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Viscosity Solutions and Standard Riemann Semigroup for Conservation Laws with Boundary.

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ABSTRACT - Two different formulations of the Initial-Boundary Value Problem for a system of conservation laws

$$(\star) \qquad u_t + [F(u)]_x = 0$$

are considered. Correspondingly, we define two Standard Riemann Semigroups (SRS) generated by the system ($\star$) plus boundary conditions. We prove that, if a SRS exists, then it is unique and its trajectories yield solutions to the Initial-Boundary Value Problem, in each of the two cases. Moreover, a proper definition of viscosity solutions allows us to characterize the solutions provided by the SRS in terms of local integral estimates.

1. - Introduction.

Consider the following system of n conservation laws in one space dimension

$$(1.1) \qquad u_t + [F(u)]_x = 0$$

where u varies in $\boldsymbol{R}^n$, F is smooth and each characteristic field in DF is either linearly degenerate or genuinely non linear. Aim of this paper is to generalize part of the theory developed in [5] for Cauchy Problems for (1.1) to the different formulations of the Initial-Boundary Problem for (1.1) considered in [1], [11], [13], [15].

More precisely, the *Standard Riemann Semigroup* (SRS) for the Cauchy Problem for (1.1) is defined in [5] as a Lipschitzean semigroup

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whose trajectories locally coincide with the standard Lax solutions in case of piecewise constant initial data. If such a SRS exists, it is unique and it yields the same solutions obtained by Glimm [12] and by a wave-front tracking procedure [3]. Similarly, in the present paper we define the SRS generated by the Initial-Boundary Problem for (1.1). We show that if the SRS exists for such problem, then it is unique and it yields the same solutions obtained in [1] by wave-front tracking.

The construction of the SRS for the Cauchy Problem is accomplished in [7] in the 2×2 case, and in [6], [9] in the $n \times n$ case. Concerning the Initial-Boundary Problem, the SRS has been constructed in [2] in the 2×2 case.

Following [1] and [2], we consider two different formulations of the Initial-Boundary Problem, referred to as the *Characteristic* (C) one and the *Non Characteristic* (NC) one. The two formulations differ in the sense given to the boundary condition and, hence, in the very definition of solution.

Separately, we introduce a concept of *viscosity solution* to the Initial-Boundary Problem for (1.1), extending the analogous definition given in [5]. Aim of this definition is to single out some intrinsic property which characterizes the solutions provided by the SRS. In fact, we show that if a SRS exists, then its trajectories are viscosity solutions of the Initial-Boundary Problem for (1.1). Conversely, a viscosity solution with small total variation coincides with the corresponding semigroup trajectory as soon as a SRS exists.

The paper is organized as follows. The next two sections are devoted to the statement of the problem in the two cases (C) and (NC). In Section 4 we prove that the semigroup trajectories provide solutions to the Initial-Boundary Problem and that the SRS is unique. Section 5 is concerned with the definition of viscosity solutions, a characterization of semigroup trajectories. The proofs are collected in the last section.

2. – The characteristic initial-boundary problem.

Fix a continuous boundary profile $\Psi: \boldsymbol{R}^+ \mapsto \boldsymbol{R}$ and define the domain $\Omega \doteq \{(t, x) \in \boldsymbol{R}^2 : t \geq 0 \text{ and } x \geq \Psi(t)\}$. The Characteristic Initial-Boundary Problem for (1.1) in Ω with boundary condition $\tilde{u}: \boldsymbol{R}^+ \mapsto \boldsymbol{R}$ is:

$$\text{(C)} \qquad \begin{cases} u_t + [F(u)]_x = 0\,, \\ u(0, x) = \overline{u}(x)\,, \\ u(t, \Psi(t)) = \tilde{u}(t)\,, \end{cases}$$

where it is assumed that the initial data $\overline{u}$ and the boundary condition $\tilde{u}$ are $\boldsymbol{L}^1$ functions with small total variation, so that $\|\overline{u}(\Psi(0)) - \tilde{u}(0)\|$ is also small.

We briefly recall here the definition of solution to (C), as stated in [11] and [1].

DEFINITION C.1. Call $u(\tau, \Psi(\tau)+) \doteq \lim_{x \to \Psi(\tau)+} u(\tau, x)$. For every $\tau \geq 0$, let w^τ be the self-similar Lax solution to the Riemann Problem

$$(2.1)\qquad \begin{cases} w_t + [F(w)]_x = 0\,, \\ w(\tau, x) = \begin{cases} \tilde{u}(\tau) & \text{if } x < \Psi(\tau)\,, \\ u(\tau, \Psi(\tau)+) & \text{if } x > \Psi(\tau)\,, \end{cases} \end{cases}$$

where $u(\tau, \Psi(\tau)+) \doteq \lim_{x \to \Psi(\tau)+} u(\tau, x)$. A function $u\colon \Omega \mapsto \boldsymbol{R}^n$ is a solution to (C) if

(i) u is a weak entropic solution to (1.1) and satisfies the initial condition, in the sense that

$$(2.2)\qquad \int_0^{+\infty}\int_{-\infty}^{+\infty} (u(t, x)\cdot\phi_t(t, x) + F(u(t, x))\cdot\phi_x(t, x))\, dx\, dt + \\ + \int_{-\infty}^{+\infty} \overline{u}(x)\cdot\phi(0, x)\, dx = 0$$

for any $\boldsymbol{C}^1$ function ϕ with compact support contained in the set $\{(t, x) \in \boldsymbol{R}^2\colon t < 0 \text{ or } x > \Psi(t)\}$.

(ii) u satisfies the boundary condition in the sense that for all but countably many $\tau \geq 0$

$$(2.3)\qquad w^\tau(t, x) = u(\tau, \Psi(\tau)+) \\ \text{for all } (t, x) \text{ such that } \begin{cases} x - \Psi(\tau) > D_-\Psi(\tau)\cdot(t - \tau)\,, \\ t > \tau\,, \end{cases}$$

where $D_-\Psi(t) \doteq \liminf_{s \to t-} (\Psi(s) - \Psi(t))/(s - t)$ is the lower left Dini derivative.

The role that Riemann Problems have in the theory of Cauchy Problems for (1.1) is here played by the

Characteristic Riemann problem with boundary.

Fix a continuous boundary profile $\Psi: \boldsymbol{R}^+ \mapsto \boldsymbol{R}$ and choose two constant states $\overline{u}, \tilde{u} \in \boldsymbol{R}^n$, with $\|\overline{u} - \tilde{u}\|$ sufficiently small. The Characteristic Riemann Problem with Boundary is

$$\begin{cases} u_t + [F(u)]_x = 0 & \text{for } (t, x) \in \{(t, x) \in \Omega : x > \Psi(t)\}, \\ u(0, x) = \overline{u} & \text{for } x > 0, \\ u(t, x) = \tilde{u} & \text{for } x = \Psi(t). \end{cases} \tag{2.4}$$

We shall now provide an explicit formula for the solution of (2.4). For $(t, x) \in \Omega$, let

$$\overline{\lambda}(t, x) \doteq \min_{s \in [0, t]} \frac{x - \Psi(s)}{t - s} \tag{2.5}$$

denote the slowest speed with which a wave exiting the boundary can reach (t, x). Call w_R the Lax solution to the Riemann Problem (Figure 1)

$$\begin{cases} u_t + [F(u)]_x = 0, \\ u(x, 0) = \begin{cases} \tilde{u} & \text{if } x < 0, \\ \overline{u} & \text{if } x > 0. \end{cases} \end{cases} \tag{2.6}$$

By direct verification, the solution w to (2.4) in the sense of Definition C.1 is found to be (Figure 2)

$$w(t, x) \doteq w_R(t, \overline{\lambda}(t, x) \cdot t). \tag{2.7}$$

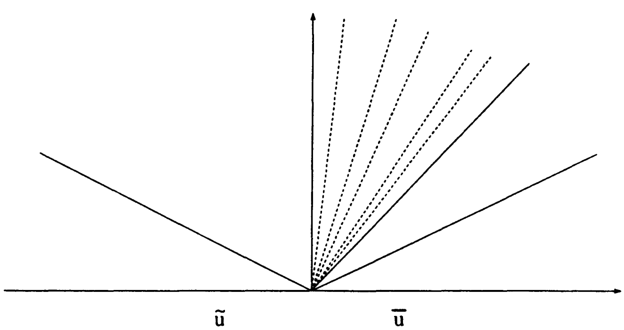

Figure 1.

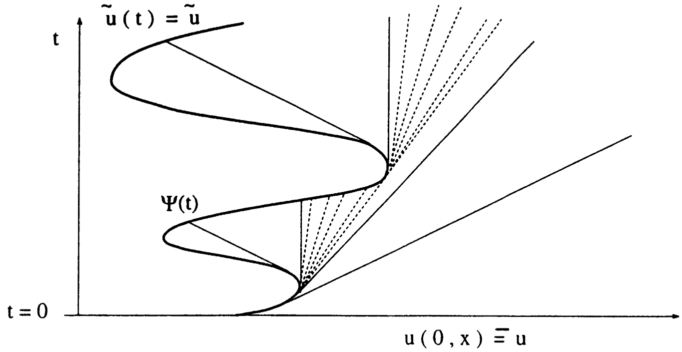

Figure 2.

We remark that in the case $\Psi(t) = m\cdot t$, the above procedure amounts to define w simply as the restriction of w_R to Ω. In particular, if $\Psi(t) = 0$, then Definition C.1 coincides with the one introduced in [11].

