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**On the Wellposedness in the Gevrey Classes  
of the Cauchy Problem for Weakly Hyperbolic Equations  
whose Coefficients are Hölder Continuous in  $t$   
and Degenerate in  $t = T$ .**

TAMOTU KINOSHITA (\*)

**1. Introduction.**

In 1983 F. Colombini, E. Jannelli and S. Spagnolo got the result concerned with the relation between the Hölder continuity of the coefficients and the Gevrey well-posedness for weakly hyperbolic equations of second order (see [3]). In their paper the coefficients of the principle part are Hölder continuous and degenerate in infinite number of points. Therefore the order of the degeneration is automatically determined by the regularity of the coefficients. Pay attention to this fact, we shall restrict the coefficients which degenerate in only one (or finite number of) point(s) and distinguish the condition of the degeneration from the condition of the regularity. Our aim is to deduce the relation among the Hölder continuity of the coefficients and the order of the degeneration and the Gevrey well-posedness (see Theorem).

While many people studied the influence of the lower term for the Gevrey well-posedness(see [6], [7], [9], etc.). In this paper, imposing the condition that the lower terms degenerate in the same point with the principal part, we shall find the influence of the lower term (see Corollary). We remark that this result can be easily compared with Ivrii's result.

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We shall consider in  $[0, T] \times \mathbf{R}_x^n$ ,

$$(P) \quad \begin{cases} u_{tt} - \sum_{i,j=1}^n a_{i,j}(t) u_{x_i x_j} + \sum_{i=1}^n b_i(t) u_{x_i} = 0, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \end{cases}$$

where  $A(t) \equiv (a_{i,j})_{1 \leq i,j \leq n}$  is a real symmetric matrix whose components  $a_{i,j}(t)$  ( $1 \leq i, j \leq n$ ) belong to  $C^{k+\alpha}[0, T]$  ( $k \in \mathbf{N}^1$ ,  $0 \leq \alpha \leq 1$ ).  $B(t) \equiv (b_i)_{1 \leq i \leq n}$  is a real vector whose components  $b_i(t)$  ( $1 \leq i \leq n$ ) belong to  $C^0[0, T]$ .

Now we assume the followings.

For  $a(t, \xi) \equiv (A(t) \xi, \xi)$  ( $= \sum a_{i,j} \xi_i \xi_j$ ) and  $b(t, \xi) \equiv (B(t), \xi)$  ( $= \sum b_i(t) \xi_i$ ), there exist  $C_1, C_2 > 0$  independent of  $T$ , such that

$$(1) \quad C_1 \left(1 - \frac{t}{T}\right)^\beta |\xi|^2 \leq a(t, \xi) \leq C_1^{-1} \left(1 - \frac{t}{T}\right)^\beta |\xi|^2 \quad (0 \leq \beta \leq \infty)$$

$$(2) \quad |b(t, \xi)| \leq C_2 \left(1 - \frac{t}{T}\right)^\gamma |\xi| \quad (-1 < \gamma \leq \infty)$$

for  $\forall t \in [0, T]$ ,  $\forall \xi \in \mathbf{R}_\xi^n \setminus \{0\}$ .

In this paper we don't assume the conditions concerned with  $\partial_t a(t, \xi)$ , for example the condition such that  $\partial_t a(t, \xi)$  admits finite number of zeros for  $\forall \xi \in \mathbf{R}_\xi^n \setminus \{0\}$  (see [3], [5]).

**THEOREM.** *Let  $T > 0$ . The coefficients satisfy (1), (2). Then for any  $u_0$  and  $u_1 \in G^s$ , the Cauchy problem (P) has a unique solution  $u \in C^2([0, T], G^s)$ , provided*

$$(3) \quad 1 \leq s < 1 + \frac{\alpha(\beta + 2)}{2(\beta - \alpha + 1)} \quad \text{if } k = 0,$$

$$(4) \quad 1 \leq s < 1 + \frac{(k + \alpha)(\beta + 2)}{2\beta} \quad \text{if } k = 1 \text{ or } k = 2, 3, \dots, \beta - 2\gamma - 2 < 0,$$

$$(5) \quad 1 \leq s < 1 + \min \left\{ \frac{(k + \alpha)(\beta + 2)}{2\beta}, \frac{\beta + 2}{\beta - 2\gamma - 2} \right\}$$

$$\text{if } k = 2, 3, \dots, \beta - 2\gamma - 2 \geq 0.$$

We remark that (3) when  $k = 0$ ,  $\alpha = 1$  coincides (4) when  $k = 1$ ,  $\alpha = 0$ . We shall compare our theorem with the results of others.

