RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

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Rendiconti del Seminario Matematico della Università di Padova, tome 100 (1998), p. 97-122
<http://www.numdam.org/item?id=RSMUP_1998__100__97_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 100 (1998)

Extrinsic Symmetric Submanifolds Contained in Quaternionic Symmetric Spaces.

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ABSTRACT - In this paper imbeddings into R^n, for the irreducible Quaternionic symmetric spaces, are constructed. These imbeddings have the following property. For some maximal Hermitian symmetric subspace H there is an affine subspace $P \subset R^n$ such that H is an extrinsic symmetric submanifold of P in the Ferus' sense.

1. - Introduction.

Symmetric submanifolds of Euclidean spaces, or extrinsic symmetric submanifolds were studied by D. Ferus [6]. These submanifolds are essentially compact symmetric spaces M with an injective isometric immersion into a Euclidean space R^n such that, for each $x \in M$, the symmetry s_x extends to an isometry of the ambient space R^n, so that its action on the normal space to M at x is the identity. On that paper, Ferus deter-

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1991 Mathematics Subject Classification. Primary 53C30; Secondary 53C35.
This research was partially supported by CONICET and CONICOR, Argentina.

mines these submanifolds and proves that they are exactly, the symmetric R-spaces.

Quaternionic symmetric spaces were studied and characterized by J. Wolf in [8]. It is well known that these spaces are not symmetric R-spaces; hence, they are not extrinsic symmetric submanifold of any Euclidean space. In the present paper we study Quaternionic symmetric spaces and our main result could be roughly described by saying that Quaternionic symmetric spaces are just as extrinsic symmetric as they can possibly be.

To explain the meaning of this phrase we must notice first that any Quaternionic symmetric space Q contains, as a totally geodesic submanifold, a Hermitian symmetric space H which has maximal dimension (these subspaces are described in [3] for the Quaternionic symmetric spaces of the exceptional groups). The present paper is devoted to show that for each Quaternionic symmetric spaces Q there are an isometric immersion $g\colon Q \to R^n$ into a Euclidean space and an affine subspace $P \subset \subset R^n$ such that the image of the composition $H \xrightarrow{i} Q \xrightarrow{g} R^n$ is contained in P and is an extrinsic symmetric submanifold of $P \subset R^n$. In the majority of the cases, the immersion $g\colon Q \to R^n$ has minimal dimension, more precisely, there are no *equivariant* immersions of Q in R^p with $p < n$.

The paper is organized as follows. In section 2 we recall the definition of extrinsic symmetric submanifolds of R^n from [6] and give a proof of the well known fact that every Hermitian symmetric space is extrinsic symmetric. We also recall the definition of Quaternionic symmetric space and describe the maximal Hermitian symmetric subspace for each of them.

In section 3 we give the general construction of the immersion for the Quaternionic symmetric spaces in Theorem 3.2. In section 4 we describe the immersions for the Quaternionic symmetric spaces of the exceptional Lie groups and in section 5 those of the spaces corresponding to the classical Lie groups.

2. – Extrinsic symmetric submanifolds, Quaternionic symmetric spaces and maximal Hermitian subspaces.

Let M be a connected compact n-dimensional Riemannian manifold and let $f\colon M \to R^{n+p}$ be an isometric immersion into an Euclidean $(n+p)$-space. For each $x \in M$, let σ_x be the affine linear transformation of R^{n+p} which fixes the affine normal space to $df_x(T_x M)$ at $f(x)$ and reflects $f(x)+df_x(T_x M)$ at $f(x)$.

As in [6] we shall say that $f\colon M \to R^{n+p}$ is an *extrinsic symmetric submanifold*, if for every $x \in M$ there is an isometry $\theta_x\colon M \to M$ such that $\theta_x(x) = x$ and $f \circ \theta_x = \sigma_x \circ f$.

The following theorem shows that every Hermitian symmetric space is an extrinsic symmetric submanifold.

THEOREM 2.1. *Let $H = G/K$ be a compact irreducible Hermitian symmetric space. Let $\mathcal{G}$ and $\mathcal{K}$ be the respective Lie algebras of G and K; $\mathcal{M}$ the orthogonal complement of $\mathcal{K}$ with respect to the Killing form B of $\mathcal{G}$ (B is negative definite on $\mathcal{G}$). Considering on $\mathcal{G}$ the inner product $\langle X, Y\rangle = -B(X, Y)$, $(\mathcal{G}, \langle,\rangle)$ is an Euclidean space. Then*

(i) *there is a nonzero vector* $\mathrm{v}_0 \in \mathcal{K}$, *such that*

$$K = \{g \in G\colon \operatorname{Ad}(g)\, \mathrm{v}_0 = \mathrm{v}_0\},$$

this defines an imbedding $f\colon H \to \mathcal{G}$ given by $f(gK) = \operatorname{Ad}(g)\, \mathrm{v}_0$;

(ii) *we consider H with the Riemannian metric induced by $\langle\, ,\, \rangle$, i.e we make f an isometric imbedding, then H is an extrinsic symmetric submanifold.*

PROOF. (i) Since H is irreducible, $Z(K)$ is analytically isomorphic to the group of the unit complex numbers T^1 [4, p.382 (6.2)]. Then $z(\mathcal{K})$, the center of $\mathcal{K}$, is isomorphic to R.

Take $\mathrm{v}_0 \neq 0$ in $z(\mathcal{K})$.

If $g \in K$, $\operatorname{Ad}(g)\mathrm{v}_0 = \mathrm{v}_0$.

Conversely, if $g \in G$ and $\operatorname{Ad}(g)\mathrm{v}_0 = \mathrm{v}_0$, then

$$\exp \operatorname{Ad}(g)\, t\mathrm{v}_0 = \exp t\mathrm{v}_0 \qquad \forall t \in R\,,$$
$$g(\exp t\mathrm{v}_0) = (\exp t\mathrm{v}_0)\, g\,, \quad \forall t \in R\,.$$

Consequently, $g \in C_G(T^1)$, the centralizer of T^1 in G. Since, K is maximal connected in G [4, p. 381-2 (6.1)] $K = C_G(T^1)$.

Thus, $K = \{g \in G\colon \operatorname{Ad}(g)\, \mathrm{v}_0 = \mathrm{v}_0\}$.

(ii) We observe that if $o = [K]$, then $df_o(T_o H) = \mathcal{M}$.

Since $df_o(T_oH) = [\mathcal{G}, v_0] = [\mathcal{M}, v_0]$ and for each $X \in \mathcal{K}$, $Y \in \mathcal{M}$

$$B(X, [Y, v_0]) = -B(X, [v_0, Y]) = -B([X, v_0], Y) = 0$$

we have as a result $(T_oH)^{\perp} = \mathcal{K}$ and $[\mathcal{M}, v_0] \subset \mathcal{M}$.

Let s_o be the geodesic symmetry of the symmetric space H at $o = [K]$. We know that $s_o \in T^1 = Z(K)$ [4, p. 376 (4.5) (ii)] and its action on H is given by $s_o(gK) = (s_o g)K$.

Let us call $\sigma_0 = \mathrm{Ad}(s_o)$. σ_0 is an isometry of the Euclidean space $(\mathcal{G}, \langle\,,\,\rangle)$.

If gK is a point in H, we have:

(1) $$f(s_o(gK)) = \mathrm{Ad}(s_o g)\, v_0 = \mathrm{Ad}(s_o)\, \mathrm{Ad}(g)\, v_0 = \sigma_0 f(gK).$$

We must see that σ_0 fixes every element in $\mathcal{K}$ and is -Id on $\mathcal{M}$.

Write $s_o = \exp t_0 v_0$ for some $t_0 \in R$. If $X \in \mathcal{K}$, then

(2) $$\sigma_0(X) = \mathrm{Ad}(\exp t_0 v_0)\, X = \exp(t_0\, ad(v_0))\, X = X.$$

Since G acts by isometries on H, if $x = gK$ the symmetry at x is $s_x = g s_o g^{-1} \in G$.

By [4, p. 208 (3.3)] the geodesics through o are

$$\gamma(t) = (\exp tY)K, \qquad \text{for } Y \in \mathcal{M}.$$

Hence,

(3) $$s_o(\exp tY)\, K = (\exp(-tY))K, \ -\varepsilon < t < \varepsilon.$$

From (3) we obtain

$$f(s_o(\exp tY)\, K) = \mathrm{Ad}(\exp(-tY))v_0, \qquad -\varepsilon < t < \varepsilon$$

and by (1)

$$f(s_o(\exp tY)\, K) = \sigma_0(\mathrm{Ad}(\exp(tY))v_0), \qquad -\varepsilon < t < \varepsilon.$$

It follows that

$$\sigma_0(\mathrm{Ad}(\exp(tY))\, v_0) = \mathrm{Ad}(\exp(-tY))\, v_0, \qquad -\varepsilon < t < \varepsilon,$$

then

(4) $$\sigma_0([Y, v_0]) = -[Y, v_0].$$

In that way, (1), (2) and (4) prove (ii). QED ■

We adopt the notation from [8] concerning Quaternionic symmetric spaces. We recall the basic facts from [8] for the convenience of the reader.