Problem (C) is in general not time-homogeneous, due to the boundary condition $\tilde{u}$ and to the boundary profile Ψ. In order to define a semigroup, we thus need to incorporate both the boundary condition and the boundary profile in the domain of the semigroup, as in [2]. Let $\mathcal{D}^*$ be the set of those triples $\boldsymbol{p} \doteq (\overline{u}, \tilde{u}, \Psi)$ such that

$$\overline{u} \in \boldsymbol{L}^1(\boldsymbol{R}, \boldsymbol{R}^n) \cap \boldsymbol{BV}(\boldsymbol{R}, \boldsymbol{R}^n) \text{ with } \overline{u}(x) = 0 \text{ for } x < \Psi(0),$$
$$\tilde{u} \in \boldsymbol{L}^1(\boldsymbol{R}^+, \boldsymbol{R}^n) \cap \boldsymbol{BV}(\boldsymbol{R}^+, \boldsymbol{R}^n),$$
$$\Psi \in \boldsymbol{C}^0(\boldsymbol{R}^+, \boldsymbol{R}).$$

Definition C.2. A Standard Riemann Semigroup generated by (C) is a continuous semigroup

$$\begin{array}{cccccc} S: & \boldsymbol{R}^+ & \times & \mathcal{D} & \mapsto & \mathcal{D}, \\ & t & , & \boldsymbol{p} & \mapsto & S_t\boldsymbol{p}, \end{array}$$

such that

(1) $\mathcal{D}$ is a subset of $\mathcal{D}^*$ containing all triples $\boldsymbol{p} = (\overline{u}, \tilde{u}, \Psi)$ with $\mathrm{TV}(\overline{u}) + \mathrm{TV}(\tilde{u})$ sufficiently small.

(2) If $\boldsymbol{p} = (\overline{u}, \tilde{u}, \Psi)$, then $S_t\boldsymbol{p} = (E_t\boldsymbol{p}, \mathcal{T}_t\tilde{u}, \mathcal{T}_t\Psi)$. The evolution operator $E: \boldsymbol{R}^+ \times \mathcal{D} \mapsto \mathcal{D}$ is such that, for any two triples $\boldsymbol{p}' \doteq (\overline{u}', \tilde{u}', \Psi')$

and $\boldsymbol{p}'' \doteq (\overline{u}'', \tilde{u}'', \Psi'')$, if Ψ' and Ψ'' have Lipschitz constant L' and L'', then

$$(2.8)\qquad \|E_{t'}\boldsymbol{p}' - E_{t''}\boldsymbol{p}''\|_{L^1} \leq L\cdot(\|\overline{u}'' - \overline{u}'\|_{L^1} + \|\Psi'' - \Psi'\|_{C^0}) + \\ + L\cdot(1 + L' + L'')\cdot(\|\tilde{u}'' - \tilde{u}'\|_{L^1} + |t'' - t'|)$$

for some fixed constant L and for all t', t'' in $\boldsymbol{R}^+$. $\mathcal{T}_t$ is the time-translation operator, i.e. $(\mathcal{T}_t\tilde{u})(s) \doteq \tilde{u}(t+s)$ and $(\mathcal{T}_t\Psi)(s) \doteq \Psi(t+s)$, for any $s \geq 0$.

(3) If $\overline{u}$ and $\tilde{u}$ are piecewise constant and if Ψ is piecewise linear and continuous, then $E_t\boldsymbol{p}$ coincides for t small with the glueing of the Lax solutions to Riemann Problems in the points of jumps of $\overline{u}$, and of the solution to the Characteristic Riemann Problem with Boundary at $(0, \Psi(0))$.

At present, such a semigroup for Initial-Boundary Problems has been constructed in the case $n = 2$ in [2].

Denoting $\boldsymbol{p}' = (\overline{u}', \tilde{u}', \Psi')$ and $\boldsymbol{p}'' = (\overline{u}'', \tilde{u}'', \Psi'')$, introduce the quantity

$$(2.9)\qquad d(\boldsymbol{p}', \boldsymbol{p}'') \doteq \|\overline{u} - \overline{u}'\|_{L^1} + \|\tilde{u}'' - \tilde{u}'\|_{L^1} + \|\Psi'' - \Psi'\|_{C^0}\,.$$

Note that the continuity of S and (2) imply the Lipschitz estimate

$$(2.10)\qquad d(S_t\boldsymbol{p}'', S_t\boldsymbol{p}') \leq L\cdot d(\boldsymbol{p}', \boldsymbol{p}'')$$

for any pair of initial data $\overline{u}'$, $\overline{u}''$ and of continuous boundary profiles Ψ' and Ψ'', provided $\tilde{u}' = \tilde{u}''$.

3. – The non characteristic initial-boundary problem.

Fix a boundary profile $\Psi: \boldsymbol{R}^+ \mapsto \boldsymbol{R}$ and define the domain $\Omega \doteq \{(t, x) \in \boldsymbol{R}^2 : t \geq 0 \text{ and } x \geq \Psi(t)\}$. The Non Characteristic Initial-Boundary Problem for (1.1) in Ω with boundary condition g is:

$$(\mathrm{NC})\qquad \begin{cases} u_t + [F(u)]_x = 0\,, \\ u(0, x) = \overline{u}(x)\,, \\ b(u(t, \Psi(t))) = g(t)\,, \end{cases}$$

where the initial data $\overline{u}$ and the boundary condition g are $\boldsymbol{L}^1$ functions with small total variations. Call $\lambda_i(u)$ and $r_i(u)$ the i-th eigenvalue and the corresponding i-th right eigenvector of the matrix $DF(u)$. We assume that

$$\lambda_i(u) \in [\lambda_i^{\min}, \lambda_i^{\max}] \quad \text{for all } u \text{ and } i = 1, \ldots, n,$$

for n suitable pairwise disjoint bounded intervals $[\lambda_i^{\min}, \lambda_i^{\max}]$ chosen so that $\lambda_i^{\max} < \lambda_{i+1}^{\min}$. We require that Ψ is Lipschitz continuous and satisfies

$$(3.1) \qquad \lambda_{n-q}^{\max} < \dot{\Psi}(t) < \lambda_{n-q+1}^{\min} \quad \text{for a.e. } t$$

for some fixed $q \in \{1, \ldots, n\}$. By (3.1), all characteristic lines cross the boundary transversally, motivating the denomination *Non Characteristic.*

On the function b, following [15], we assume:

(b.1) $b: \boldsymbol{R}^n \mapsto \boldsymbol{R}^q$ is smooth;

(b.2) at the point $u = 0$, the differential $Db(0)$ restricted to the span of $\{r_{n-q+1}, \ldots, r_n\}$ is injective.

We recall here the definition of solution to (NC), as stated in [13], [15].

DEFINITION NC.1. A function $u: \Omega \mapsto \boldsymbol{R}^n$ is a solution to (NC) if

(i) u satisfies condition (2.2), for any $\boldsymbol{C}^1$ function ϕ with compact support contained in the set $\{(t, x) \in \boldsymbol{R}: t < 0 \text{ or } x > \Psi(t)\}$;

(ii) it satisfies the boundary condition in the sense that for all but countably many $\tau \geq 0$

$$(3.2) \qquad \lim_{\substack{(t,x) \to (\tau, \Psi(\tau)) \\ (t,x) \in \Omega}} b(u(t, x)) = g(\tau).$$

The following is the equivalent to Riemann Problems in the present case, and will be referred to as the

Non Characteristic Riemann problem with boundary.

Fix q in $\{1, \ldots, n\}$ and a Lipschitzean $\Psi: \boldsymbol{R}^+ \mapsto \boldsymbol{R}$ with $\lambda_{n-q}^{\max} < \dot{\Psi}(t) < \lambda_{n-q+1}^{\min}$ a.e. Let a constant initial data $\overline{u} \in \boldsymbol{R}^n$ and a constant boundary condition $g \in \boldsymbol{R}^q$ be given, both sufficiently close to the origin. Let b be any smooth function satisfying (b.1), (b.2) and

(b.3) $\|b(\overline{u}) - g\|$ is sufficiently small.

The Non Characteristic Riemann Problem with Boundary is

$$(3.3)\qquad \begin{cases} u_t + [F(u)]_x = 0 & \text{for } (t, x) \in \{(t, x) \in \Omega : x > \Psi(t)\}, \\ u(x, 0) = \overline{u} & \text{for } x > 0, \\ b(u(t, x)) = g & \text{for } x = \Psi(t). \end{cases}$$

As introduced in [13] and [15], the solution to (3.3) according to Definition NC.1 is the restriction to the set $\{(x, t) \in \Omega : x > \Psi(t)\}$ of the Lax solution to the Riemann Problem

$$(20)\qquad \begin{cases} u_t + [F(u)]_x = 0, \\ u(x, 0) = \begin{cases} u^+ & \text{if } x < 0, \\ \overline{u} & \text{if } x > 0, \end{cases} \end{cases}$$

where u^+ is defined by the conditions

(*a*) $b(u^+) = g$, and

(*b*) u^+ is connected to $\overline{u}$ by means of the shock-rarefaction curves of the families $n - q + 1, \ldots, n$, in increasing order.

Because of the assumptions (*b*.1), (*b*.2) and (*b*.3), such a state u^+ exists and is unique.