i) When  $\beta = 0$ , i.e., (P) is equivalent to the strictly hyperbolic equation, (3) becomes  $1 \leq s < 1/(1 - \alpha)$  which is quite same with the result of F. Colombini, E. De Giorgi, and S. Spagnolo (see [2] and see also [1]).

ii) When  $\beta = \infty$ , i.e., (P) is a weakly hyperbolic equation whose coefficients are identically equal to zero, (3) and (4) become  $1 \leq s < 1 + (k + \alpha)/2$  which is quite same with the result of F. Colombini, E. Jannelli, and S. Spagnolo (see [3] and see also [4], [10]). Hence we can see that our results connect with their result, in which the coefficients of the equations may be identically zero in some intervals.

iii) The coefficients of our equation degenerate in  $t = T$ . From the time-reversal of hyperbolic equations, we can see that our results also holds in the case that the coefficients degenerate in  $t = 0$ . On the other hand Ivrii treated the equations whose coefficients degenerate in  $t = 0$  and investigated the influence of the lower terms (see [6]). Then we shall introduce the similar corollary as the Ivrii's result, which is obtained easily by our theorem( use (5)).

**COROLLARY.** *Let  $T > 0$ ,  $\beta - 2\gamma - 2 \geq 0$ , and  $a_{ij}(t)$  ( $1 \leq i, j \leq n$ )  $\in C^{2\beta/(\beta - 2\gamma - 2)}[0, T]$ . The coefficients satisfy (1), (2). Then the Cauchy problem (P) is well-posed in  $G^s$ , provided*

$$1 \leq s < \frac{\beta + 2}{\beta - 2\gamma - 2}.$$

We find that the Gevrey order of this corollary coincides the one of the Ivrii's result. Moreover the condition  $\beta - 2\gamma - 2 \geq 0$  also coincides the Ivrii's condition. In Ivrii's case the coefficients are given concretely as the power of  $t$  which naturally belong  $C^\infty$ , while in our case the coefficients are admitted to belong  $C^{2\beta/(\beta - 2\gamma - 2)}$ .

Finally we mention that since generally the lower terms don't always degenerate in the only one (or finite number of) point(s) where the principal part also degenerates, the another type of condition is needed for the general lower terms (see [3], [7], [11]).

**NOTATIONS.**  $C^{k+\alpha}[0, T]$  ( $k \in \mathbb{N}^1$ ,  $0 \leq \alpha \leq 1$ ) is the space of functions

$f$  having  $k$ -derivatives continuous, and the  $k$ -th derivative locally Hölder continuous with exponent  $\alpha$  on  $[0, T]$ .

$G^s$  is the Gevrey functions  $f$  satisfying for any compact set  $K \subset \mathbf{R}^n$ ,

$$\sup_{x \in K} |D^\alpha f(x)| \leq C_K \varrho_K^{|\alpha|} |\alpha|^s! \quad \text{for } \forall \alpha \in \mathbf{N}^n.$$

$G_0^s$  is the space of  $\gamma^s$  with compact support.

## 2. Proof of Theorem.

When  $s = 1$ , the problem (P) is well-posed in  $G^1$  with the coefficient  $A(t)$  which belongs to  $C^0[0, T]$  or even to  $L^1[0, T]$  (see [2]). Therefore we can suppose  $s > 1$  for the proof.

By Holmgren's theorem we get the uniqueness of solutions to (P) and can suppose that  $u_0(x)$  and  $u_1(x)$  belong to  $G_0^s$ . Moreover Ovciannikov theorem gives the existence of solutions (see [3]). Our task is to deduce the regularity for  $x$  of solutions.