Let V be a real vector space. A *quaternion algebra* on V is an algebra of linear transformations on V which is isomorphic to the algebra of real quaternions, and whose unit element is the identity transformation of V. If A is a quaternion algebra on V, then V is a quaternion vector space with the structure given by A.

Let M be a Riemannian manifold. Given $x \in M$, let Ψ_x be the *linear holonomy group* consisting of all linear transformations of the tangent space M_x obtained from parallel translation along curves from x to x. A set A_x of linear transformations of M_x is called Ψ_x-invariant if $gA_xg^{-1} = A_x$ for every $g \in \Psi_x$. A set A of fields of linear transformations of all tangent spaces of M is called *parallel* if, given x, $y \in M$ and a curve σ from x to y, the parallel translation τ_σ satisfies $\tau_\sigma A_x \tau_\sigma^{-1} = A_y$. A_x extends to a parallel set of fields of linear transformations of tangent spaces, if and only if A_x is Ψ_x -invariant; in that case, the extension is unique, being defined by $A_y = \tau_\sigma A_x \tau_\sigma^{-1}$.

A *quaternionic structure* on a Riemannian manifold M is a parallel field A of quaternion algebras A_x on the tangent spaces M_x, such that every unimodular element of A_x is an orthogonal linear transformation on M_x (compare [8]).

Choose $x \in M$ and let A be the quaternion structure of M. Then A_x is a Ψ_x-stable quaternion algebra on M_x, so $\Psi_x = \Phi_x \cdot A_x'$, where Φ_x is the centralizer of A_x and A_x' is the intersection with A_x. Φ_x and A_x' are the A-linear and A-scalar parts of Ψ_x. We say that the holonomy group have *quaternion scalar part* if A_x' spans A_x.

In [8] the following fact is proved. Let G be a compact centerless simple Lie group. Let T be a maximal torus and let $\mathcal{G}$ and $\mathcal{T}$ be the respective Lie algebras. Let $\mathcal{G}^C$ and $\mathcal{H}$ be the respective complexifications of $\mathcal{G}$ and $\mathcal{T}$; $\mathcal{H}$ is a Cartan subalgebra of the complex simple Lie algebra $\mathcal{G}^C$. Let Ψ be the set of root of $\mathcal{G}^C$ for $\mathcal{H}$ and $\mathcal{G}_\alpha$ the one-dimensional subspace characterized by

$$[h, E] = \alpha(h)E \quad \text{for all} \quad h \in \mathcal{H}, \quad E \in \mathcal{G}_\alpha .$$

The Killing form on $\mathcal{G}^C$ is denoted $\langle , \rangle$, and let h_α $(\alpha \in \Psi)$ be the element of $\mathcal{H}$ characterized by:

$$\langle h_\alpha, h \rangle = \alpha(h) \quad \text{for all} \quad h \in \mathcal{H} .$$

Let π be a base of simple roots and let β be the maximal root with respect to the order induced by π. We define

$$\mathcal{L}_1 = \{h \in \mathcal{T}: \beta(h) = 0\} + \sum_{\substack{\alpha > 0 \\ \alpha(h_\beta) = 0}} \mathcal{G} \cap (\mathcal{G}_\alpha + \mathcal{G}_{-\alpha}),$$

$$\mathcal{A}_1 = \mathcal{G} \cap (\{h_\beta\} + \mathcal{G}_\alpha + \mathcal{G}_{-\alpha}),$$

$$\mathcal{K} = \mathcal{L}_1 + \mathcal{A}_1 .$$

$\mathcal{K} = \mathcal{L}_1 \oplus \mathcal{A}_1$ is direct sum of ideals. Let L_1, A_1, and $K = L_1 A_1$ be the corresponding analytic subgroups of $\mathcal{G}$. In [8] Wolf proves the following theorem.

THEOREM 2.2. *G/K is a compact simply connected irreducible Riemannian symmetric space. $A_1 \cong Sp(1)$ generates a quaternion algebra on the tangent space on $[K]$. This quaternion algebra parallel translated over G/K gives a quaternionic structure A in which the holonomy has quaternion scalar part.*

Conversely, if M is a compact simply connected Riemannian symmetric space with a quaternionic structure A_M in which the holonomy has quaternion scalar part, then there is an isometry of M onto a manifold G/K as above, which carries A_M to A.

A Quaternionic symmetric space will be a compact simply connected Riemannian symmetric space M with a quaternionic structure A_M in which the holonomy has quaternion scalar part.

REMARK 2.3. *In* [1, p. 408-9] *the so called «quaternionic-Kähler Riemannian symmetric spaces» are defined. They coincide with the «quaternionic symmetric» of J. Wolf, such as is indicated.*

Hereafter we adopt the following notation. Q will be a Quaternionic symmetric space as in Theorem 2. Inside $Q = G/K$ we want to consider the compact irreducible Hermitian symmetric subspaces M defined by subgroups L of G as follows. $M = L/U$ and $U = L \cap K$.

If the Hermitian symmetric subspace M has maximal dimension among those contained in Q we denote it by H and coinsider it written as $H = L/U$ with $U = L \cap K$.

We shall denote by o the point $[K] \in Q$. Clearly, $H = L \cdot o \subset Q$.

When G is an exceptional Lie group, the maximal Hermitian symmetric subspaces of the corresponding Quaternionic symmetric spaces are listed in [3]. They are:

Q	H
$G_2/SO(4)$, $(SO(4)\cong Sp(1)Sp(1))$	CP^2
$F_4/Sp(3)\,Sp(1)$	CP^2
$E_6/SU(6)\,Sp(1)$	CP^4
$E_7/Spin(12)\,Sp(1)$	CP^6
$E_8/E_7Sp(1)$	CP^7

The following list shows the other Quaternionic symmetric spaces (i.e those with G classical) (compare [1, pag. 409]) and the corresponding maximal Hermitian subspaces:

Q	H
$SU(n+2)/S(U(n)\times U(2))$	$SU(n+2)/S(U(n)\times U(2))$
$(n\geqslant 1)$	
$SO(n+4)/SO(n)\times SO(4)$	$SO(n+2)/SO(n)\times SO(2)$
$(n\geqslant 3,\ \ n=2m)$	$SU(m+2)/S(U(m)\times U(2))$
$SO(n+4)/SO(n)\times SO(4)$	$SO(n+2)/SO(n)\times SO(2)$
$(n\geqslant 3,\ \ n=2m+1)$	
$Sp(n+1)/Sp(n)\times Sp(1)$	$SU(n+1)/S(U(n)\times U(1))$
$(n\geqslant 1)$	

REMARK 2.4. *The space* $Q=SU(n+2)/S(U(n)\times U(2))$ *is itself a Hermitian symmetric space.*

REMARK 2.5. *Observe that there are two subspaces in the Quaternionic symmetric space* $SO(n+4)/SO(n)\times SO(4)$ *when* $n=2m$.

The subspaces H are totally geodesic submanifolds of the corresponding Q. This is straightforward in all cases except for the Hermitian space $H=SU(m+2)/S(U(m)\times U(2))$ in the Quaternionic metric space $Q=SO(n+4)/SO(n)\times SO(4)$ when $n=2m$. We give a proof of this fact in the following lemma.

LEMMA 2.6. *The space* $H=SU(m+2)/S(U(m)\times U(2))$ *is totally geodesic submanifold of* $Q=SO(n+4)/SO(n)\times SO(4)$ *when* $n=2m$.

PROOF. Let Ψ: $SU(m+2) \to SO(n+4)$ be given by the correspondence

$$A = [a_{ij}] = [b_{ij} + \sqrt{-1}c_{ij}] \xrightarrow{\Psi} [A_{ij}^{2\times 2}], \qquad \text{where} \qquad A_{ij}^{2\times 2} = \begin{bmatrix} b_{ij} & c_{ij} \\ -c_{ij} & b_{ij} \end{bmatrix}.$$

Let $\Phi = d\Psi_e$.