As in case (C), we incorporate the boundary condition g and the boundary profile Ψ in the domain of the flow. Thus we obtain a semigroup acting on the set $\mathcal{D}^*$ of triples $\boldsymbol{p} = (\overline{u}, g, \Psi)$, where

$$\begin{aligned} &\overline{u} \in \boldsymbol{L}^1(\boldsymbol{R}, \boldsymbol{R}^n) \cap \boldsymbol{BV}(\boldsymbol{R}, \boldsymbol{R}^n) \quad \text{with } \overline{u}(x) = 0 \text{ for } x < \Psi(0), \\ &g \in \boldsymbol{L}^1(\boldsymbol{R}^+, \boldsymbol{R}^q) \cap \boldsymbol{BV}(\boldsymbol{R}^+, \boldsymbol{R}^q), \\ &J \in \boldsymbol{C}^0(\boldsymbol{R}^+, \boldsymbol{R}) \text{ Lipschitzean and satisfying (3.1)}. \end{aligned}$$

Similarly to the previous case, define

$$(3.4)\qquad \begin{cases} \mathrm{TV}(\boldsymbol{p}) \doteq \mathrm{TV}(\overline{u}) + \mathrm{TV}(g) + \|b(\overline{u}(\Psi(0))) - g(0)\|, \\ d(\boldsymbol{p}', \boldsymbol{p}'') \doteq \|\overline{u}'' - \overline{u}'\|_{L^1} + \|g'' - g'\|_{L^1} + \|\Psi'' - \Psi'\|_{C^0}. \end{cases}$$

DEFINITION NC.2. A Standard Riemann Semigroup generated by (NC) is a continuous map

$$\begin{array}{rccccc} S\colon & \boldsymbol{R}^+ & \times & \mathcal{D} & \mapsto & \mathcal{D} \\ & t & , & \boldsymbol{p} & \mapsto & S_t\boldsymbol{p}\,, \end{array}$$

such that

(1) $\mathcal{D}$ is a subset of $\mathcal{D}^*$ containing all triples $\boldsymbol{p}$ with $\mathrm{TV}(\boldsymbol{p})$ sufficiently small.

(2) S is Lipschitzean, i.e.

$$d(S_{t''}\boldsymbol{p}'', S_{t'}\boldsymbol{p}') \leq L\cdot(d(\boldsymbol{p}', \boldsymbol{p}'') + |t''-t'|) \tag{3.5}$$

for a fixed L, for all $\boldsymbol{p}'$, $\boldsymbol{p}''$ in $\mathcal{D}$ and for all t', t'' in $\boldsymbol{R}^+$, d being the distance defined at (3.4). Moreover, if $\boldsymbol{p} = (\bar{u}, \tilde{u}, \Psi)$, then $S_t\boldsymbol{p} = (E_t\boldsymbol{p}, \mathcal{T}_t\tilde{u}, \mathcal{T}_t\Psi)$, where E is the evolution operator and $\mathcal{T}_t$ is the time-translation operator.

(3) If $\bar{u}$ and g are piecewise constant and if Ψ is piecewise linear and continuous, then $E_t\boldsymbol{p}$ coincides for t small with the glueing of the Lax solutions to Riemann Problems in the points of jumps of $\bar{u}$, and of the solution to the Non Characteristic Riemann Problem with Boundary at $(0, \Psi(0))$.

Note that (2) above implies that for all $\boldsymbol{p}'$, $\boldsymbol{p}''$ in $\mathcal{D}$ and for all t', t'' in $\boldsymbol{R}^+$

$$\|E_{t'}\boldsymbol{p}' - E_{t''}\boldsymbol{p}''\|_{L^1} \leq L\cdot(d(\boldsymbol{p}', \boldsymbol{p}'') + |t''-t'|)$$

where L is the same constant as in (3.5).

4. – Uniqueness of the standard Riemann semigroup.

Assume that the Characteristic Initial-Boundary Problem for (1.1) generates a SRS with associated evolution operator E. Call $(\bar{u}, \tilde{u}, \Psi)$ the triple of the initial data, the boundary condition and the boundary profile in (C). Aim of this section is to prove that the SRS associated to the Characteristic Problem (C) is unique. Furthermore, the SRS yields weak entropic solutions. To this end, it will be proved that the function $t \mapsto E_t(\bar{u}, \tilde{u}, \Psi)$ coincides with any solution to (C) constructed in [1]. We consider only case (C), since the other one is entirely analogous.

All the proofs are deferred to Section 6.

THEOREM 4.1. *Assume that* (1.1) *generates a SRS with associated evolution operator* $E: \boldsymbol{R}^+ \times \mathcal{D} \mapsto \boldsymbol{L}^1$ *and that the triple* $(\bar{u}, \tilde{u}, \Psi)$ *in problem* (C) *belongs to* $\mathcal{D}$. *Let* $u: \boldsymbol{R}^+ \mapsto \boldsymbol{L}^1$ *be a solution to* (C) *constructed in* [1] *as limit of wave-front tracking approximations,* u_n. *Then*

$$u(t, x) = E_t(\bar{u}, \tilde{u}, \Psi)(x) \qquad \textit{for all } (t, x) \in \Omega .$$

As a consequence, if the SRS generated by the initial-boundary problem for (1.1) exists, then

- it yields weak entropic solutions to (C);
- it is unique, up to the domain.

In particular, the above properties are enjoied by the semigroup constructed in [2], for $n = 2$.

A key point in the proof of Theorem 4.1 is the following estimate on piecewise constant approximate solutions. This result can be interpreted as an analog for Conservation Laws of the Gronwall Lemma for O.D.E.s.

PROPOSITION 4.2. *Let* $E: \boldsymbol{R}^+ \times \mathcal{D} \mapsto \boldsymbol{L}^1$ *be the evolution operator associated with a SRS generated by* (C). *Let* $\tilde{v}: \boldsymbol{R}^+ \mapsto \boldsymbol{R}^n$ *be piecewise constant and* $\Psi: \boldsymbol{R}^+ \mapsto \boldsymbol{R}$ *be piecewise linear and continuous. Let* $v: [0, T] \mapsto \boldsymbol{L}^1(\boldsymbol{R}; \boldsymbol{R}^n)$ *be a continuous map, piecewise constant in the* (t, x)*-plane, vanishing outside* Ω, *with discontinuities occurring along finitely many polygonal lines, and such that*

$$(v(t, \cdot), \mathcal{C}_t \tilde{v}, \mathcal{C}_t \Psi) \in \mathcal{D}, \qquad \textit{for all } t \in [0, T[$$

Then, for all $t \in [0, T]$

$$\|v(t) - E_t(v(0, \cdot), \tilde{v}, \Psi)\|_{L^1} \leq \tag{4.1}$$

$$\leq L \cdot \int_0^t \limsup_{h \to 0+} \frac{\|v(\tau\| + h) - E_h(v(\tau), \mathcal{C}_\tau \tilde{v}, \mathcal{C}_\tau \Psi)\|_{L^1}}{h} \, d\tau .$$

The proof is almost identical to that of Lemma 4 in [5].

We conclude this Section with two results concerning the dependency domain of the solution provided by the SRS near the boundary. Away from the boundary, the same results as in [5] still hold.

PROPOSITION 4.3. *Assume that there exists a* SRS *generated by the Characteristic Initial-Boundary Problem for* (1.1). *Call E the associated evolution operator. Let* $\hat{\lambda}$ *be an upper bound for all characteristic* *speeds. Fix some* $\varrho \in \mathbf{R}$ *and two triples* $\mathbf{p}' \doteq (\overline{u}', \tilde{u}', \Psi')$ *and* $\mathbf{p}'' \doteq (\overline{u}'', \tilde{u}'', \Psi'')$ *in* $\mathcal{D}$.

$$\text{If } \begin{cases} \overline{u}'(x) = \overline{u}''(x) & \forall x \in]\Psi'(0), \varrho] \\ \tilde{u}'(t) = \tilde{u}''(t) & \forall t \in [0, \tau] \\ \Psi'(t) = \Psi''(t) & \forall t \in [0, \tau] \end{cases}$$

$$\text{then } (E_t\mathbf{p}')(x) = (E_t\mathbf{p}'')(x), \qquad \forall (t, x) \in \mathbf{R}^2 \text{ with } \begin{cases} t \leqslant \tau, \\ x \leqslant \varrho - \hat{\lambda} t. \end{cases}$$

The proof is as in [5] and relies on the construction of the solution as a limit of the approximate solutions defined in [1], by means of a wave front tracking technique. In fact, the equality above is satisfied by all those approximate solutions.

REMARK. The distance defined at (2.9) may become infinite. However, by the Proposition above, the solution $u(t, \cdot)$ to (C) computed at time t depends only on the restrictions of the boundary profile and of the boundary data to $[0, t]$. Thus, the whole construction may well be carried out in an arbitrary bounded time interval $[0, T]$, making the distance (2.9) finite for any pair $\mathbf{p}', \mathbf{p}''$ in $\mathcal{D}^*$. The choice of such a T has no influence on the construction.

PROPOSITION 4.4. *Let the Characteristic Initial-Boundary Problem for* (1.1) *admit a SRS defined on* $\mathcal{D}$ *with associated evolution operator* E. *Let* $\hat{\lambda}$ *be an upper bound for all characteristic speeds. Fix some* $\varrho \in \mathbf{R}$ *and two triples* $\mathbf{p}' \doteq (\overline{u}', \tilde{u}', \Psi')$ *and* $\mathbf{p}'' \doteq (\overline{u}'', \tilde{u}'', \Psi'')$ *in* $\mathcal{D}$.