We change (P) by Fourier transform.

$$(6) \quad \begin{cases} v_{tt} + a(t, \xi) v + ib(t, \xi) v = 0, \\ v(0, \xi) = v_0(\xi), \quad v_t(0, \xi) = v_1(\xi), \end{cases}$$

where

$$v = \frac{1}{\sqrt{2\pi}} \int e^{-ix \cdot \xi} u(x) dx, \quad v_i = \frac{1}{\sqrt{2\pi}} \int e^{-ix \cdot \xi} u_i(x) dx \quad (i = 0, 1),$$

and define

$$A_\delta(t) = \begin{cases} A(0) & -\infty < t < 0, \\ A(t) & 0 \leq t \leq T - \delta, \\ A(T - \delta) & T - \delta < t < \infty, \end{cases}$$

$(0 < \delta \leq T)$ . We shall first consider the case  $k = 0$ . Putting

$$A_\delta^\varepsilon(t) = \frac{1}{\varepsilon} \int_{-\infty}^{\infty} \varphi\left(\frac{s-t}{\varepsilon}\right) A_\delta(s) ds \left( = \frac{1}{\varepsilon} \int_{-\infty}^{\infty} \varphi\left(\frac{s}{\varepsilon}\right) A_\delta(t+s) ds \right)$$

$(0 < \varepsilon < 1)$ , where  $\varphi(t) \in C_0^\infty(\mathbf{R}_t^1)$  satisfies  $0 \leq \varphi(t) < \infty$ ,  $\int_{-\infty}^{\infty} \varphi(t) dt = 1$ ,

we define

$$a_\delta^{(\varepsilon)}(t, \xi) \equiv (A_\delta^{(\varepsilon)}(t) \xi, \xi).$$

Moreover using this, we shall also define the approximate energy

$$(7) \quad E_\delta^{(\varepsilon)}(t, \xi) = e^{\varrho(t)\langle \xi \rangle_\nu^k} (a_\delta^{(\varepsilon)}(t, \xi) |v|^2 + |v'|^2)^{1/2},$$

where  $\langle \xi \rangle_\nu = (|\xi|^2 + \nu^2)^{1/2}$  ( $\nu > 0$ ),  $\varrho(t) \in C^1[0, T]$  is determined later. Defferentiating  $(E_\delta^{(\varepsilon)})^2$  in  $t$ , by (6) we get

$$\begin{aligned} d(E_\delta^{(\varepsilon)})^2/dt &= 2\varrho' \langle \xi \rangle_\nu^k (E_\delta^{(\varepsilon)})^2 + e^{2\varrho\langle \xi \rangle_\nu^k} \times \\ &\quad \times \{ (da_\delta^{(\varepsilon)}/dt) |v|^2 + a_\delta^{(\varepsilon)} \cdot 2 \operatorname{Re}(v, v') + 2 \operatorname{Re}(v', v'') \} = \\ &= 2\varrho' \langle \xi \rangle_\nu^k (E_\delta^{(\varepsilon)})^2 + \frac{(da_\delta^{(\varepsilon)}/dt)}{a_\delta^{(\varepsilon)}} e^{2\varrho\langle \xi \rangle_\nu^k} a_\delta^{(\varepsilon)} |v|^2 + \\ &\quad + \frac{a_\delta^{(\varepsilon)} - a}{(a_\delta^{(\varepsilon)})^{1/2}} e^{2\varrho\langle \xi \rangle_\nu^k} \cdot 2 \operatorname{Re}((a_\delta^{(\varepsilon)})^{1/2} v, v') + \\ &\quad + \frac{-ib}{(a_\delta^{(\varepsilon)})^{1/2}} e^{2\varrho\langle \xi \rangle_\nu^k} \cdot 2 \operatorname{Im}((a_\delta^{(\varepsilon)})^{1/2} v, v') \leq \\ &\leq 2\varrho' \langle \xi \rangle_\nu^k (E_\delta^{(\varepsilon)})^2 + \frac{|da_\delta^{(\varepsilon)}/dt|}{a_\delta^{(\varepsilon)}} (E_\delta^{(\varepsilon)})^2 + \\ &\quad + \frac{|a_\delta^{(\varepsilon)} - a|}{(a_\delta^{(\varepsilon)})^{1/2}} (E_\delta^{(\varepsilon)})^2 + \frac{|b|}{(a_\delta^{(\varepsilon)})^{1/2}} (E_\delta^{(\varepsilon)})^2, \end{aligned}$$