Take the descompositions $so(n+4) = \mathcal{K} \oplus \mathcal{M}$, $su(m+2) = \mathcal{U} \oplus \mathcal{T}$, where

$$\mathcal{K} = \left\{ \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : A_1 \in so(n),\ A_2 \in so(4) \right\},$$

$$\mathcal{M} = \left\{ \begin{bmatrix} 0 & X \\ -X^t & 0 \end{bmatrix} : X \in M(n \times 4, R) \right\},$$

$$\mathcal{U} = \left\{ \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \quad A_1 \in u(n),\ A_2 \in u(2) \right\},$$

$$\mathcal{T} = \left\{ \begin{bmatrix} 0 & X \\ -\overline{X}^t & 0 \end{bmatrix} : X \in M(n \times 2, C) \right\},$$

and $\mathcal{M} \cong T_o(Q)$, $\mathcal{T} \cong T_o(H)$, $o = [K]$.

We shall see that

$$\Phi(\mathcal{T}) = \left\{ \begin{bmatrix} 0 & \widetilde{X} \\ \widetilde{(-\overline{X}^t)} & 0 \end{bmatrix} : X \in M(n \times 2, C) \right\}$$

is a Lie triple system, where if

$$X = [\alpha_{ij} + \sqrt{-1}\beta_{ij}], \qquad \widetilde{X} = [X_{ij}^{2\times 2}] \qquad \text{and} \qquad X_{ij}^{2\times 2} = \begin{bmatrix} \alpha_{ij} & \beta_{ij} \\ -\beta_{ij} & \alpha_{ij} \end{bmatrix}.$$

A simple computation shows that $\widetilde{(-\overline{X}^t)} = -\widetilde{X}^t$. Then,

$$\Phi(\mathcal{T}) = \left\{ \begin{bmatrix} 0 & \widetilde{X} \\ -\widetilde{X}^t & 0 \end{bmatrix} : X \in M(n \times 2, C) \right\} \subset \mathcal{M}.$$

Let X, W, Z $\in M(n \times 2, C)$,

$$X = [\alpha_{ij} + \sqrt{-1}\beta_{ij}], \qquad W = [\gamma_{ij} + \sqrt{-1}\delta_{ij}], \qquad Z = [\eta_{ij} + \sqrt{-1}\mu_{ij}],$$

thus

$$X_1 = \begin{bmatrix} 0 & X \\ -\overline{X}^t & 0 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0 & W \\ -\overline{W}^t & 0 \end{bmatrix} \quad \text{and} \quad Z_1 = \begin{bmatrix} 0 & Z \\ -\overline{Z}^t & 0 \end{bmatrix} \in \mathcal{T}.$$

It is easy to see that

$$[[\Phi(X_1), \Phi(W_1)], \Phi(Z_1)] = \begin{bmatrix} 0 & A \\ -A^t & 0 \end{bmatrix},$$

where $A = -\tilde{X}\tilde{W}^t\tilde{Z} + \tilde{W}\tilde{X}^t\tilde{Z} + \tilde{Z}\tilde{X}^t\tilde{W} - \tilde{Z}\tilde{W}^t\tilde{X}$.

The first term of this sum is

$$-\tilde{X}\tilde{W}^t\tilde{Z} = -[[X_{ij}^{2\times 2}][B_{ij}^{2\times 2}][Z_{ij}^{2\times 2}]] = -[C_{ij}^{2\times 2}],$$

where $B_{ij}^{2\times 2} = (W_{ji}^{2\times 2})^t$, and the others terms of A are similar.

Now we compute $C_{ij}^{2\times 2}$:

$$C_{ij}^{2\times 2} = \sum_{k=1}^{m+2}\sum_{h=1}^{m+2}(X_{ih}^{2\times 2}B_{hk}^{2\times 2})(Z_{kj}^{2\times 2}) = \sum_{k,h} X_{ih}^{2\times 2}(W_{kh}^{2\times 2})^t Z_{kj}^{2\times 2} =$$

$$= \sum_{k,h}\begin{bmatrix} \alpha_{ih} & \beta_{ih} \\ -\beta_{ih} & \alpha_{ih} \end{bmatrix}\begin{bmatrix} \gamma_{kh} & -\delta_{kh} \\ \delta_{kh} & \gamma_{kh} \end{bmatrix}\begin{bmatrix} \eta_{kj} & \mu_{kj} \\ -\mu_{kj} & \eta_{kj} \end{bmatrix} = \begin{bmatrix} a_{ij} & b_{ij} \\ -b_{ij} & a_{ij} \end{bmatrix},$$

where

$$a_{ij} = (\alpha_{ih}\gamma_{kh} + \beta_{ih}\delta_{kh})\eta_{kj} - (-\alpha_{ih}\delta_{kh} + \beta_{ih}\gamma_{kh})\mu_{kj},$$

$$b_{ij} = (\alpha_{ih}\gamma_{kh} + \beta_{ih}\delta_{kh})\mu_{kj} + (-\alpha_{ih}\delta_{kh} + \beta_{ih}\gamma_{kh})\eta_{kj}.$$

It follows that, $[[\Phi(X_1), \Phi(W_1)], \Phi(Z_1)] \in \Phi(\mathcal{T})$ and the lemma is proven. QED ■

3. – General construction of the immersion.

To construct an immersion of the Quaternionic symmetric space Q such that the maximal Hermitian simmetric space H is an extrinsic symmetric submanifold of the ambient Euclidean space, we shall use some results from representation theory which we now recall.

Let $\mathcal{G}$ be a compact semisimple Lie algebra, $\mathcal{G}^C$ the complexification of $\mathcal{G}$.

Let $\mathcal{H}$ be a Cartan subalgebra of $\mathcal{G}^C$ and Ψ a set of root of $\mathcal{G}^C$ relative to $\mathcal{H}$. Let $\pi = \{\alpha_1, \dots, \alpha_n\}$ be a base of simple roots of Ψ, and Ψ^+ and Ψ^- denote the sets of positive and negative roots with respect to the order induced by π.

We denote by h_α and x_α the elements of $\mathcal{G}^C$ usually defined as in section 2 and [4, p. 176] respectively. We shall always call β the maximal root of $\mathcal{G}^C$.

Let Φ: $\mathcal{G}^C \to \mathcal{G}l(V)$ be a irreducible complex representation with highest weight λ. We write $2\lambda = \sum_{j=1}^{n} q_j \alpha_j$ and define $\varepsilon(\lambda) = \prod_{j=1}^{n} (-1)^{q_j}$.

Let ω_0 be the element of the Weyl group such that $\omega_0(\Psi^+) = = \Psi^-$.

The following theorem is a well known criterion to decide if a complex representation induces a real representation [7, pag. 305].

THEOREM 3.1. *There is a descomposition* $V = V_1 + iV_2$ *with* V_1 *and* V_2 $\mathcal{G}$*-invariant vector spaces over* R *if and only if* $\varepsilon(\lambda) = 1$ *and* $\omega_0 \lambda = -\lambda$.

Let Q be an irreducible Quaternionic symmetric space and H a maximal compact irreducible Hermitian subspace of Q. Write $Q = G/K$ and $H = L/U$ as in section 2.

Let $\mathcal{G}$, $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{U}$ be the Lie algebras of G, K, L and U respectively. Then

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{M}, \qquad \mathcal{M} \cong T_o Q; \qquad o = [K];$$

$$\mathcal{L} = \mathcal{U} \oplus \mathcal{T}, \qquad \mathcal{T} \cong T_o H;$$

$$\mathcal{U} = \mathcal{K} \cap \mathcal{L}, \ \mathcal{T} \subset \mathcal{M} \quad \text{and} \quad \mathcal{T} \text{ a Lie triple system}.$$

Let $\mathcal{G}^C$, $\mathcal{K}^C$ and $\mathcal{M}^C$ be the complexifications of $\mathcal{G}$, $\mathcal{K}$ and $\mathcal{M}$, respectively.

THEOREM 3.2. *Let* $Q = G/K$ *be an irreducible Quaternionic symmetric space,* $H = L/U$ *a maximal compact irreducible Hermitian subspace of* Q*. Let* $\mathcal{G}$, $\mathcal{K}$, $\mathcal{L}$ *and* $\mathcal{U}$ *as above and* V^C *a nontrivial irreducible complex representation of* $\mathcal{G}^C$ *such that its restriction to* $\mathcal{K}^C$ *contains the trivial representation* $V^C(0)$ *with multiplicity 1. Let us assume that it induces a real representation* Φ *of* $\mathcal{G}$ *on* V *with* $V^C = V + iV$ *and* V $\mathcal{G}$-

invariant as in Theorem 3.1. *Furthermore, assume that* Φ *induces a representation* $\overline{\Phi}$: $G \to Gl(V)$.

Let $V^C(0) = V(0) + iV(0)$ *and* $v_0 \in V(0)$*be a nonzero vector. Then the orbit* $\overline{\Phi}(G)\, v_0$ *gives an immersion of* Q *into* $V \cong R^n$.