If Ψ' (resp. Ψ'') *is Lipschitzean with constant* L' (resp. L''), *then for all* $t \geqslant 0$

$$\int_{-\infty}^{\varrho - \hat{\lambda} t} \|(E_t\mathbf{p}')(x) - (E_t\mathbf{p}'')(x)\| \, dx \leqslant \tag{4.2}$$

$$\leqslant L \cdot \left(\int_{-\infty}^{\varrho} \|\overline{u}'(x) - \overline{u}''(x)\| \, dx + \|\Psi' - \Psi''\|_{C^0([0,t])} \right) +$$

$$+ L \cdot (1 + L' + L'') \int_0^t \|\tilde{u}'(x) - \tilde{u}''(x)\| \, dx .$$

If the boundary data coincide $\tilde{u}' = \tilde{u}''$ *and the boundary profiles*

Ψ', Ψ'' are (arbitrary) continuous functions, then for all $t \geq 0$

$$\int_{-\infty}^{\varrho - \hat{\lambda} t} \| (E_t \boldsymbol{p}')(x) - (E_t \boldsymbol{p}'')(x) \| \, dx \leq \tag{4.3}$$

$$\leq L \cdot \left(\int_{-\infty}^{\varrho} \| \bar{u}'(x) - \bar{u}''(x) \| \, dx + \| \Psi' - \Psi'' \|_{C^0([0,\, t])} \right).$$

5. - Viscosity solutions.

In this section we introduce a suitable definition of *viscosity solution* for (C) and (NC), which characterizes the solutions provided by the corresponding SRS. This will extend the results of Section 4 in [5] to the case of problems (C) or (NC).

Let $u : [0, T] \times \boldsymbol{R} \mapsto \boldsymbol{R}^n$ be a locally integrable function with $u(t, \cdot) \in \boldsymbol{BV}$ for all $t \in [0, T]$, and fix any point (τ, ξ) in the domain of u. In order to give a meaning to the pointwise values of u, we shall consider the $\boldsymbol{L}^1$-representative of $u(t, \cdot)$ which is right continuous.

Assume first that $\xi > \Psi(\tau)$. As in [5], call $\widehat{A} \doteq DF(u(\tau, \xi))$ the Jacobian matrix of F computed at $u(\tau, \xi)$. For $t > \tau$, define $U^{\flat}_{(u;\, \tau,\, \xi)}(t, x)$ as the solution of the linear hyperbolic Cauchy Problem with constant coefficients

$$\begin{cases} w_t + \widehat{A} w_x = 0\,, \\ w(\tau, x) = u(\tau, x)\,. \end{cases}$$

Next, call $\omega^{\#}$ the self-similar solution, centered at (τ, ξ), to the Riemann Problem

$$\begin{cases} \omega_t + [F(\omega)]_x = 0\,, \\ \omega(\tau, x) = \begin{cases} u(\tau, \xi-) & \text{if } x < \xi\,, \\ u(\tau, \xi+) & \text{if } x > \xi\,. \end{cases} \end{cases} \tag{5.1}$$

Let $\hat{\lambda}$ be an upper bound for all characteristic speeds. For $t > \tau$, define

$$U^{\#}_{(u;\, \tau,\, \xi)}(t, x) \doteq \begin{cases} \omega^{\#}(t, x) & \text{if } |x - \xi| \leq \hat{\lambda}(t - \tau)\,, \\ u(\tau, x) & \text{if } |x - \xi| > \hat{\lambda}(t - \tau)\,, \end{cases} \tag{5.2}$$

The function $t \mapsto U^{\#}_{(u;\,\tau,\,\xi)}(t,\,\cdot)$ is Lipschitz continuous w.r.t. the L^1 distance, and approaches $u(\tau,\,\cdot)$ as $t \to \tau +$.

Assume now that $\xi = \Psi(\tau)$. Then, similarly to above, define $\omega^{\#}$ to be the standard solution to the Characteristic Riemann Problem with Boundary, see (2.7)

$$(5.3)\qquad \begin{cases} \omega_t + [F(x)]_x = 0 & (t,\,x) \in \Omega\,, \\ \omega(\tau,\,x) = u(\tau,\,\xi +) & x > \xi\,, \\ \omega(t,\,\Psi(t)) = \tilde{u}(\tau +) & t > \tau\,, \end{cases}$$

and for $t > \tau$, $(t,\,x) \in \Omega$ define

$$(5.4)\qquad U^{\#}_{(u;\,\tau,\,\xi)}(t,\,x) \doteq \begin{cases} \tilde{u}(\sup\{s \in [\tau,\,t];\ \Psi(s) = x\}) & \text{if } x \in [\Psi(t),\,\check{\Psi}_\tau(t)[\,, \\ w^{\#}(t,\,x) & \text{if } x \in [\check{\Psi}_\tau(t),\,\widehat{\Psi}_\tau(t)]\,, \\ u(\tau,\,x) & \text{if } x > \widehat{\Psi}_\tau(t)\,, \end{cases}$$

where

$$(5.5)\qquad \check{\Psi}_\tau(t) \doteq \mathop{sup}_{s \in [\tau,\,t]} (\Psi(s) - \hat{\lambda}\cdot(t - s))$$

$$\text{and}\qquad \widehat{\Psi}_\tau(t) \doteq \sup_{s \in [\tau,\,t]} (\Psi(s) + \hat{\lambda}\cdot(t - s))$$

and $\hat{\lambda}$ is as above an upper bound for all characteristic speeds. A few comments are in order. The functions $\widehat{\Psi}$ and $\check{\Psi}$ are defined so that the region $\{(t,\,x)\colon t \geq \tau,\ x \in [\check{\Psi}_\tau(t),\,\widehat{\Psi}_\tau(t)]\}$ contains all the waves originating from the jump at $(\tau,\,\Psi(\tau))$.

Concerning the first line in (5.4), the value of $U^{\#}_{(u;\,\tau,\,\xi)}$ at some $(t_*,\,x_*)$ in Ω with x_* belonging to the interval $[\Psi(t_*),\,\check{\Psi}_\tau(t_*)]$ can be obtained as follows. Trace from $(t_*,\,x_*)$ the vertical half-line $x = x_*$, $t \leq t_*$. The interval $[\Psi(t_*),\,\check{\Psi}_\tau(t_*)]$ is not empty, hence this half-line intersects the boundary for the first time at some $(\bar{t},\,\Psi(\bar{t}))$, with $\bar{t} < t_*$. Then $U^{\#}_{(u;\,\tau,\,\xi)}(t_*,\,x_*) = \tilde{u}(\bar{t})$, see Figure 3.

Note also that the map $t \mapsto U^{\#}_{(u;\,\tau,\,\xi)}$ is continuous in L^1. Moreover, if $\bar{u}$ and $\tilde{u}$ are both constant, then $U^{\#}_{(u;\,\tau,\,\xi)}(t,\,x) = \omega^{\#}(t,\,x)$.

In the following, $\mathrm{TV}\{u(\tau);\,I\}$ denotes the total variation of the function $u(\tau,\,\cdot)$ over the set I.

DEFINITION C.3. Let $u\colon [0,\,T] \mapsto \boldsymbol{BV}$ be continuous w.r.t. the $\boldsymbol{L}^1_{\mathrm{loc}}$-topology. We say that u is a *viscosity solution* of the Characteristic Initial-Boundary problem (C) if there exists a constant $C > 0$ such that, at

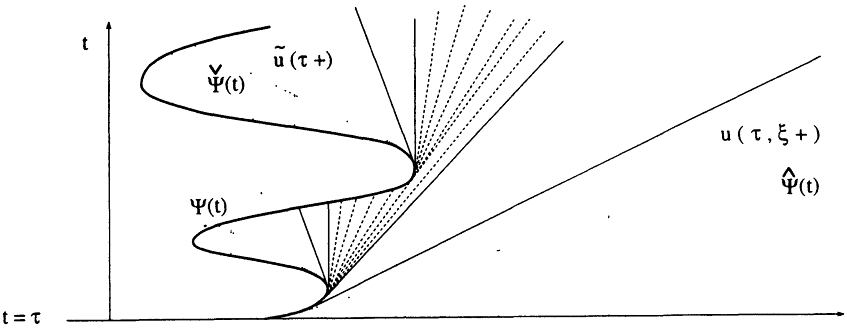

Figure 3.

each point $(\tau, \xi) \in \Omega$ with $\tau < T$, for all $\varrho, \varepsilon > 0$ sufficiently small one has

- if $\xi > \Psi(\tau)$:

$$\frac{1}{\varepsilon} \int_{\xi-\varrho+\varepsilon\hat{\lambda}}^{\xi+\varrho-\varepsilon\hat{\lambda}} \left\| u(\tau+\varepsilon, x) - U^{\#}_{(u;\,\tau,\,\xi)}(\tau+\varepsilon, x) \right\| dx \leqslant \tag{5.6}$$

$$\leqslant C \cdot \mathrm{TV}\{u(\tau);]\xi-\varrho, \xi[\cup]\xi, \xi+\varrho[\},$$

$$\frac{1}{\varepsilon} \int_{\xi-\varrho+\varepsilon\hat{\lambda}}^{\xi+\varrho-\varepsilon\hat{\lambda}} \left\| u(\tau+\varepsilon, x) - U^{\flat}_{(u;\,\tau,\,\xi)}(\tau+\varepsilon, x) \right\| dx \leqslant \tag{5.7}$$

$$\leqslant C \cdot (\mathrm{TV}\{u(\tau);]\xi-\varrho, \xi+\varrho[\})^2 ;$$

- if $\xi = \Psi(\tau)$:

$$\frac{1}{\varepsilon} \int_{\Psi(\tau+\varepsilon)}^{\xi+\varrho-\varepsilon\hat{\lambda}} \left\| u(\tau+\varepsilon, x) - U^{\#}_{(u;\,\tau,\,\xi)}(\tau+\varepsilon, x) \right\| dx \leqslant \tag{5.8}$$

$$\leqslant C \cdot (\mathrm{TV}\{u(\tau);]\xi, \xi+\varrho[\} + \mathrm{TV}\{\tilde{u};]\tau, \tau+\varepsilon[\}) .$$

The next result shows that the solutions provided by the SRS are indeed viscosity solutions in the sense defined above.

