$, since otherwise some subpaths of these sections are compatible or anticompatible in Δ, by Lemmas 4.14 and 4.13, and, using Lemma 24.2 [6] and the argument in the proof of Lemma 4.17, we arrive at a contradiction to the choice of T. By the same reason, there are no contiguity submaps Γ_1 and Γ_2 of long sections s_1 and s_2 of q_1 to a section p of q_k, $k \in \{2, 3\}$, such that $|\Gamma_j \wedge s_j| > \zeta |s_j|$, $j = 1, 2$, since otherwise either some subpaths of s_j and p are compatible in Δ and therefore $T \in \langle A \rangle$ or $T\langle A \rangle$ contains an involution, or, by Lemma 24.2 [6] and Theorem 22.4 [6], there are sections of q_2 and q_3 which are compatible in Δ, and we arrive at a contradiction to our assumption. They by Lemmas 24.2 [6], 4.14, 4.13 and 4.2, Theorem 22.4 [6] and our assumption, there exists the only possibility when some subpaths of long sections of q_1 are compatible with sections of distinct contours q_k, $k = 2, 3$. But in this case, we have that $A \in P_i'$ and $\langle A \rangle T \langle A \rangle$ contains STS^{-1}, which contradicts the choice of T.

17) Let ψ be an automorphism of a subgroup L_C, where $C \not\subseteq G_\mu$ for each $\mu \in I$. Repeating the proof of Lemma 3.8 [5] and using Lemma 4.16, we obtain that $\psi(A)$ is a power of a period of some rank for each period A, but it might be possible that $\psi(b)$ is not conjugate in G to an element of Ω for some $b \in \Omega_1 \cap C$. In this case, b is an involution and there are $c, d \in C$ such that $\{b, c\} \not\subseteq G_\mu$, $\{b, d\} \not\subseteq G_\nu$ for each $\mu, \nu \in I$ and $c \neq d$. We have (after multiplying ψ by an inner automorphism of R_C) that $\psi([b, c]^k) = A^r$ and $\psi([b, d]) = W$, where A is a period, $k > n^7$, (or $|r| = 2k$ if c is an involution), $|W| < \iota^2(|r||A|)^{1/3}$ and W is a minimal word in G such that $WAW^{-1} \neq A^{\pm 1}$, since $[b, d][b, c][b, d]^{-1} \neq [b, c]^{\pm 1}$ in G.

Suppose that A is a non-dihedral period. Then $\psi(b)A^t = \psi(cbc^{-1})$ for some t, $|t| \leq 2$, hence $\psi(b)A^t\psi(b)^{-1} = A^{-t}$, and it follows from Lemma 4.15 that $\psi(b) = SA^k$ in G for some integer k, where S is the subword $S_{j+p-1,C}aS_{j+p-1,C}^{-1}$ of the relator (2.9) for A. Therefore, $\psi(c)SA^k\psi(c)^{-1} = SA^{k+t}$, and repeating the argument in 2), we obtain that $\psi(c)SA^l$ for some integer l. Then there exists an element $e \in C$ such that $\{b, e\} \not\subseteq G_\mu$ for each $\mu \in I$, $e \neq c$ and $\psi([b, e])$ is not conjugate to a power of a non-dihedral period, since otherwise it follows from the previous considerations that the cyclic subgroups $\langle bc \rangle$ and $\langle be \rangle$ have a nontrivial intersection, which contradicts assertion 3 of Theorem A.

Thus we may assume that A is a product of two involutions in rank $|A| - 1$.

By Lemmas 3.2 [5] and 3 [4], we obtain (after multiplying ψ by an inner automorphism of R_C) that $\psi((bd)B_0(bd)^{-1}) = E^l$ and $\psi([b, c]^k) = B$, where E is a period, $BE^lB^{-1} \neq E^{\pm l}$ and $F(\{B\}) \subseteq F(\{E\})$.