Assume that $\Phi|_{\mathcal{L}}$ *splits as a sum* $\Phi|_{\mathcal{L}} = r\mathrm{Ad}(\mathcal{L}) \oplus sW(0) \oplus W$ *where* $\mathrm{Ad}(\mathcal{L})$ *is the adjoint representation of* $\mathcal{L}$, $W(0)$ *stands for the trivial one dimensional representation of* $\mathcal{L}$ *and* W *is some, not necessarily irreducible,* $\mathcal{L}$*-module which contains neither the adjoint representation of* $\mathcal{L}$ *nor the trivial one.*

If $v_0 \in r\,\mathrm{Ad}(\mathcal{L}) \oplus sW(0)$, *then* H *is a extrinsic symmetric submanifold of an affine subspace* $P \subset V$.

PROOF. Let us consider the orbit $\Phi(\mathcal{G})\, v_0$.

We have $\Phi(\mathcal{K})v_0 = 0$ and an isomorphism between $\mathcal{M}$ and $\Phi(\mathcal{M})v_0$ given by $X \to \Phi(X)v_0$. In fact, $\mathcal{N} = \{X \in \mathcal{M}\colon \Phi(X)\, v_0 = 0\}$ is a $\mathcal{K}$-submodule of $\mathcal{M}$ because if $X \in \mathcal{M}$, $Y \in \mathcal{K}$,

$$\Phi(Y)\, \Phi(X)\, v_0 = \Phi(X)\, \Phi(Y) v_0 + \Phi([X, Y]) v_0 \,.$$

Since the action of $\mathcal{K}$ on $\mathcal{M}$ is irreducible, we either have $\mathcal{N} = \{0\}$ or $\mathcal{N} = \mathcal{M}$. This last case is not possible because Φ is irreducible and so $\mathcal{N} = \{0\}$. This shows that the orbit has the same dimension as Q. Then, it defines the desired immersion. Clearly, $\overline{\Phi}(G)\, v_0$ gives an induced immersion of Q.

To see the other part assume that $v_0 \in \mathrm{Ad}(\mathcal{L}) \oplus sW(0)$ $(r = 1)$ and write $v_0 = w_1 + w_0$ with $w_1 \in \mathrm{Ad}(\mathcal{L})$ and $w_0 \in sW(0)$. We have for $X \in \mathcal{L}$, $\Phi(X)\, v_0 = 0$ if and only if $X \in \mathcal{U}$ and this is equivalent to $\Phi(X)\, w_1 = 0$ and in turn, to $w_1 \in z(\mathcal{U})$ (center of $\mathcal{U}$). Now Theorem 2.2 yields the result.

If $v_0 \in r\,\mathrm{Ad}(\mathcal{L}) \oplus sW(0)$ with $r > 1$, v_0 may have nonzero components in more than one of the copies of $\mathrm{Ad}(\mathcal{L})$. Let us write $v_0 = (v_1, \ldots, v_r)$ and take the first nonzero component, say v_j. The isotropy subalgebra of v_j by $\mathcal{L}$ is clearly $\mathcal{U}$ (because it is maximal) and this is so for each nonzero component. Furthermore, if we identify $\mathcal{L}$ and $\mathrm{Ad}(\mathcal{L})$ as usual, each component of v_0 lies in the center of $\mathcal{U}$, which is one dimensional. Then v_0 can be written as $v_0 = (\lambda_1 v_j, \ldots, \lambda_r v_j)$ (some λ_i may be 0) and $X \to$ $\to (\lambda_1 X, \ldots, \lambda_r X)$ defines an invariant subspace whose representation is $\mathrm{Ad}(\mathcal{L})$. This subspace contains H and we can act as in the case $r = 1$. QED ■

4. – Immersions of the exceptional irreducible Quaternionic symmetric spaces.

We said that a Quaternionic symmetric space $Q = G/K$ is *exceptional* if G is an exceptional Lie group; analogously, $Q = G/K$ will be called *classical* if G is a classical Lie group.

It is a well known fact of representation theory that if G is simply connected, then any representation of the Lie algebra $\mathcal{G}$ induces a representation of the Lie group G. In this section we shall assume that our exceptional Quaternionic symmetric space is written as $Q = G/K$ with G simply connected.

From [7, p. 307-310] we obtain the number ε indicated in Theorem 3.1 (the compact case corresponds to $j = 0$) and the element w_0 in the corresponding Weyl group can be found in [2, p. 251-275].

For the numbering of the roots and basic weights, we adopt the notation from [5, p. 64-65-69].

Let Ω be a subset of the set Ψ of roots of $\mathcal{G}^C$. We say that a subalgebra $\mathcal{A} \subset \mathcal{G}^C$ is «the subalgebra generated by Ω» if $\mathcal{A}$ is generated by the set $\{h_\alpha,\ \alpha \in \Omega\} \cup \{x_\alpha \alpha \in \mathrm{span}_Z(\Omega) \cap \Psi\}$.

In the rest of this section we describe the results obtained for the exceptional irreducible Quaternionic symmetric spaces.

$A)$ $Q = G_2/Sp(1)\,Sp(1)$, $H = CP^2 = SU(3)/S(U(2) \times U(1))$.

Representation of g_2^C: $V = V(20)$, $\dim V = 27$,

$$\varepsilon = 1 \quad \text{and} \quad \omega_0 = -\mathrm{id}\,,$$

$$\mathcal{K} = sp(1) \oplus sp(1), \qquad \mathcal{K}^C = A_1 \oplus A_1\,,$$

$$V|_{\mathcal{K}^C} = V(2.2) \oplus V(0.4) \oplus V(1.3) \oplus V(1.1) \oplus V(0.0)\,.$$

$$\mathcal{L} = su(3)\,, \qquad \mathcal{L}^C = A_2\,,$$

$$V|_{\mathcal{L}^C} = V(20) \oplus V(02) \oplus V(10) \oplus V(01) \oplus V(11) \oplus V(00)\,.$$

Thus, $v_0 \in V(11) \oplus V(00)$.

B) $Q = F_4/Sp(1)\,Sp(3)$, $H = CP^2 = SU(3)/S(U(2)\times U(1))$.

Representation of f_4^C: $V = V(0002)$, $\dim V = 324$.

$\varepsilon = 1$ and $\omega_0 = -\mathrm{id}$,

$\mathcal{K} = sp(1)\oplus sp(3)$, $\mathcal{K}^C = A_1\oplus C_3$,

$V|_{\mathcal{K}^C} = V(2.200)\oplus V(1.110)\oplus V(1.001)\oplus V(0.020)\oplus V(0.010)\oplus V(0.000)$.

1) $\mathcal{L} = su(3)$, $\mathcal{L}^C = A_2$, we take $\mathcal{L}^C$ generated by the roots $\{-\beta, \alpha_1\}$.

$$V|_{\mathcal{L}^C} = 3V(21)\oplus 3V(12)\oplus 9V(20)\oplus 9V(02)\oplus 3V(10)\oplus 3V(01)\oplus 9V(11)\oplus$$
$$\oplus 9V(00)\oplus V(22).$$

Then,

$$\mathrm{v}_0 \in 9V(11)\oplus 9V(00)\oplus V(22).$$

But $\mathrm{v}_0 \notin V(22)$ since $\mathcal{U}^C = \mathcal{L}^C \cap \mathcal{K}^C = \{h_{\alpha_1}, h_\beta, x_\beta, x_{-\beta}\}\subset \mathcal{K}^C$ and $V(22)|_{A_1} = 3V(4)\oplus 3V(2)$ where A_1 is generated by $\{-\beta\}$.

Observe that v_0 does not have component in $V(22)$, because the component is a vector fixed by $\mathcal{U}$, then $\mathrm{v}_0 \in 9V(11)\oplus 9V(00)$.

2) $\mathcal{L} = su(3)$, $\mathcal{L}^C = A_2$, generated by the root $\{\alpha_1, \alpha_2\}$.

$$V|_{\mathcal{L}^C} = 6V(02)\oplus 6V(20)\oplus 24V(01)\oplus 24V(10)\oplus 9V(11)\oplus 36V(00).$$

$\mathrm{v}_0 \in 9V(11)\oplus 36V(00)$.

C) $Q = E_6/Sp(1)SU(6)$, $H = CP^4 = SU(5)/S(U(4)\times U(1))$.

Representation of e_6: $V = V(100001)$, $\dim V = 650$.