THEOREM 5.1. *Assume that the Characteristic Initial-Boundary Problem for* (1.1) *generates a SRS on the domain* $\mathcal{D}$. *Call E the* *associated evolution operator. If* $(\overline{u}, \tilde{u}, \Psi) \in \mathcal{D}$, *then the map* $t \mapsto E_t(\overline{u}, \tilde{u}, \Psi)$ *is a viscosity solution to* (C).

For the proof, see Section 6.

The following is slightly more general than the converse to the above Theorem 5.1.

THEOREM 5.2. *Assume that the Characteristic Initial-Boundary Problem for* (1.1) *generates a SRS. Call E the associated evolution operator. Let* $u : [0, T] \mapsto \boldsymbol{L}^1$ *be continuous with* $u(0) = \overline{u}$ *and* $(u(t), \mathcal{T}_t \tilde{u}, \mathcal{T}_t \Psi) \in \mathcal{D}$ *for* $t \in [0, T]$. *Assume that for all but countably many times* $\tau \in [0, T]$, *there exist two positive Radon measures* μ_τ *and* $\tilde{\mu}_\tau$ *on* $\boldsymbol{R}$ *such that*

- *if* $\xi > \Psi(\tau)$:

$$\frac{1}{\varepsilon} \int_{\xi - \varrho + \varepsilon \hat{\lambda}}^{\xi + \varrho - \varepsilon \hat{\lambda}} \| u(\tau + \varepsilon, x) - U^{\#}_{(u; \tau, \xi)}(\tau + \varepsilon, x) \| \, dx \leqslant$$

$$\leqslant \mu_\tau(]\xi - \varrho, \xi[\cup]\xi, \xi + \varrho[),$$

$$\frac{1}{\varepsilon} \int_{\xi - \varrho + \varepsilon \hat{\lambda}}^{\xi + \varrho - \varepsilon \hat{\lambda}} \| u(\tau + \varepsilon, x) - U^{\flat}_{(u; \tau, \xi)}(\tau + \varepsilon, x) \| \, dx \leqslant$$

$$\leqslant (\mu_\tau(]\xi - \varrho, \xi + \varrho[))^2 ;$$

- *if* $\xi = \Psi(\tau)$:

$$\frac{1}{\varepsilon} \int_{\Psi(\tau + \varepsilon)}^{\xi + \varrho - \varepsilon \hat{\lambda}} \| u(\tau + \varepsilon, x) - U^{\#}_{(u; \tau, \xi)}(\tau + \varepsilon, x) \| \, dx \leqslant \tag{5.9}$$

$$\leqslant \mu_\tau(]\xi, \xi + \varrho[) + \tilde{\mu}_\tau(]\tau, \tau + \varepsilon[),$$

for all $\varrho, \varepsilon > 0$ *sufficiently small. Then, for all t in* $[0, T]$

$$u(t) = E_t(\overline{u}, \tilde{u}, \Psi).$$

The proof of this Theorem is entirely similar to the proof of Theorem 4 in [5]. Observe that Theorem 5.2 can be used, in particular, with μ_τ = TV measure of $u(\tau, \cdot)$, $\tilde{\mu}_\tau$ = TV measure of $\tilde{u}$ (independent of τ).

All the previous results remain valid in the Non Characteristic Initial-Boundary value problem, with only a few minor modifications. If $\omega^{\#}$ is the solution to

$$\begin{cases} \omega_t + [F(x)]_x = 0 & (t, x) \in \Omega , \\ \omega(\tau, x) = u(\tau, \xi +) & x > \xi , \\ b(\omega(t, \Psi(t))) = g(\tau +) & t > \tau , \end{cases}$$

then define

$$U^{\#}_{(u;\, \tau,\, \xi)}(t, x) \doteq \begin{cases} w^{\#}(t, x) & \text{if } \Psi(t) < x < \Psi(\tau) + \hat{\lambda}(t-\tau) , \\ u(\tau, x) & \text{if } x > \Psi(\tau) + \hat{\lambda}(t-\tau) , \end{cases}$$

in place of (5.3). The statements and proofs of the analogous to Theorems 5.1 and 5.2 in the Non Characteristic case are entirely similar.

6. – Technical proofs.

We begin by showing that the function introduced at (2.7) is indeed a solution of the Characteristic Riemann Problem with Boundary (2.4), in the sense of Definition C.1.

We briefly recall the construction of piecewise constant $\{u^{\nu}: \nu \geq 1\}$ to the Riemann Problem (2.6), according to [1]. Given a state $u \in \boldsymbol{R}^n$, let $\sigma \mapsto \psi_i(u, \sigma)$ denote the i-th shock-rarefaction curve through u, parametrized by means of the arc-length σ. Given two nearby states $\bar{u}$ and $\tilde{u}$ as in (2.6), introduce the intermediate values $\omega_0, \ldots, \omega_n$ defined inductively by

$$\omega_0 \doteq \tilde{u}, \qquad \omega_i \doteq \psi_i(\omega_{i-1}, \sigma_i) \quad \text{for } i = 1, \ldots, n \qquad \omega_n \doteq \bar{u} .$$

If $\sigma_i > 0$ and the i-th characteristic field is genuinely non linear, let

$$\begin{cases} \omega_{i,l} \doteq \psi_i(\omega_{i-1}, l/\nu\, \sigma_i) & \text{for } l = 0, \ldots, \nu , \\ \Lambda_{i,l} \doteq \lambda_i(\omega_{i,l}) & \text{for } l = 1, \ldots, \nu , \\ l_i \doteq \nu , \end{cases}$$

otherwise, set

$$\omega_{i,0} \doteq \omega_i , \qquad \Lambda_{i,0} \doteq \lambda_i(\omega_{i-1}, \omega_i) , \qquad l_i \doteq 0 .$$

Define

$$
(6.1) \qquad u_R^\nu(t, x) \doteq \begin{cases} \tilde{u} & \text{if } x < \Lambda_{1,0} \cdot t\,, \\ \omega_{i,0} & \text{if } l_i > 0 \text{ and } x \in]\Lambda_{i-1, l_{i-1}} \cdot t, \Lambda_{i,1} \cdot t[\,, \\ \omega_{i,l} & \text{if } x \in]\Lambda_{i,l} \cdot t, \Lambda_{i,l+1} \cdot t[,\ l = 1, \ldots, l_i - 1\,, \\ \omega_{i,l_i} & \text{if } x \in]\Lambda_{i,l_i} \cdot t, \Lambda_{i+1,0} \cdot t[\,, \\ \bar{u} & \text{if } x > \Lambda_{n,l_n} \cdot t\,. \end{cases}
$$

Observe that

$$
(6.2) \qquad \lim_{\nu \to +\infty} u^\nu(t, x) = u(t, x) \qquad \text{uniformly for } t > 0\,, \quad x \in \mathbf{R}\,.
$$

The following Proposition states a sufficient condition for a sequence of approximate solutions to converge to the solution provided by the SRS.

PROPOSITION 6.1. *Let* $E: \mathbf{R}^+ \times \mathcal{D} \mapsto \mathbf{L}^1$ *be the evolution operator associated to a SRS for the Characteristic Initial-Boundary Problem for* (1.1). *For* $\nu \in \mathbf{N}$, *let* $\tilde{u}_\nu: \mathbf{R}^+ \mapsto \mathbf{R}$ *be piecewise constant,* $\Psi_\nu: \mathbf{R}^+ \mapsto \mathbf{R}$ *be piecewise linear and continuous. Let* $u_\nu: [0, T] \mapsto \mathbf{L}^1(\mathbf{R}; \mathbf{R}^n)$ *be continuous, piecewise constant in the* (t, x)*-plane with discontinuities occuring along finitely many polygonal lines. Assume that*

(*a*) *For all* $t \in [0, T]$, $(u_\nu(t, \cdot), \mathcal{T}_t \tilde{u}_\nu, \mathcal{T}_t \Psi_\nu) \in \mathcal{D}$.

(*b*) *The instantaneous rate of error is uniformly bounded w.r.t.* ν *and tends to zero as* $\nu \to +\infty$, *for a.e.* t:

$$
\lim_{\nu \to +\infty} \left(\limsup_{h \to 0+} \frac{\|u_\nu(t+h) - E_h(u_\nu(t), \mathcal{T}_t \tilde{u}_\nu, \mathcal{T}_t \Psi_\nu)\|_{\mathbf{L}^1}}{h} \right) = 0\,.
$$

(*c*) *There exist functions* $u: [0, T] \mapsto \mathbf{L}^1(\mathbf{R}; \mathbf{R}^n)$, $\tilde{u}: [0, T] \mapsto \mathbf{R}^n$ *and* $\Psi: [0, T] \mapsto \mathbf{R}$ *such that for all* $t \in [0, T]$ $(u(t, \cdot), \mathcal{T}_t \tilde{u}, \mathcal{T}_t \Psi) \in \mathcal{D}$ *and*

$$
\lim_{\nu \to +\infty} d((u_\nu(t, \cdot), \mathcal{T}_t \tilde{u}_\nu, \mathcal{T}_t \Psi_\nu), (u(t, \cdot), \mathcal{T}_t \tilde{u}, \mathcal{T}_t \Psi)) = 0\,.
$$

Then, for all $t \in [0, T]$

$$
u(t, \cdot) = E_t(\bar{u}, \tilde{u}, \Psi)\,.
$$

PROOF. The continuity of the semigroup implies that

$$\lim_{\nu \to +\infty} \|E_t(\bar{u}, \tilde{u}, \Psi) - E_t(u_\nu(0, \cdot), \tilde{u}_\nu, \Psi_\nu)\|_{L^1} = 0 \tag{6.3}$$

thus, by (c) and the triangle inequality, it is sufficient to prove that

$$\lim_{\nu \to +\infty} \|u_\nu(t, \cdot) - E_t(u_\nu(0, \cdot), \tilde{u}_\nu, \Psi_\nu)\|_{L^1} = 0 .$$

By Proposition 4.2 one has

$$\|u_\nu(t, \cdot) - E_t(u_\nu(0, \cdot), \tilde{u}_\nu, \Psi_\nu)\|_{L^1} \leq$$

$$\leq L \int_0^t \limsup_{h \to 0+} \frac{\|u_\nu(\tau + h) - E_h(u_\nu(\tau), \mathcal{T}_\tau \tilde{u}_\nu, \mathcal{T}_\tau \Psi_\nu)\|_{L^1}}{h} d\tau .$$

The assumption (b) ensures that the last r.h.s. above tends to zero as $\nu \to +\infty$, concluding the proof. ■

PROOF OF THEOREM 4.1. It is enough to prove that the sequence of approximate solutions defined in [1] satisfy the assumptions (a), (b) and (c) of Proposition 6.1.