here we used  $|(x, y)| \leq |x||y| \leq (1/2)(|x|^2 + |y|^2)$ .

Hence we get

$$dE_\delta^{(\varepsilon)}/dt \leq \frac{1}{2} \left\{ 2\varrho' \langle \xi \rangle_\nu^k + \frac{|da_\delta^{(\varepsilon)}/dt|}{a_\delta^{(\varepsilon)}} + \frac{|a_\delta^{(\varepsilon)} - a|}{(a_\delta^{(\varepsilon)})^{1/2}} + \frac{|b|}{(a_\delta^{(\varepsilon)})^{1/2}} \right\} E_\delta^{(\varepsilon)}.$$

Thus Gronwall's inequality yields

$$\begin{aligned}
 (8) \quad E_\delta^{(\varepsilon)}(t) &\leq E_\delta^{(\varepsilon)}(0) \cdot \\
 &\cdot \exp \left\{ \frac{1}{2} \int_0^t 2\varrho' \langle \xi \rangle_\nu^k + \frac{|da_\delta^{(\varepsilon)}/dt|}{a_\delta^{(\varepsilon)}} + \frac{|a_\delta^{(\varepsilon)} - a|}{(a_\delta^{(\varepsilon)})^{1/2}} + \frac{|b|}{(a_\delta^{(\varepsilon)})^{1/2}} ds \right\} \leq \\
 &\leq E_\delta^{(\varepsilon)}(0) \exp \left\{ (\varrho(t) - \varrho(0)) \langle \xi \rangle_\nu^k + \right. \\
 &\left. + \frac{1}{2} \int_0^t \frac{|da_\delta^{(\varepsilon)}/dt|}{a_\delta^{(\varepsilon)}} ds + \frac{1}{2} \int_0^t \frac{|a_\delta^{(\varepsilon)} - a|}{(a_\delta^{(\varepsilon)})^{1/2}} ds + \frac{1}{2} \int_0^t \frac{|b|}{(a_\delta^{(\varepsilon)})^{1/2}} ds \right\} \\
 &\qquad \qquad \qquad \text{for } t \in [0, T].
 \end{aligned}$$

Picking up each term, we shall estimate. From the definition of  $a_\delta^{(\varepsilon)}$ , we can get

$$\begin{aligned}
 (9) \quad |da_\delta^{(\varepsilon)}(t)/dt| &= |((dA_\delta^{(\varepsilon)}(t)/dt) \xi, \xi)| \leq \\
 &\leq \left| \frac{1}{\varepsilon} \int_{-\infty}^{\infty} -\frac{1}{\varepsilon} \varphi' \left( \frac{s-t}{\varepsilon} \right) A_\delta(s) ds \right| |\xi|^2 = \\
 &= \left| -\frac{1}{\varepsilon} \int_{-\infty}^{\infty} \varphi'(s) A_\delta(t + \varepsilon s) ds \right| |\xi|^2 = \\
 &= \left| \frac{1}{\varepsilon} \int_{-\infty}^{\infty} \varphi'(s) (A_\delta(t) - A_\delta(t + \varepsilon s)) ds \right| |\xi|^2 \leq \\
 &\leq C_3 \frac{1}{\varepsilon} \int_{-\infty}^{\infty} \varphi'(s) (\varepsilon s)^\alpha ds |\xi|^2 \leq C_3 C_4 \varepsilon^{\alpha-1} |\xi|^2,
 \end{aligned}$$

here we used  $(1/\varepsilon) \int_{-\infty}^{\infty} \varphi'(s) ds = 0$  for  $0 < \forall \varepsilon < 1$ .