$\varepsilon = 1$ and ω_0 is the permutation sending $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ in $-\alpha_6, -\alpha_2, -\alpha_5, -\alpha_4, -\alpha_3, -\alpha_1$ respectively, as

$$(100001) = \lambda_1 + \lambda_6 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6,$$

a simple computation shows that $\omega_0(100001) = -(100001)$,

$$\mathcal{K} = sp(1)\oplus su(6), \qquad \mathcal{K}^C = A_1\oplus A_5,$$

$$V|_{\mathcal{K}^C} = V(1.\,11000)\oplus V(1.\,00011)\oplus V(2.10001)\oplus V(1.00100)\oplus$$
$$\oplus V(0.01010)\oplus \oplus V(0.10001)\oplus V(0.00000).$$

1) $\mathcal{L} = su(5)$, $\mathcal{L}^C = A_4$, generated by the roots $\{\alpha_1, \alpha_3, \alpha_4, \alpha_2\}$

$$V|_{\mathcal{L}^C} = V(0110) \oplus V(0101) \oplus V(1010) \oplus 2V(1100) \oplus 2V(0011) \oplus 2V(0002) \oplus$$
$$\oplus 2V(2000) \oplus 4V(0010) \oplus 4V(0100) \oplus 6V(0001) \oplus 6V(1000) \oplus 5V(1001) \oplus$$
$$\oplus 5V(0000).$$

$v_0 \in V(0110) \oplus 5V(1001) \oplus 5V(0000)$.

$\mathcal{U}^C = \mathcal{L}^C \cap \mathcal{K}^C$ contains the subalgebra generated by the roots $\{\alpha_2, \alpha_3, \alpha_4\}$ of type A_3,

$$V(0110)|_{A_3} = V(011) \oplus V(110) \oplus V(020) \oplus V(101),$$

then $v_0 \notin V(0110)$ because v_0 is a vector fixed by $\mathcal{U}$.

Thus, $v_0 \in 5V(1001) \oplus 5V(0000)$.

2) Choosing A_4 generated by $\{\alpha_2, \alpha_4, \alpha_5, \alpha_6\}$ the descomposition in irreducible component of $V|_{\mathcal{L}^C}$ is the same.

D) $Q = E_7/Sp(1)\, Spin(12)$, $H = CP^6 = SU(7)/S(U(6) \times U(1))$.

Representation of e_7^C: $V = V(0000010)$, $\dim V = 1539$.

$$\varepsilon = 1 \quad \text{and} \quad \omega_0 = -\mathrm{id}.$$

$$\mathcal{K} = sp(1) \oplus spin(12), \qquad \mathcal{K}^C = A_1 \oplus D_6,$$

$$V|_{\mathcal{K}^C} = V(0.000100) \oplus V(1.100001) \oplus V(2.010000) \oplus V(0.200000) \oplus$$
$$\oplus V(1.000010) \oplus V(0.000000).$$

$\mathcal{L} = su(7)$, $\mathcal{L}^C = A_6$ generated by the roots $\{\alpha_i, i = 1, 3, 4, 5, 6, 7\}$,

$$V|_{\mathcal{L}^C} = V(000010) \oplus V(010000) \oplus V(010001) \oplus V(100010) \oplus V(101000) \oplus$$
$$\oplus V(000101) \oplus V(000011) \oplus V(110000) \oplus V(100000) \oplus V(000001) \oplus$$
$$\oplus V(001000) \oplus V(000100) \oplus V(010010) \oplus 2V(100001) \oplus V(000000).$$

$v_0 \in V(010010) \oplus 2V(100001) \oplus V(000000)$.

$\mathcal{U}^C = \mathcal{L}^C \cap \mathcal{K}^C = A_5 \oplus C$, A_5 generated by the roots $\{\alpha_i, i = = 3, 4, 5, 6, 7\}$

$v_0 \notin V(010010)$ since $V(010010)|_{A_5} = V(01001) \oplus V(10010) \oplus$
$$\oplus 4V(10000) \oplus 4V(00001) \oplus V(01010) \oplus V(10001).$$

Thus, $v_0 \in 2V(100001) \oplus V(000000)$.

E) $Q = E_8/Sp(1)\,E_7$, $H = CP^7 = SU(8)/S(U(7) \times U(1))$.

Representation of e_8^C: $V = V(00000001)$, $\dim V = 3875$.

$\mathcal{K} = sp(1) \oplus e_7$, $\quad \mathcal{K}^C = A_1 \oplus e_7$,

$$V|_{\mathcal{K}^C} = V(0.0000010) \oplus$$

$$\oplus V(1.0100000) \oplus V(2.1000000) \oplus V(1.0000001) \oplus V(0.0000000).$$

$\mathcal{L} = su(8)$, $\mathcal{L}^C = A_7$ generated by the root $\{\alpha_i,\ i = 1, 3, 4, 5, 6, 7, 8\}$, then,

$$V|_{\mathcal{L}^C} = 2V(1000000) \oplus 2V(0000001) \oplus V(1000010) \oplus V(0100001) \oplus$$

$$\oplus V(0010001) \oplus V(1000100) \oplus V(0000002) \oplus V(2000000) \oplus V(1001000) \oplus$$

$$\oplus V(0001001) \oplus V(0000011) \oplus V(1100000) \oplus V(0100000) \oplus V(0000010) \oplus$$

$$\oplus V(0000100) \oplus V(0010000) \oplus V(0100010) \oplus 2V(1000001) \oplus$$

$$\oplus 2V(0001000) \oplus V(0000000).$$

$v_0 \in V(0100010) \oplus 2V(1000001) \oplus V(0000000)$.

$\mathcal{U}^C = \mathcal{L}^C \cap \mathcal{K}^C = A_6 \oplus C$, A_6 generated by the roots $\{\alpha_i,\ i = 1, 3, 4, 5, 6, 7\}$. But $V(0100010)|_{A_6} = V(100010) \oplus V(010001) \oplus V(010010) \oplus V(100001)$, so $v_0 \notin V(0100010)$.

Then, $v_0 \in 2V(1000001) \oplus V(0000000)$.

REMARK 4.1. *In all cases the vector* v_0 *can be taken in the corresponding real* $\mathcal{G}$*-module and the representations of the involved subalgebras are also real in the following sense; either they induce themselves a real representation (as defined before) or appear, in the sum, the contragredient representation in such a way that they lumped together, form the complexification of a real irreducible one.*

5. – Immersions of the classical irreducible symmetric spaces.

In this section we construct the immersions for the spaces of classical «type». In B1 and D we work with a realization of the representation, while in B2 and C we need some results from representation theory.

A) $Q = SU(n+2)/S(U(n) \times U(2))$. This case is trivial since Q itself is a Hermitian symmetric space.

B) $Q = SO(n+4)/SO(n) \times SO(4)$, $n = 2m$, $m \geqslant 2$ (see Lemma 2.6).

1) $H_1 = SU(m+2)/S(U(m) \times U(2))$.
We take the usual Cartan decomposition of $sl(n+4)$:

$$sl(n+4) = so(n+4) \oplus \mathcal{P},$$

where

$$\mathcal{P} = \{A \in M(n+4, R) : A = A^t, \operatorname{tr} A = 0\}.$$

Give to $\mathcal{P}$ structure of $so(n+4)$-module as usual

$$A \cdot v = (ad|_{sl(n+4)} A)(v) = [A, v],\ A \in so(n+4),\ v \in \mathcal{P}.$$

This is a realization of the real representation induced by the complex representation of highest weight $(20 \ldots 0)$.

Let $v_0 = \begin{bmatrix} I_n & 0 \\ 0 & -\lambda I_4 \end{bmatrix}$, where $\lambda = \dfrac{n}{4}$.

For $so(n+4)$ we have the decomposition, $so(n+4) = \mathcal{K} \oplus \mathcal{M}$, where

$$\mathcal{K} = \left\{ \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : A_1 \in so(n),\ A_2 \in so(4) \right\},$$

$$\mathcal{M} = \left\{ \begin{bmatrix} 0 & X \\ -X^t & 0 \end{bmatrix} : X \in M(n \times 4, R) \right\}.$$

Let

$$A \in so(n+4), \quad A \cdot v_0 = [A, v_0] = (1+\lambda) \begin{bmatrix} 0 & X \\ -X^t & 0 \end{bmatrix}.$$

Then, $A \cdot v_0 = 0$ if and only if $A \in \mathcal{K}$. Thus, $SO(n+4) \cdot v_0$ gives a immersion of Q into $\mathcal{P}$.

The induced action of the group $SO(n+4)$ is given by

$$\mathrm{A}\cdot \mathrm{v}_0 = \mathrm{A}\mathrm{v}_0\mathrm{A}^{-1} = \mathrm{A}\mathrm{v}_0\mathrm{A}^t, \qquad \mathrm{A}\in SO(n+4).$$

We want to show that v_0 is contained in the adjoint representation of $SU(m+2)$.