By the definition of the approximate boundary condition, boundary profile and by the bounds (4.2) in [1] on the total variation of the approximate solution, (a) and (c) hold. Concerning (b), we refer to the construction in Section 3 of [1].

Fix $T > 0$. For ν in N, choose t in $[0, T]$ so that at t no wave-front in the approximate solution $u_\nu(t, \cdot)$ interacts with the (approximate) boundary, the approximate boundary condition $\tilde{u}_\nu$ is locally constant and the approximate boundary profile Ψ_ν is locally linear. The analysis in [1] ensures that the above choice excludes at most a finite number of times on any bounded time interval.

Let $x_\alpha(t)$, $\alpha = 1, \ldots, N$ be locations of the discontinuities in $u_\nu(t, \cdot)$ corresponding to wave-fronts with *generation order* (see [1] Section 3) less or equal to ν; denote with ε_α the corresponding size.

Call $y_\beta(t)$, $\beta = 1, \ldots, N'$, the (locations of the) *non-physical waves* with speed $\hat{\lambda}$.

Let $\mathcal{S}$ be the set of indexes α such that $u_\nu(t, x_\alpha -)$ and $u_\nu(t, x_\alpha +)$ are connected by a shock or by a contact discontinuity. Denote by ω^α and respectively by w^β the Lax (exact) solution to the Riemann problems

$$\begin{cases} u_t + [F(u)]_x = 0 \,, \\ u(\bar t, x) = \begin{cases} u_\nu(\bar t, x_\alpha -) & \text{if } x < x_\alpha \,, \\ u_\nu(\bar t, x_\alpha +) & \text{if } x > x_\alpha \,, \end{cases} \end{cases}$$

$$\text{and} \quad \begin{cases} u_t + [F(u)]_x = 0 \,, \\ u(\bar t, x) = \begin{cases} u_\nu(\bar t, y_\beta -) & \text{if } x < y_\beta \,, \\ u_\nu(\bar t, y_\beta +) & \text{if } x > y_\beta \,, \end{cases} \end{cases}$$

respectively. Call $\mathcal{R}$ the set of indexes α corresponding to rarefaction waves of a genuinely nonlinear family.

Recalling Remark 4.1 in [1], the jumps at x_α for $\alpha \in S$ satisfy the Rankine-Hugoniot conditions, while the jumps at x_α for $\alpha \in \mathcal{R}$ approximate them with an error that vanishes for $\nu \to +\infty$, and the maximum size of a rarefaction wave in u_ν tends to zero as $\nu \to +\infty$. On the other hand, the total amplitude of the jumps at y_β tends to zero as $\nu \to +\infty$, i.e.

$$\lim_{\nu \to +\infty} \sum_{\beta=1}^{N'} \|u_\nu(\bar t, y_\beta -) - u_\nu(\bar t, y_\beta +) = 0 \,. \tag{6.4}$$

By property (3) in the definition of S, Lemma 3 in [5] and the above (6.4), one has

$$\limsup_{h \to 0+} \frac{\|u_\nu(t+h) - E_h(u_\nu(t), \mathcal{C}_t \tilde u_\nu, \mathcal{C}_t \Psi_\nu)\|}{h} =$$

$$= \sum_{\alpha \in \mathcal{R} \cup S} \left(\lim_{h \to 0+} \frac{1}{h} \int_{x_\alpha - \varrho}^{x_\alpha + \varrho} \|u_\nu(t+h, x) - \omega^\alpha(t+h, x)\| \, dx \right) +$$

$$+ \sum_{\beta=1}^{N'} \left(\lim_{h \to 0+} \frac{1}{h} \int_{y_\beta - \varrho}^{y_\beta + \varrho} \|u_\nu(t+h, x) - w^\beta(t+h, x)\| \, dx \right) \leq$$

$$\leq C \sum_{\alpha \in \mathcal{R}} \varepsilon_\alpha^2 + C \sum_{\beta=1}^{N'} \|u_\nu(t, y_\beta(t)+) - u_\nu(t, y_\beta(t)-)\| \leq$$

$$\leq C \max_{\alpha \in \mathcal{R}} \varepsilon_\alpha \cdot \left(\sum_{\alpha \in \mathcal{R}} \varepsilon_\alpha \right) + C \cdot 2^{-\nu} \,,$$

for $\varrho > 0$ suitably small and for some positive C, independent from ν. Since the last term is uniformly bounded w.r.t. ν and approaches zero as $\nu \to \infty$, the proof is completed. ■

PROOF OF PROPOSITION 4.4. Fix $t \geq 0$ and define

$$\begin{cases} \widehat{\boldsymbol{p}}' \doteq (\overline{u}'\chi_{]-\infty,\varrho]}, \widetilde{u}'\chi_{[0,t]}, \widehat{\Psi}') & \text{where } \widehat{\Psi}'(\tau) \doteq \begin{cases} \Psi'(\tau) & \text{if } \tau \leq t, \\ \Psi'(t) & \text{if } \tau > t; \end{cases} \\ \widehat{\boldsymbol{p}}'' \doteq (\overline{u}'\chi_{]-\infty,\varrho]}, \widetilde{u}''\chi_{[0,t]}, \widehat{\Psi}'') & \text{where } \widehat{\Psi}''(\tau) \doteq \begin{cases} \Psi''(\tau) & \text{if } \tau \leq t, \\ \Psi''(t) & \text{if } \tau > t. \end{cases} \end{cases}$$

Note that $\widehat{\boldsymbol{p}}'$, $\widehat{\boldsymbol{p}}''$ both belong to $\mathcal{D}$. Moreover, by Proposition 6.3 and by the above definition of $\widehat{\boldsymbol{p}}'$, $\widehat{\boldsymbol{p}}''$

$$\int_{-\infty}^{\varrho - \hat{\lambda} t} \|(E_t\boldsymbol{p}')(x) - (E_t\boldsymbol{p}'')(x)\|\,dx =$$

$$= \int_{-\infty}^{\varrho - \hat{\lambda} t} \|(E_t\widehat{\boldsymbol{p}}')(x) - (E_t\widehat{\boldsymbol{p}}'')(x)\|\,dx \leq \|(E_t\widehat{\boldsymbol{p}}') - (E_t\widehat{\boldsymbol{p}}'')\|_{L^1}.$$

To conclude the proof of (4.2), it is now sufficient to use (6.5) and the Lipschitz type estimate (2.8). The proof of (4.3) is entirely similar. ■

Before passing to the proofs relative to Section 5, we remark the following useful consequences of the Lipschitzean property (2.8). Assume that only one of the boundary profiles is Lipschitzean, say Ψ', with Lipschitz constant L'. Combining (2.8) and the triangle inequality

$$(6.6) \qquad \|E_t\boldsymbol{p}'' - E_t\boldsymbol{p}'\|_{L^1} \leq$$

$$\leq L\cdot(\|\overline{u}'' - \overline{u}'\|_{L^1} + \|\Psi'' - \Psi'\|_{C^0} + (1 + 2L')\cdot\|\widetilde{u}'' - \widetilde{u}'\|_{L^1}).$$

A further consequence of (2.8) is the following. Given a continuous boundary profile Ψ, consider two triples $\boldsymbol{p}' = (\overline{u}', \widetilde{u}', \Psi)$ and $\boldsymbol{p}'' = (\overline{u}'', \widetilde{u}'', \Psi)$. If $\Phi: \boldsymbol{R}^+ \mapsto \boldsymbol{R}$ is a function with Lipschitz constant L_Φ, then

$$(6.7) \qquad \|E_t\boldsymbol{p}'' - E_t\boldsymbol{p}'\|_{L^1} \leq$$

$$\leq L\cdot(\|\overline{u}'' - \overline{u}'\|_{L^1} + 2\cdot\|\Psi - \Phi\|_{C^0} + (1 + 2L_\Phi)\cdot\|\widetilde{u}'' - \widetilde{u}'\|_{L^1}).$$

PROOF OF THEOREM 5.1. Fix $T > 0$. It will be proved that $t \mapsto E_t(\overline{u}, \tilde{u}, \Psi)$ is a viscosity solution on $[0, T]$. Note first that the continuity requirement in definition C.3 is clearly satisfied, due to (2.8). Choose now a point (τ, ξ) in Ω, with $\tau < T$.

If $\xi > \Psi(\tau)$, then the same proof as in [5] still holds.