Now we show that E is a non-dihedral period. First of all we note that $A \in N$, since $[b, c] \in N$ and the group H is either torsion-free or trivial. As it was shown above, $WAW^{-1} \neq A^{\pm 1}$, hence $W \notin \langle A \rangle$. Suppose that $W\langle A \rangle$ contains an involution, then $V \equiv [b, d][b, c]^t$ (or $V \equiv [b, d](bc)^t$ if c is an involution) is an involution for some integer t. By Lemmas 4.16 and 4.19, there exists a reduced annular diagram Δ with contours $q_1 = p_1 p_2$ and q_2 for the conjugacy of V and an involution X, where either $X \in \Omega$ or X is the subword $S_{j+p-1, C}\, a S_{j+p-1, C}^{-1} A_p^l$ of a relator of the form (2.10), such that Δ is a N-map. Pasting some cells of the second type corresponding to relators (2.10) for some powers of A_p to q_2, we may assume that the contour q_2 is either smooth or geodesic in Δ. Moreover, $[b, c]$ (or bc if c is an involution) is a product of two involutions, then by Lemma 4.20, we have that p_2 is a smooth section in $\partial\Delta$. If $X \in \Omega$, then it follows from Corollary 22.2 [6], Lemma 21.7 and the definition of the relations of G that $r(\Delta) = 0$, which contradicts the choice of b, c and d, otherwise we may assume that the cyclic section q_1 with the label V is geodesic in Δ. Then by Lemmas 4.20 and 4.21, Δ is a C-map in which q_1 is the long section of the second kind and the long sections of the first kind are chosen in q_2 in the obvious way. We arrive at a contradiction to Lemmas 23.15 [6], 4.12 and 4.2.

Suppose that $\langle A \rangle W \langle A \rangle$ contains an element XWX^{-1}, where X is an involution such that $XAX^{-1} = A^{-1}$. Then $V \equiv [b, c]^{t_1}[b, d][b, c]^{t_2}$ (or $V \equiv (bc)^{t_1}[b, d](bc)^{t_2}$ if c is an involution) is equal in G to $Y[b, d]Y^{-1}$ for some integers t_1 and t_2 and an involution Y such that $Y[b, c]Y^{-1} = [b, c]^{-1}$ (or $Y(bc)Y^{-1} = (bc)^{-1}$ if c is an involution). Let Δ be a reduced annular diagram without self-compatible cells for the conjugacy of V and $[b, d]$. (Such a diagram Δ exists by Lemma 4.16, since $[b, d]$ is a product of two involutions.) Hence, by Lemma 4.19, Δ is a N-map. It follows from the previous considerations, Corollary 22.2 [6] and the definition of the relations of G that $r(\Delta) = 0$. Then $t_1 = -t_2$, and by assertion 3 of Theorem A, $Y \in [b, c]^{t_1}\langle [b, d] \rangle$ (or $Y \in (bc)^{t_1}\langle [b, d] \rangle$ if c is an involution), but it follows from Lemma 4.15 that $Y \in b\langle [b, c] \rangle$ (or $Y \in b\langle bc \rangle$ if c is an involution), hence a power of $[b, d]$ is an involution, which contradicts Lemma 4.20 and Theorem 22.4 [6].

Thus 16) enables us to assert that E is a non-dihedral period. We choose an integer m such that $ml > 0$, $|m| > k^2 n^3$ and $|B| < \iota^2 (ml|E|)^{1/3}$. It follows from Lemmas 3.6 [5] and 3 [4] and the definition of relations of the first type (after multiplying ψ by an inner autmorphism of R_C) that either $\psi([b, c]^k) = [u, vy]^t$ in G for some integer t, $u \in \Omega_1 \cap C$, and $v \in C \cup \{1\}$ such that $\{u, y\} \not\subseteq G_\mu$ for each $\mu \in I$ and $vy \in \Omega_2$, or $\psi((bd)B_m(bd)^{-1}) = L$ and $\psi([b, c]^k) = T$, where L is a non-dihedral period, $L \in L_C$, $|T| < 3|L|$, $TLT^{-1} \neq L^{\pm 1}$ and $F(\{T\}) \subseteq F(\{L\})$. In the first case, it follows from the standard considerations that $\psi(b)$ is conjugate to an element of Ω. In the second case, it follows from Lemmas 1 [4], 4.14 and 4.13 that (after multiplying ψ by an inner automorphism of R_C) there is $T_1 \in Y_L$ such that $T = T_1 L^p$, where $|p| < n$. Applying the automorphism ψ to both sides of the defining relation (2.2) (conjugated by an element bd) for $(bd)B_m(bd)^{-1}$, $[b, c]^k$ and $l = 4n - p$, we obtain that

$$\psi([b, c]^{-k})L^n T_1 L^{5n} \dots T_1 L^{5n + 6n(h-2)} = 1\,,$$

and it follows from the definition of the relations (2.2), (2.6) and (2.7) for L and T_1 that we again obtain the first case which has already been considered.