Let Ψ: $SU(m+2)\to SO(n+4)$ as in the lemma 2.6 and let Θ: $su(m+2)\to \mathcal{P}$ be defined by: if

$$\mathrm{Z} = [x_{ij} + \sqrt{-1}y_{ij}] \in su(m+2), \qquad \Theta(\mathrm{Z}) = \tilde{\mathrm{Z}} = [\mathrm{Z}_{ij}^{2\times 2}],$$

where

$$\mathrm{Z}_{ij}^{2\times 2} = \begin{bmatrix} y_{ij} & -x_{ij} \\ x_{ij} & y_{ij} \end{bmatrix}.$$

To show that the subspace $\Theta(su(m+2))$ is $SU(m+2)$-invariant and the representation of the group $SU(m+2)$ on that space is the adjoint one, it suffices to see that

$$\Psi(\mathrm{A})\cdot\Theta(\mathrm{Z}) = \Theta(\mathrm{Ad}\,|_{\mathrm{SU}(m+2)}(\mathrm{A})\mathrm{Z}) \text{ for } \mathrm{A}\in SU(m+2), \quad \mathrm{Z}\in su(m+2).$$

Let

$$\mathrm{A} = [b_{ij} + \sqrt{-1}c_{ij}] \in SU(m+2),$$

$$\mathrm{Z} = [x_{ij} + \sqrt{-1}y_{ij}] \in su(m+2),$$

$$\mathrm{Ad}\,|_{SU(m+2)}(\mathrm{A})\,\mathrm{Z} = \mathrm{AZA}^{-1} = \mathrm{AZ}\bar{\mathrm{A}}^t = \mathrm{W},$$

where

$$\mathrm{W}_{ij} = \sum_{k,h}((b_{ih}x_{hk} - c_{ih}y_{hk}) + \sqrt{-1}(b_{ih}y_{hk} - c_{ih}x_{hk}))(b_{jk} - \sqrt{-1}c_{jk}) =$$

$$= \alpha_{ij} + \sqrt{-1}\beta_{ij},$$

with

$$\alpha_{ij} = \sum_{k,h}[(b_{ih}x_{hk} - c_{ih}y_{hk})\,b_{jk} + (b_{ih}y_{hk} + c_{ih}x_{hk})\,c_{jk}],$$

$$\beta_{ij} = \sum_{k,h}[(b_{ih}y_{hk} + c_{ih}x_{hk})\,b_{jk} - (b_{ih}x_{hk} - c_{ih}y_{hk})\,c_{jk}].$$

On the other hand,

$$\Psi(\mathrm{A})\cdot\Phi(\mathrm{Z})=\tilde{\mathrm{A}}\cdot\tilde{\mathrm{Z}}\cdot\tilde{\mathrm{A}}^{\mathrm{t}}=[[\mathrm{A}_{ij}^{2\times 2}][\mathrm{Z}_{ij}^{2\times 2}][\mathrm{B}_{ij}^{2\times 2}]]=[\mathrm{C}_{ij}^{2\times 2}],\quad \mathrm{B}_{ij}^{2\times 2}=(\mathrm{A}_{ji}^{2\times 2})^{\mathrm{t}},$$

and

$$\mathrm{C}_{ij}^{2\times 2}=\sum_{k,h}\begin{bmatrix} b_{ih} & c_{ih}\\ -c_{ih} & b_{ih}\end{bmatrix}\begin{bmatrix} y_{hk} & -x_{hk}\\ x_{hk} & y_{hk}\end{bmatrix}\begin{bmatrix} b_{jk} & -c_{jk}\\ c_{jk} & b_{jk}\end{bmatrix}=$$

$$=\begin{bmatrix} \beta_{ij} & -\alpha_{ij}\\ \alpha_{ij} & \beta_{ij}\end{bmatrix}=\mathrm{W}_{ij}^{2\times 2}.$$

Consequently, $\Psi(\mathrm{A})\cdot\Theta(\mathrm{Z})=\Theta(\mathrm{Ad}\,|_{SU(m+2)}(\mathrm{A})\mathrm{Z})$.
Let

$$\mathrm{Z}_0=\begin{bmatrix}\sqrt{-1}\mathrm{I}_m & 0\\ 0 & -\sqrt{-1}\lambda\mathrm{I}_2\end{bmatrix}\in su(m+2).$$

Clearly, $\Theta(\mathrm{Z}_0)=\mathrm{v}_0$; then, H_1 is a extrinsic symmetric submanifold of $\Theta(su(m+2))$.

2) $H_2=SO(n+2)/SO(n)\times SO(2)$, $n\geqslant 4$.

If $n=4$, $Q=SO(8)/SO(4)\times SO(4)$ and $H_2=SO(6)/SO(4)\times\times SO(2)\cong SU(4)/S(U(2)\times U(2))=H$ [4, p. 519] is the above case.

If $n=6$, $Q=SO(10)/SO(6)\times SO(4)$; $H_2=SO(8)/SO(6)\times SO(2)$. $\mathcal{G}=so(10)$, $\mathcal{G}^C=D_5$.

As representation of D_5 we take $V=V(00011)$, $\dim V=210$.

$$\varepsilon=1 \quad\text{and}\quad \omega_0(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)=-(\alpha_1,\alpha_2,\alpha_3,\alpha_5,\alpha_4).$$

Since $(00011)=\lambda_4+\lambda_5=\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+2\alpha_5$, is easy to see that $\omega_0(00011)=-(00011)$.

$$\mathcal{K}=so(6)\oplus so(4)\cong so(6)\oplus so(3)\oplus so(3),$$

$$\mathcal{K}^C=D_3\oplus D_2=B_3\oplus A_1\oplus A_1,$$

generated by the roots $\{\{\alpha_3,\alpha_4,\alpha_5\},\{\alpha_1\},\{-\beta\}\}$.

$$V|_{\mathcal{K}^C}=V(101.0.2)\oplus V(101.2.0)\oplus V(101.0.0)\oplus V(002.1.1)\oplus$$

$$\oplus V(200.1.1)\oplus V(010.1.1)\oplus V(000.0.0).$$

$\mathcal{L} = so(8)$, $\mathcal{L}^C = D_4$, generated by the roots $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$.

$$V|_{\mathcal{L}^C} = 2V(0011) \oplus V(0002) \oplus V(0020) \oplus V(0100).$$

$$v_0 \in V(0002) \oplus V(0020) \oplus V(0100).$$

$\mathcal{U}^C = \mathcal{L}^C \cap \mathcal{K}^C = D_3 \oplus C$, D_3 generated by $\{\alpha_3, \alpha_4, \alpha_5\}$.

$$V(0002)|_{\mathcal{U}^C} = V(002) \oplus V(011) \oplus V(020) = V(0020)|_{\mathcal{U}^C},$$

then $v_0 \notin V(0002) \oplus V(0020)$. Thus, $v_0 \in V(0100)$ which is the adjoint representation of D_4.

Now we assume $n \geqslant 8$, $n = 2m$, $g^C = D_{m+2}$.

We take as representation of D_{m+2}: $V = V(00010\ldots0) = V(\lambda_4)$, $\dim V = \binom{n+4}{4}$.

Here $\varepsilon = 1$ and $\omega_0 = -\mathrm{id}$, if m is even, and $\omega_0(\alpha_1, \ldots, \alpha_m, \alpha_{m+1}, \alpha_{m+2}) = -(\alpha_1, \ldots, \alpha_m, \alpha_{m+2}, \alpha_{m+1})$, if m is odd so, in any case, $\omega_0(\lambda_4) = -\lambda_4$.

The representation is not spin, so it induces a representation of $SO(n+4)$. A realization of this complex representation of the group is $V = \wedge^4(C^{n+4})$, and the induced real representation is $V = \wedge^4(R^{n+4})$.

Let $v_0 = e_{n+1} \wedge e_{n+2} \wedge e_{n+3} \wedge e_{n+4}$, where $\{e_1, \ldots, e_{n+4}\}$ is the standard basis of C^{n+4}, $\mathcal{K} = so(n) \oplus so(4)$.

$$SO(n) \times SO(4) = \left\{ T = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \quad A \in SO(n) \text{ y } B \in SO(4) \right\}.$$

Let $T \in SO(n) \times SO(4)$,

$$T(v_0) = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} v_0 = Be_{n+1} \wedge Be_{n+2} \wedge Be_{n+3} \wedge Be_{n+4} = \det B v_0 = v_0.$$

Then, $(so(n) \oplus so(4))v_0 = 0$, so $V|_{\mathcal{K}}$ contains the trivial representation of $\mathcal{K}$ and clearly $so(n) \oplus so(4)$ is the isotropy subalgebra of v_0.

Now we consider the restriction to $\mathcal{L} = so(n+2)$, $\mathcal{L}^C = D_{m+1}$.