Assume that $\xi = \Psi(\tau)$. If $(\overline{u}, \tilde{u}, \Psi)$ is the triple of the initial data, boundary data and boundary profile in (C), call $\boldsymbol{p} \doteq S_\tau(\overline{u}, \tilde{u}, \Psi)$ and $u(t, \cdot \cdot) \doteq E_t(\overline{u}, \tilde{u}, \Psi)$. Since $\overline{u}, \tau$ and ξ are kept fixed, to simplify the notation we let $U_\Psi^\# \doteq U^\#_{(\overline{u};\, \tau,\, \xi)}$, $\breve{\Psi} \doteq \breve{\Psi}_\tau$ and $\widehat{\Psi} \doteq \widehat{\Psi}_\tau$.

First fix $\varepsilon > 0$ and $\varrho > 0$ with $\tau + \varrho < T$.

Moreover, choose an arbitrary $\varepsilon' > 0$ and a continuous function $\Phi: [0, T] \mapsto \boldsymbol{R}$, piecewise linear on $[\tau, T]$, with Lipschitz constant L_Φ on $[\tau, T]$, such that

$$(6.8) \qquad \begin{cases} \forall t \in [0, \tau] \ \ \Phi(t) = \Psi(t), & \|\Psi - \Phi\|_{C^0([\tau, T])} \leq \varepsilon', \\ \forall t \in [\tau, T] \ \ |\dot{\Phi}(t+)| \geq \hat{\lambda}, & \forall t \in]\tau, T] \ \ |\dot{\Phi}(t-) \geq \hat{\lambda}. \end{cases}$$

Moreover, define $U_\Phi^\#$ as $U_\Psi^\#$ but replacing Ψ in (5.3) and (5.4) with Φ. Then

$$(6.9) \qquad \int_{\Psi(\tau+\varepsilon)}^{\xi + \varrho - \varepsilon\hat{\lambda}} \|u(\tau + \varepsilon, x) - U_\Psi^\#(\tau + \varepsilon, x)\| \, dx \leq$$

$$\leq \int_{\Psi(\tau+\varepsilon)}^{\xi + \varrho - \varepsilon\hat{\lambda}} \|E_{\tau+\varepsilon}(\overline{u}, \tilde{u}, \Psi)(x) - E_{\tau+\varepsilon}(\overline{u}, \tilde{u}, \Phi)(x)\| \, dx +$$

$$+ \int_{\Psi(\tau+\varepsilon)}^{\xi + \varrho - \varepsilon\hat{\lambda}} \|E_{\tau+\varepsilon}(\overline{u}, \tilde{u}, \Phi)(x) - U_\Phi^\#(\tau + \varepsilon, x)\| \, dx +$$

$$+ \int_{\Psi(\tau+\varepsilon)}^{\xi + \varrho - \varepsilon\hat{\lambda}} \|U_\Phi^\#(\tau + \varepsilon, x) - U_\Psi^\#(\tau + \varepsilon, x)\| \, dx.$$

By (6.6) applied to the time interval $[\tau, T]$, the first summand in the r.h.s. above is bounded by $\mathcal{O}(1) \cdot \varepsilon'$. Moreover, by construction, $U^\#$ is a continuous function in $\boldsymbol{L}^1$ of the boundary profile in $\boldsymbol{C}^0$. Thus, using the dominated convergence theorem, the third summand is $o(1)$ as $\varepsilon' \to 0$ (i.e. it tends to 0 as ε' tends to 0).

Concerning the second term in (6.14), one has

$$\int_{\Psi(\tau+\varepsilon)}^{\xi+\varrho-\varepsilon\hat{\lambda}} \|E_{\tau+\varepsilon}(\overline{u}, \tilde{u}, \Phi)(x) - U_{\Phi}^{\#}(\tau+\varepsilon, x)\| \, dx =$$

$$= \int_{\Psi(\tau+\varepsilon)}^{\Phi(\tau+\varepsilon)} \|E_{\tau+\varepsilon}(\overline{u}, \tilde{u}, \Phi)(x) - U_{\Phi}^{\#}(\tau+\varepsilon, x)\| \, dx +$$

$$+ \int_{\Phi(\tau+\varepsilon)}^{\xi+\varrho-\varepsilon\hat{\lambda}} \|E_{\tau+\varepsilon}(\overline{u}, \tilde{u}, \Phi)(x) - U_{\Phi}^{\#}(\tau+\varepsilon, x)\| \, dx \, .$$

The first term above is bounded by $\mathcal{O}(1)\cdot\varepsilon'$, due to (6.8). For the second term, we will prove below that

$$\int_{\Phi(\tau+\varepsilon)}^{\xi+\varrho-\varepsilon\hat{\lambda}} \|E_{\tau+\varepsilon}(\overline{u}, \tilde{u}, \Phi)(x) - U_{\Phi}^{\#}(\tau+\varepsilon, x)\| \, dx \leqslant \tag{6.10}$$

$$\leqslant \varepsilon \cdot C \cdot (\mathrm{TV}\{u(\tau);]\xi, \xi+\varrho[\} + \mathrm{TV}\{\tilde{u};]\tau, \tau+\varepsilon[\}) + o(1)$$

where $\lim_{\varepsilon'\to 0} o(1) = 0$.

Indeed, introduce a triple $\boldsymbol{q} \doteq (\overline{v}, \tilde{v}, \mathcal{C}_\tau \Phi)$ in $\mathcal{D}$ such that

(i) $\overline{v}$ and $\tilde{v}$ are piecewise constant,

(ii) $\|\overline{v} - u(\tau, \cdot)\|_{L^1} \leqslant \varepsilon'$, $\|\tilde{v} - \mathcal{C}_\tau \tilde{u}\|_{L^1} \leqslant \varepsilon'/(1 + L_\Phi)$,

(iii) $\overline{v}(\xi+) = u(\tau, \xi+)$ and $\tilde{v}(0+) = \tilde{u}(\tau+)$,

(iv) $\mathrm{TV}\{\overline{v}:]\xi, \xi+\varrho[\} \leqslant \mathrm{TV}\{u(\tau, \cdot):]\xi, \xi+\varrho[\}$ and $\mathrm{TV}\{\tilde{v}:]0, \varepsilon[\} \leqslant \leqslant \mathrm{TV}\{\tilde{u}:]\tau, \tau+\varepsilon[\}$,

and define $\boldsymbol{q}(t) = E_{t-\tau}\boldsymbol{q}$. Let $\omega(t, \cdot)$ be the solution to the Characteristic Riemann Problem

$$\begin{cases} w_t + [F(w)]_x = 0 & \text{for } t > \tau \text{ and } x \geqslant \Phi(t), \\ w(\tau, x) = u(\tau, \xi+) & \text{for } x > \Phi(\tau), \\ w(t, \Phi(t)) = \tilde{u}(\tau+) & \text{for } t > \tau\,. \end{cases} \tag{6.11}$$

Finally, for $t > \tau$ and $(t, x) \in \Omega$, define

$$(6.12)\qquad v(t, x) \doteq \begin{cases} \tilde{v}(\sup([\tau, t] \cap \Phi^{-1}(x)) - \tau) & \text{if } x \in [\Phi(t), \check{\Phi}_\tau(t)[\,, \\ w(t, x) & \text{if } x \in [\check{\Phi}_\tau(t), \widehat{\Phi}_\tau(t)]\,, \\ \bar{v}(x) & \text{if } x > \widehat{\Phi}_\tau(t)\,. \end{cases}$$

Then

$$\int_{\Phi(\tau+\varepsilon)}^{\xi+\varrho-\varepsilon\hat{\lambda}} \|u(\tau+\varepsilon, x) - U_\Phi^\#(\tau+\varepsilon, x)\|\, dx \leqslant$$

$$(6.13)\qquad \leqslant \int_{\Phi(\tau+\varepsilon)}^{\xi+\varrho-\varepsilon\hat{\lambda}} \|u(\tau+\varepsilon, x) - (E_\varepsilon \boldsymbol{q})(x)\|\, dx +$$

$$(6.14)\qquad + \int_{\Phi(\tau+\varepsilon)}^{\xi+\varrho-\varepsilon\hat{\lambda}} \|(E_\varepsilon \boldsymbol{q})(x) - v(\tau+\varepsilon, x)\|\, dx +$$

$$(6.15)\qquad + \int_{\Phi(\tau+\varepsilon)}^{\xi+\varrho-\varepsilon\hat{\lambda}} \|v(\tau+\varepsilon, x) - U_\Phi^\#(\tau+\varepsilon, x)\|\, dx\,.$$

Consider the three summands above separately. (6.13) can be estimated using (2.8) on $[\tau, T]$:

$$(6.16)\qquad \int_{\Phi(\tau+\varepsilon)}^{\xi+\varrho-\varepsilon\hat{\lambda}} \|E_\varepsilon \boldsymbol{p} - E_\varepsilon \boldsymbol{q}\|\, dx \leqslant$$

$$\leqslant L \cdot (\|\bar{v} - u(\tau, \cdot)\|_{L^1} + (1 + 2L_\Phi) \cdot \|\tilde{v} - \mathcal{C}_\tau \tilde{u}\|_{L^1}) = \mathcal{O}(1) \cdot \varepsilon'\,.$$

Passing to (6.14), by (4.1) it follows that

$$(6.17)\qquad \int_{\Phi(\tau+\varepsilon)}^{\xi+\varrho-\varepsilon\hat{\lambda}} \|E_\varepsilon(\bar{v}, \tilde{v}, \mathcal{C}_\tau \Phi) - v(\tau+\varepsilon, x)\|\, dx \leqslant$$

$$\leqslant L \cdot \int_\tau^{\tau+\varepsilon} \limsup_{h \to 0+} \frac{1}{h} \|E_h(\boldsymbol{q}(t)) - v(t+h, \cdot)\|_{L^1}\, dt\,.$$