While from the definition of  $a_\delta^{(\varepsilon)}$  and (1), we can also get

$$(10) \quad a_\delta^{(\varepsilon)}(t) \geq a(T - \delta) \geq C_1 \left(1 - \frac{T - \delta}{T}\right)^\beta |\xi|^2 = C_1 \left(\frac{\delta}{t}\right)^\beta |\xi|^2$$

for  $0 < \delta \leq T$ .

By (9), (10) we get

$$(11) \quad \frac{1}{2} \int_0^t \frac{|da_\delta^{(\varepsilon)}/dt|}{a_\delta^{(\varepsilon)}} ds \leq \frac{1}{2} \int_0^t \frac{C_3 C_4 \varepsilon^{\alpha-1} |\xi|^2}{C_1 (\delta/T)^\beta |\xi|^2} ds = C_5 T^\beta t \delta^{-\beta} \varepsilon^{\alpha-1},$$

where  $C_5 = (1/2) C_1^{-1} C_3 C_4$ .

From the definition of  $a_\delta^{(\varepsilon)}$  again, we can get

$$(12) \quad |a_\delta^{(\varepsilon)}(t) - a(t)| \leq |((A_\delta^{(\varepsilon)}(t) - A_\delta(t)) \xi, \xi)| + |((A_\delta(t) - A(t)) \xi, \xi)| \leq$$

$$\leq \left| \frac{1}{\varepsilon} \int_{-\infty}^{\infty} \varphi\left(\frac{s}{\varepsilon}\right) (A_\delta(t+s) - A_\delta(t)) ds \right| |\xi|^2 + |A_\delta(t) - A(t)| |\xi|^2 \leq$$

$$\leq \frac{1}{\varepsilon} \int_{-\infty}^{\infty} \varphi\left(\frac{s}{\varepsilon}\right) s^\alpha ds |\xi|^2 + |A_\delta(t) - A(t)| |\xi|^2 \leq$$

$$\leq C_6 \varepsilon^\alpha |\xi|^2 + |A_\delta(t) - A(t)| |\xi|^2.$$

Noting that  $A_\delta(t) = A(t)$  on  $0 \leq t \leq T - \delta$ , and  $|A_\delta(t) - A(t)| \leq C_1^{-1} (\delta/T)^\beta$  on  $T - \delta \leq t < T$ , by (10), (12) we get

$$(13) \quad \frac{1}{2} \int_0^t \frac{|a_\delta^{(\varepsilon)} - a|}{(a_\delta^{(\varepsilon)})^{1/2}} ds \leq \frac{1}{2} \int_0^t \frac{C_6 \varepsilon^\alpha |\xi|^2}{\{C_1 (\delta/T)^\beta |\xi|^2\}^{1/2}} ds +$$

$$+ \begin{cases} \frac{1}{2} \int_{T-\delta}^t \frac{C_1^{-1} (\delta/T)^\beta |\xi|^2}{\{C_1 (\delta/T)^\beta |\xi|^2\}^{1/2}} ds & \text{if } t \geq T - \delta, \\ 0 & \text{if } t < T - \delta, \end{cases}$$

$$\leq C_7 T^{\beta/2} t \delta^{-\beta/2} \varepsilon^\alpha |\xi| + \begin{cases} C_8 T^{-\beta/2} (t - T + \delta) \delta^{\beta/2} |\xi| & \text{if } t \geq T - \delta, \\ 0 & \text{if } t < T - \delta, \end{cases}$$

where  $C_7 = (1/2) C_1^{-1/2} C_6$ ,  $C_8 = (1/2) C_1^{-3/2}$ .