To find $V|_{\mathcal{L}^C}$ we use the reference [9, p. 378-379] where there are formulas to restrict a representation from $so(\nu)$ to $so(\nu - 1)$. We have used this formulas traslated into our notation. The conclussion is the following.

If $n=8$, $\mathcal{L}=so(10)$, $\mathcal{L}^C=D_5$.

$$V|_{D_5}=V(00011)\oplus 2V(00100)\oplus V(01000).$$

$$\mathcal{U}=\mathcal{L}\cap\mathcal{K}=so(8)\oplus R, \qquad \mathcal{U}^C=D_4\oplus C.$$

But,

$$V(00011)|_{D_4}=V(0002)\oplus 2V(0011)\oplus V(0020)\oplus V(0100)$$

and

$$V(00100)|_{D_4}=V(0011)\oplus 2V(0100)\oplus V(1000).$$

Then, $v_0\notin V(00011)\oplus 2V(00100)$ because v_0 is a vector fixed by $\mathcal{U}$. Hence, $v_0\in V(01000)$, the adjoint representation of D_5.
If $n=10$, $\mathcal{L}=so(12)$, $\mathcal{L}^C=D_6$.

$$V|_{D_6}=V(000100)\oplus 2V(001000)\oplus V(010000).$$

$$\mathcal{U}=\mathcal{L}\cap\mathcal{K}=so(10)\oplus R, \qquad \mathcal{U}^C=D_5\oplus C.$$

But,

$$V(000100)|_{D_5}=V(00011)\oplus 2V(00100)\oplus V(01000)$$

and

$$V(001000)|_{D_5}=V(00100)\oplus 2V(01000)\oplus V(10000).$$

Thus, $v_0\in V(010000)$, the adjoint representation of D_6.

If $n\geqslant 12$, $\mathcal{L}=so(n+2)$, $\mathcal{L}^C=D_{m+1}$.

$$V|_{D_{m+1}}=V_{D_{m+1}}(00010\ldots 0)\oplus 2V_{D_{m+1}}(0010\ldots 0)\oplus V_{D_{m+1}}(010\ldots 0).$$

$$\mathcal{U}=\mathcal{L}\cap\mathcal{K}=so(n)\oplus R, \qquad \mathcal{U}^C=D_m\oplus C.$$

Now,

$$V_{D_{m+1}}(00010\ldots 0)|_{D_m}=$$
$$=V_{D_m}(00010\ldots 0)\oplus 2V_{D_m}(0010\ldots 0)\oplus V_{D_m}(010\ldots 0)$$

and

$$V_{D_{m+1}}(0010\ldots 0)|_{D_m}=V_{D_m}(0010\ldots 0)\oplus 2V_{D_m}(010\ldots 0)\oplus V_{D_m}(10\ldots 0).$$

Then, $v_0\in V(010\ldots 0)=V(\lambda_2)$, the adjoint representation of D_{m+1}.

C) $Q\cong SO(n+4)/SO(n)\times SO(4)$, $n=2m+1$, $H\cong SO(n+2)/SO(n)\times SO(2)$.

If $n=3$, $Q=SO(7)/SO(3)\times SO(4)$; $H=SO(5)/SO(3)\times SO(2)$.

$$\mathcal{G}=so(7), \qquad \mathcal{G}^C=B_3 .$$

Representation of B_3: $V=V(002)$, dim $V=35$.

$$\varepsilon=1 \quad \text{and} \quad \omega_0=-\mathrm{id} .$$

The representation is not spin so induces a representation of $SO(7)$.

A realization of the representation space is $V=\wedge^3(C^7)$.

Let $v_0=e_1\wedge e_2\wedge e_3$, where $\{e_1, e_2, e_3, \dots, e_7\}$ is the canonical basis of C^7. Here $\mathcal{K}=so(3)\oplus so(4)$.

$$SO(3)\times SO(4)=\left\{T=\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \ A\in SO(3) \ \text{y} \ B\in SO(4)\right\}.$$

$$T(v_0)=\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} v_0 = Ae_1\wedge Ae_2\wedge Ae_3=\det A \ v_0=v_0 .$$

Then, $(so(3)\oplus so(4))v_0=0$ and $SO(7)\cdot v_0$ gives an immersion of Q. We take

$$SO(5)=\left\{T=\begin{bmatrix} A & 0 \\ 0 & I_2 \end{bmatrix}, \ A\in SO(5)\right\}\subset SO(7).$$

It is easy to see that if we call W the real span of the orbit $SO(5)\cdot v_0$; hence, $W=\wedge^3(C^5)$ because any element $e_i\wedge e_j\wedge e_k$ $(i<j<k\leqslant 5)$ of the basis of $\wedge^3(C^5)$ is contained in W.

It is well known that $\wedge^3(C^5)\cong\wedge^2(C^5)$, so the representation of $SO(5)$ in W is just the adjoint representation. Since $v_0\in W$, we obtain the desired result.

If $n=5$, $Q=SO(9)/SO(5)\times SO(4)$; $H=SO(7)/SO(5)\times SO(2)$.

$$\mathcal{G}=so(9), \qquad \mathcal{G}^C=B_4 .$$

We take the representation of B_4: $V=V(0002)$, dim $V=126$.

$$\varepsilon=1 \quad \text{and} \quad \omega_0=-\mathrm{id} .$$

$$\mathcal{K}=so(5)\oplus so(4)\cong so(5)\oplus so(3)\oplus so(3), \qquad \mathcal{K}^C=B_2\oplus A_1\oplus A_1$$

generated by the roots $\{\{\alpha_3, \alpha_4\}, \{\alpha_1\}, \{-\beta\}\}$.
$V|_{\mathcal{K}^C} = V(02.0.2) \oplus V(02.2.0) \oplus V(10.0.0) \oplus V(10.1.1) \oplus V(02.1.1) \oplus$

$$\oplus V(00.0.0).$$

$\mathcal{L} = so(7)$, $\mathcal{L}^C = B_3$, generated by the roots $\{\alpha_2, \alpha_3, \alpha_4\}$

$$V|_{\mathcal{L}^C} = 3V(002) \oplus V(010).$$

$\mathcal{U}^C = \mathcal{L}^C \cap \mathcal{L}^C = B_2 \oplus C$, B_2 generated by $\{\alpha_3, \alpha_4\}$

$$V(002)|_{\mathcal{U}^C} = V(02) \oplus V(01).$$

Then, $v_0 \notin V(002)$.

Thus, $v_0 \in V(010)$ which is the adjoint representation.

Now we assume $n \geqslant 7$, $n = 2m+1$, $\mathcal{G}^C = B_{m+2}$.

We take the representation of B_{m+2}: $V = V(00010\ldots0) = V(\lambda_4)$, $\dim V = \binom{n+4}{4}$.

$$\varepsilon = 1 \quad \text{and} \quad \omega_0 = -\mathrm{id}.$$

The representation is not spin, so it gives a representation of $SO(n+4)$.

A realization of this representation is $V = \wedge^4(C^{n+4})$. As before let $e_1, \ldots, e_{n+4}$ be the canonical basis. Let $v_0 = e_{n+1} \wedge e_{n+2} \wedge e_{n+3} \wedge \wedge e_{n+4}$.

$$\mathcal{K} = so(n) \oplus so(4).$$

As in the case n even we see that $x \cdot v_0 = 0$ if and only if $x \in so(n) \oplus \oplus so(4)$.

Thus, $V|_{so(n)\times so(4)}$ contains the trivial representation and again we get an immersion of Q.

Now consider the Hermitian symmetric subspace.

$$\mathcal{L} = so(n+2), \qquad \mathcal{L}^C = B_{m+1}.$$

To find $V|_{\mathcal{L}^C}$ we resort to [9, p. 378-379] again. The conclussion is the following.

If $n = 7$, $\mathcal{L} = so(9)$, $\mathcal{L}^C = B_4$.

$$V|_{B_4} = V(0002) \oplus 2V(0010) \oplus V(0100).$$

To see that $V(0002) \oplus 2V(0010)$ does not have vectors fixed by $\mathcal{U}$, we observe that

$$\mathcal{U} = \mathcal{L} \cap \mathcal{K} = so(7) \oplus R\,, \qquad \mathcal{U}^C = B_3 \oplus C\,.$$

But, $V(0002)\,|_{B_3} = 3V(002) \oplus V(010)$ and

$$V(0010)\,|_{B_3} = V(002) \oplus 2V(010) \oplus V(100)\,.$$

Then, $\mathrm{v}_0 \in V(0100)$ the adjoint representation of B_4.