Rewrite the last integrand as

$$\limsup_{h\to 0+} \frac{1}{h} \|E_h(\boldsymbol{q}(t)) - v(t+h,\cdot)\|_{L^1} = \tag{6.18}$$

$$= \limsup_{h\to 0+} \frac{1}{h} \int_{\Phi(t+h)}^{\check{\Phi}(t+h)} \|E_h(\boldsymbol{q}(t)) - v(t+h,\cdot)\|\, dx +$$

$$+ \frac{1}{h} \int_{\widehat{\Phi}(t+h)}^{\xi+\varrho-(t+h)\hat{\lambda}} \|E_h(\boldsymbol{q}(t)) - v(t+h,\cdot)\|\, dx .$$

Call $x_1 < \ldots < x_N$ the location of the jumps in $\bar{v}$ and $0 \leq \tau_1 < \ldots < \tau_M$ the location of the jumps in $\tilde{v}$. Then, using Lemma 3 in [5]

$$\limsup_{h\to 0+} \frac{1}{h} \int_{\widehat{\Phi}(t+h)}^{\xi+\varrho-(t+h)\hat{\lambda}} \|E_h(\boldsymbol{q}(t)) - v(t+h,\cdot)\|\, dx = \tag{6.19}$$

$$= \mathcal{O}(1)\cdot\left(\|\bar{v}(\xi+) - \bar{v}(\widehat{\Phi}(t))\| + \sum_{x_\alpha > \widehat{\Phi}(t)} \|\bar{v}(x_\alpha+) - \bar{v}(x_\alpha-)\|\right) =$$

$$= \mathcal{O}(1)\cdot \mathrm{TV}\,\{u(\tau,\cdot) \colon]\xi, \xi+\varrho[\} .$$

An entirely similar argument allows us to obtain

$$\limsup_{h\to 0+} \frac{1}{h} \int_{\Phi(t+h)}^{\check{\Phi}(t+h)} \|E_h(\boldsymbol{q}(t)) - v(t+h,\cdot)\|\, dx = \tag{6.20}$$

$$= \mathcal{O}(1)\cdot(\sum_\alpha \|\tilde{v}(\tau_\alpha+) - \tilde{v}(\tau_\alpha-)\| +$$

$$+ \|\tilde{v}(0+) - \tilde{v}(\sup([\tau, t] \cap \Phi^{-1}(\Phi(t))) - \tau)\|)$$

$$= \mathcal{O}(1)\cdot \mathrm{TV}\{\tilde{u} \colon]\tau, \tau+\varepsilon[\} \tag{6.21}$$

where the sum in (6.20) is extended to all α such that, denoting $t_\alpha = \tau + \tau_\alpha$

$$\tau < t_\alpha \leq t \quad \text{and} \quad \Phi(s) \leq \Phi(t_\alpha) \leq \check{\Phi}(s) \quad \text{for all } s \in [t_\alpha, t] .$$

Inserting (6.19) and (6.21) in (6.18) and then in (6.17), the summand (6.14) is bounded by

$$\text{(6.22)} \qquad \int_{\Phi(\tau+\varepsilon)}^{\xi+\varrho-\varepsilon\hat{\lambda}} \|(E_\varepsilon \boldsymbol{q})(x) - v(\tau+\varepsilon, x)\|\, dx =$$

$$= \mathcal{O}(1)\cdot\varepsilon\cdot(\mathrm{TV}\{u(\tau, \cdot):]\xi, \xi+\varrho[\} + \mathrm{TV}\{\tilde{u}:]\tau, \tau+\varepsilon[\})\,.$$

Consider now (6.15):

$$\int_{\Phi(\tau+\varepsilon)}^{\xi+\varrho-\varepsilon\hat{\lambda}} \|v(\tau+\varepsilon, x) - U^\#_\Phi(\tau+\varepsilon, x)\|\, dx =$$

$$\text{(6.23)} \qquad = \int_{\Phi(\tau+\varepsilon)}^{\check{\Phi}(\tau+\varepsilon)} \|v(\tau+\varepsilon, x) - U^\#_\Phi(\tau+\varepsilon, x)\|\, dx$$

$$\text{(6.24)} \qquad + \int_{\hat{\Phi}(\tau+\varepsilon)}^{\xi+\varrho-\varepsilon\hat{\lambda}} \|v(\tau+\varepsilon, x) - U^\#_\Phi(\tau+\varepsilon, x)\|\, dx$$

because the two functions $v(\tau+\varepsilon, \cdot)$ and $U^\#$ coincide on the interval $[\check{\Phi}(\tau+\varepsilon), \hat{\Phi}(\tau+\varepsilon)]$, due to the definitions (6.12) of v and (5.4) of $U^\#_\Phi$.

If $\check{\Phi}(\tau+\varepsilon) = \Phi(\tau+\varepsilon)$, the r.h.s. at (6.23) is zero. If, on the other hand, $\check{\Phi}(\tau+\varepsilon) > \Phi(\tau+\varepsilon)$, then $\dot{\Phi}(\tau+\varepsilon) \leqslant -\hat{\lambda}$. Denote with $\bar{t}$ the maximum between τ and the last time before $\tau+\varepsilon$, at which Φ changes slope from positive to negative, i.e.

$$\bar{t} \doteq \max\{\tau, \sup\{t \leqslant \tau+\varepsilon : \dot{\Phi}(t-) \geqslant \hat{\lambda}\}\}$$

where $\sup\emptyset \doteq -\infty$. Then

$$\text{(6.25)} \qquad \int_{\Phi(\tau+\varepsilon)}^{\check{\Phi}(\tau+\varepsilon)} \|v(\tau+\varepsilon, x) - U^\#_\Phi(\tau+\varepsilon, x)\|\, dx =$$

$$= \int_{\Phi(\tau+\varepsilon)}^{\check{\Phi}(\tau+\varepsilon)} \|\tilde{v}(\sup([\tau, \tau+\varepsilon]\cap\Phi^{-1}(x)) - \tau) - \tilde{u}(\sup([\tau, \tau+\varepsilon]\cap\Phi^{-1}(x)))\|\, dx =$$

$$= \int_{\bar{t}}^{\tau+\varepsilon} \|\tilde{v}(t-\tau) - \tilde{u}(t)\|\, \dot{\Phi}(t)\, dt \leqslant L_\Phi \cdot \|\tilde{v} - \mathcal{T}_\tau \tilde{u}\| \leqslant \mathcal{O}(1)\cdot\varepsilon'.$$

Concerning (6.24)

$$\int_{\widehat{\Phi}(\tau+\varepsilon)}^{\xi+\varrho-\varepsilon\hat{\lambda}} \|v(\tau+\varepsilon, x) - U^{\#}_{\Phi}(\tau+\varepsilon, x)\|\, dx = \tag{6.26}$$

$$= \int_{\widehat{\Phi}(\tau+\varepsilon)}^{\xi+\varrho-\varepsilon\hat{\lambda}} \|v(\tau, x) - u(\tau, x)\|\, dx = \mathcal{O}(1)\cdot\varepsilon'.$$

Summing up (6.16), (6.22), (6.25) and (6.26), the proof is completed. ■

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REFERENCES

[1] D. Amadori, *Initial-boundary value problems for nonlinear systems of conservation laws*, NoDEA, **4** (1997), pp. 1-42.
[2] D. Amadori - R. M. Colombo, *Continuous dependence for* 2×2 *conservation laws with boundary*, J. Differential Equations, **138** (1997), no. 2, pp. 229-266.
[3] A. Bressan, *Global solutions of systems of conservation laws by wave-front tracking*, J. Math. Anal. Appl., **170** (1992), pp. 414-432.
[4] A. Bressan, *Lecture notes on systems of conservation laws*, S.I.S.S.A., Trieste (1994).
[5] A. Bressan, *The unique limit of the Glimm scheme*, Arch. Rational Mech. Anal., **130** (1995), pp. 205-230.
[6] A. Bressan, *The semigroup approach to systems of conservation laws*, Matematica Contemporanea, **10** (1996), pp. 21-74.
[7] A. Bressan - R. M. Colombo, *The semigroup generated by* 2×2 *systems of conservation laws*, Arch. Rational Mech. Anal., **133** (1996), pp. 1-75.
[8] A. Bressan - R. M. Colombo, *Unique solutions of* 2×2 *conservation laws with large data*, Indiana Univ. Math. J., **44** (1995), pp. 677-725.
[9] A. Bressan - G. Crasta - B. Piccoli, *Well-posedness of the Cauchy problem for* $n \times n$ *systems of conservation laws*, Preprint S.I.S.S.A. 1996.
[10] R. M. Colombo, *Uniqueness and continuous dependence for* 2×2 *conservation laws*, Ph.D. Thesis, S.I.S.S.A. (1995).
[11] F. Dubois - P. Le Floch, *Boundary conditions for nonlinear hyperbolic systems of conservation laws*, J. Differential Equations, **71** (1988), pp. 93-122.

[12] J. GLIMM, *Solutions in the large for nonlinear hyperbolic systems of equations*, Comm. Pure Appl. Math., 18 (1965), pp. 697-715.

[13] J. GOODMAN, *Initial Boundary Value Problems for Hyperbolic Systems of Conservation Laws*, Ph. D. Thesis, California University (1982).

[14] P. LAX, *Hyperbolic systems of conservation laws II*, Comm. Pure Appl. Math., **10** (1957), pp. 537-567.

[15] M. SABLÉ-TOUGERON, *Méthode de Glimm et problème mixte*, Ann. Inst. Henri Poincaré, **10**, no. 4 (1993), pp. 423-443.

[16] J. SMOLLER, *Shock Waves and Reaction-Diffusion Equations*, Springer-Verlag, New York (1983).

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