Thus $\psi(b)$ is conjugate in G to an element of Ω for each $b \in \Omega_1 \cap C$.

18) In the proof of Lemma 3.9 [5], it is necessary to show that (using the notation of this lemma) A is a non-dihedral period. If c is an involution, then, by 16), we can repeat the argument in 17). Suppose that $c^2 \neq 1$ in G. As it was shown in the proof of Lemma 3.9 [5], $W \notin \langle S \rangle$. The fact that $W\langle S \rangle$ does not contain an involution can be proved in exactly the same way as that for $W\langle A \rangle$ in 17).

Let the double coset $\langle S \rangle W \langle S \rangle$ contains an element XWX^{-1}, where X is an involution such that $XSX^{-1} = S^{-1}$. Then $V \equiv [c, de]^{t_1}[c, fg][c, de]^{t_2}$ is equal in G to $Y[c, fg]Y^{-1}$ for some integers t_1 and t_2 and an involution Y with the property that $Y[c, de]Y^{-1} = [c, de]^{-1}$. As in 17), we obtain that $t_1 = -t_2$, and by assertion 3 of Theorem A, $Y \in [c, de]^{t_1}\langle [c, fg] \rangle$. But it follows from Lemma 4.15 that either $Y \in e\langle [c, de] \rangle$ if $d = 1$ and $e^2 = 1$, or $Y \in S_1\langle [c, de] \rangle$, where S_1 is the subword $S_{j+p-1, C}\, a S_{j+p-1, C}^{-1}$ of the relator (2.9) for $A_p \equiv [c, de]$. In any case, we have that a power of $[c, fg]$ is an involution, which contradicts Lemma 4.20 and Theorem 22.4 [6].

It remains to prove that $EW\langle S \rangle$ does not contain an involution, where E is the subword $S_{j+p-1, C}\, a S_{j+p-1, C}^{-1}$ of the relator (2.9) for $A_p \equiv S$. Assuming the contrary, we have that $V = Y[c, fg][c, de]^t$ is an involution in G for some integer t and an involution Y such that $Y[c, de]Y^{-1} = [c, de]^{-1}$. It follows from Lemma 4.15 that either 1) $d = 1, e^2 = 1$ and $Y = e[c, e]^k$ for some integer k, or 2) $Y = S_1[c, de]^k$, where k is an integer and S_1 is the subword $S_{j+p-1, C}\, a S_{j+p-1, C}^{-1}$ of the relator (2.9) for $A_p \equiv [c, de]$. The first case is impossible, by the argument in the consideration in 17) of the case when $W\langle A\rangle$ contains an involution. In the second case, by Lemmas 4.16 and 4.19, there exists a reduced annular diagram Δ' with contours $q_1' = p_1' p_2' p_3$ and q_2 for the conjugacy of $V_1 = S_1 [c, de]^m [c, fg]$ and an involution X, where m is an integer and either $X \in \Omega$ or X is the subword $S_{j+p-1, C}\, a S_{j+p-1, C}^{-1} A_p^l$ of a relator of the form (2.10), such that Δ' is a N-map. Excising cells of the second type with special or long nonspecial sections compatible with p_1' or p_2' from Δ', we obtain the diagram Δ with contours $q_1 = p_1 p_2 p_3$ and q_2 such that the sections p_1 and p_2 are smooth in Δ, by Lemmas 4.20 and 4.21. Pasting some cells of the second type corresponding to relators (2.10) for some powers of A_p to q_2, we may assume that the contour q_2 is either smooth or geodesic in Δ. If $X \in \Omega$, then Δ is a C-map in which q_2 is the long section of the second kind, which contradicts Lemmas 23.15 [6], 4.12 and 4.2, otherwise Δ is a D-map, and using Lemmas 24.2 [6], 4.12 and 4.2, we arrive at a contradiction to the choice of elements c, d, e, f and g.

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