If $n = 9$, $\mathcal{L} = so(11)$, $\mathcal{L}^C = B_5$.

$$V|_{B_5} = V(00010) \oplus 2V(00100) \oplus V(01000)\,.$$

Now,

$$\mathcal{U} = \mathcal{L} \cap \mathcal{K} = so(9) \oplus R\,, \qquad \mathcal{U}^C = B_4 \oplus C\,.$$

But, $V(00010)\,|_{B_4} = V(0002) \oplus 2V(0010) \oplus V(0100)$ and

$$V(00100)\,|_{B_4} = V(0010) \oplus 2V(0100) \oplus V(1000)\,.$$

Thus, $\mathrm{v}_0 \in V(01000)$, the adjoint representation of B_5.

If $n \geqslant 11$, $\mathcal{L} = so(n+2)$, $\mathcal{L}^C = B_{m+1}$

$$V|_{B_{m+1}} = V_{B_{m+1}}(00010\ldots0) \oplus 2V_{B_{m+1}}(0010\ldots0) \oplus V_{B_{m+1}}(010\ldots0)\,.$$

$$\mathcal{U} = \mathcal{L} \cap \mathcal{K} = so(n) \oplus R\,, \qquad \mathcal{U}^C = B_m \oplus C\,.$$

Now,

$$V_{B_{m+1}}(00010\ldots0)\,|_{B_m} =$$

$$= V_{B_m}(00010\ldots0) \oplus 2V_{B_m}(0010\ldots0) \oplus V_{B_m}(010\ldots0)\,,$$

and

$$V_{B_{m+1}}(0010\ldots0)\,|_{B_m} = V_{B_m}(0010\ldots0) \oplus 2V_{B_m}(010\ldots0) \oplus V_{B_m}(10\ldots0)\,.$$

Then, $\mathrm{v}_0 \in V(010\ldots0) = V(\lambda_2)$, which is the adjoint representation of B_{m+1}.

D) $Q = Sp(n+1)/Sp(n) \times Sp(1)$, $H = SU(n+1)/\,S(U(n) \times U(1))$.

We take the representation of C_{n+1}: $V(010\ldots0) = V(\lambda_2)$ and work with a realization of this representation.

We take the Cartan descomposition $su^*(2(n+1)) = sp(n+1) + \mathcal{P}$, where

$$\mathcal{P} = \left\{ \begin{bmatrix} \sqrt{-1}\,Z_1 & \sqrt{-1}\,Z_2 \\ \sqrt{-1}\,\overline{Z}_2 & -\sqrt{-1}\,\overline{Z}_1 \end{bmatrix} : Z_1 \in su(n+1),\ Z_2 \in so(n+1, C) \right\}.$$

$\mathcal{P}$ has the structure of $sp(n+1)$-module given by the action

$$A \cdot v = (ad|_{su^*(2n+2)} A)\, v = [A, v], \qquad A \in sp(n+1), \qquad v \in \mathcal{P}.$$

Let

$$v_0 = \begin{bmatrix} -I_n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & -I_n & 0 \\ 0 & 0 & 0 & n \end{bmatrix}.$$

Clearly, $v_0 \in \mathcal{P}$.

For $sp(n+1)$ we have the Cartan descomposition $sp(n+1) = \mathcal{K} \oplus \oplus \mathcal{M}$, where

$$\mathcal{K} = \left\{ \begin{bmatrix} X_{11} & 0 & X_{13} & 0 \\ 0 & X_{22} & 0 & X_{24} \\ -X_{13} & 0 & \overline{X}_{11} & 0 \\ 0 & -X_{24} & 0 & \overline{X}_{22} \end{bmatrix} : \begin{matrix} X_{11} \in u(n), \\ X_{22} \in u(1), \\ X_{13} \text{ simétrica } n \times n \\ X_{24} \in C \end{matrix} \right\},$$

$$\mathcal{M} = \left\{ \begin{bmatrix} 0 & Y_{12} & 0 & Y_{14} \\ -\overline{Y}^t_{12} & 0 & Y^t_{14} & 0 \\ 0 & -\overline{Y}_{14} & 0 & \overline{Y}_{12} \\ -\overline{Y}^t_{14} & 0 & -Y^t_{12} & 0 \end{bmatrix} : Y_{12}, Y_{14} \in M(n \times 1, C) \right\}.$$

If

$$A = \begin{bmatrix} X_{11} & Y_{12} & X_{13} & Y_{14} \\ \overline{Y}^t_{12} & X_{22} & -Y^t_{14} & X_{24} \\ -X_{13} & -\overline{Y}_{14} & \overline{X}_{11} & \overline{Y}_{12} \\ \overline{Y}^t_{14} & -X_{24} & Y^t_{12} & \overline{X}_{22} \end{bmatrix} \in sp(n+1)$$

a simple calculation shows that

$$A\cdot v_0 = (1+n)\begin{bmatrix} 0 & Y_{12} & 0 & Y_{14} \\ \overline{Y}^t_{12} & 0 & -Y^t_{14} & X_{24} \\ 0 & -\overline{Y}_{14} & 0 & \overline{Y}_{12} \\ \overline{Y}^t_{14} & 0 & Y^t_{12} & 0 \end{bmatrix}.$$

Thus, $A\cdot v_0 = 0$ if and only if $A \in \mathcal{K}$.

Hence, $Sp(n+1)\cdot v_0$ gives an immersion of Q.

To restrict the action to $SU(n+1)$ we need the following maps.

Let Ψ: $SU(n+1) \to Sp(n+1)$ given by

$$\Psi(A) = \begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix}, \qquad A \in SU(n+1).$$

Let Θ: $su(n+1) \to p$ given by

$$\Theta(Z) = \begin{bmatrix} \sqrt{-1}\,Z & 0 \\ 0 & -\sqrt{-1}\,\overline{Z} \end{bmatrix}, \qquad Z \in su(n+1).$$

We wish to see that if $A \in SU(n+1)$, $v \in su(n+1)$

$$\Theta(\mathrm{Ad}\,|_{SU(n+1)}(A)v) = \Psi(A)\cdot\Theta(v).$$

But,

$$\Theta(\mathrm{Ad}\,|_{SU(n+1)}(A)v) = \Theta(AvA^{-1}) = \Theta(Av\overline{A}^t) =$$

$$= \begin{bmatrix} \sqrt{-1}\,Av\overline{A}^t & 0 \\ 0 & -\sqrt{-1}\,\overline{A}\overline{v}A^t \end{bmatrix},$$

$$\Psi(A)\cdot\Theta(v) = \Psi(A)\,\Theta(v)\,\Psi(A)^{-1} =$$

$$= \begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix}\begin{bmatrix} \sqrt{-1}\,v & 0 \\ 0 & -\sqrt{-1}\,\overline{v} \end{bmatrix}\begin{bmatrix} \overline{A}^t & 0 \\ 0 & A^t \end{bmatrix} =$$

$$= \begin{bmatrix} \sqrt{-1}\,Av\overline{A}^t & 0 \\ 0 & -\sqrt{-1}\,\overline{A}\overline{v}A^t \end{bmatrix}.$$

Thus, $\Theta(su(n+1))$ is $SU(n+1)$-invariant and the action is the adjoint representation of the group.

Clearly, if

$$z_0 = \begin{bmatrix} \sqrt{-1}\, I_n & 0 \\ 0 & -n\sqrt{-1} \end{bmatrix} \in su(n+1), \qquad \Theta(z_0) = v_0 \,.$$

Then, v_0 is in the adjoint representation of $SU(n+1)$.

REFERENCES

[1] A. BESSE, *Einstein Manifolds*, Springer-Verlag, Berlin, Heidelberg (1987).

[2] N. BOURBAKI, *Éléments de mathématique, Groups et algèbres de Lie,* Chapitres 4, 5 et 6, Hermann, Paris (1968).

[3] F. BURSTALL, J. RAWNSLEY - S. SALAMON, *Stable Harmonic 2-spheres in symmetric spaces,* Bull. Amer. Math. Soc., **16** (1987), pp. 274-278.

[4] S. HELGASON, *Differential Geometry, Lie Groups and Symmetric Spaces,* Academic Press, New York, San Francisco, London (1978).

[5] J. HUMPHREYS, *Introduction to Lie Algebras and Representation Theory,* Springer-Verlag, New York (1972), third edition (1980).

[6] D. FERUS, *Symmetric submanifolds of Euclidean Space,* Math. Ann., **247** (1980), pp. 81-93.

[7] H. FREUDENTHAL - H. DE VRIES, *Linear Lie groups,* Academic Press, New York and London (1969).

[8] J. WOLF, *Complex homogeneous contact manifolds and quaternionic symmetric spaces,* J. Math. Mech., **14** (1965), pp. 1033-1047.

[9] D. ZELOBENKO, *Compact Lie groups and their representation,* Trans. Amer. Math. Soc., **40** (1973).

Manoscritto pervenuto in redazione il 13 dicembre 1996