We remark that

$$(14) \quad t - T + \delta = \frac{\delta t - \delta t + tT - T^2 + \delta t}{T} = \frac{\delta t + (\delta - T)(T - t)}{T} \leq \frac{\delta t}{T}.$$

Hence by (13) we get

$$(15) \quad \frac{1}{2} \int_0^t \frac{|a_\delta^{(\varepsilon)} - a|}{(a_\delta^{(\varepsilon)})^{1/2}} ds \leq C_7 T^{\beta/2} t \delta^{-\beta/2} \varepsilon^\alpha |\xi| + C_8 T^{-1 - \frac{\beta}{2}} t \delta^{1 + \frac{\beta}{2}} |\xi|.$$

From the hypothesis (2), by (10) we get

$$(16) \quad \begin{aligned} \frac{1}{2} \int_0^t \frac{|b|}{(a_\delta^{(\varepsilon)})^{1/2}} ds &\leq \frac{1}{2} \int_0^t \frac{C_2(1 - s/T)^\gamma |\xi|}{\{C_1(\delta/T)^\beta |\xi|^2\}^{1/2}} ds = \\ &= \frac{1}{2} C_1^{-1/2} C_2 T^{\beta/2} \delta^{-\beta/2} \int_0^t \left(1 - \frac{s}{T}\right)^\gamma ds = \\ &= \frac{1}{2} C_1^{-1/2} C_2 \frac{1}{\gamma + 1} T^{\beta/2 + 1} \delta^{-\beta/2} \left\{1 - \left(1 - \frac{t}{T}\right)^{\gamma + 1}\right\}. \end{aligned}$$

It generally holds that for  $0 \leq x \leq 1$

$$(17) \quad (1 - x)^a \geq \begin{cases} 1 - ax & \text{if } a \geq 1, \\ 1 - x^a & \text{if } 0 < a < 1. \end{cases}$$

By (16), (17), we get

$$(18) \quad \frac{1}{2} \int_0^t \frac{|b|}{(a_\delta^{(\varepsilon)})^{1/2}} ds \leq \begin{cases} C_9 T^{\beta/2} t \delta^{-\beta/2} & \text{if } \gamma \geq 0, \\ C_9 \left(\frac{1}{\gamma + 1}\right) T^{\beta/2 - \gamma} t^{\gamma + 1} \delta^{-\beta/2} & \text{if } -1 < \gamma < 0, \end{cases}$$

where  $C_9 = (1/2) C_1^{-1/2} C_2$ .

By (8), (11), (15), (18), we have the estimate

$$\begin{aligned} E_\delta^{(\varepsilon)}(t) &\leq E_\delta^{(\varepsilon)}(0) \exp\{(\varrho(t) - \varrho(0)) \langle \xi \rangle_\nu^\kappa + \\ &\quad + C_5 T^\beta t \delta^{-\beta} \varepsilon^{\alpha - 1} + C_7 T^{\beta/2} t \delta^{-\beta/2} \varepsilon^\alpha |\xi| + \end{aligned}$$

$$+C_8 T^{-1-\beta/2} t \delta^{1+\beta/2} |\xi| + \left\{ \begin{array}{ll} C_9 T^{\beta/2} t \delta^{-\beta/2} & \text{if } \gamma \geq 0, \\ C_9 \left( \frac{1}{\gamma+1} \right) T^{\beta/2-\gamma} t^{\gamma+1} \delta^{-\beta/2} & \text{if } -1 < \gamma < 0. \end{array} \right\}.$$

Here, if we take

$$(19) \quad \delta = T \langle \xi \rangle^{-2\alpha(\alpha\beta+2\beta+2)}, \quad \varepsilon = \langle \xi \rangle^{(-2\beta-2)/(\alpha\beta+2\beta+2)},$$

and put

$$\mu(t) = t + t^\lambda$$

with the parameter  $\lambda = \min \{1, \gamma + 1\}$ , we have

$$\begin{aligned} E_\delta^{(\varepsilon)}(t) &\leq E_\delta^{(\varepsilon)}(0) \exp \left\{ (\varrho(t) - \varrho(0)) \langle \xi \rangle_\nu^\kappa + \right. \\ &\quad + C_5 t \langle \xi \rangle^{(-2\alpha+2\beta+2)/(\alpha\beta+2\beta+2)} + C_7 t \langle \xi \rangle^{(-2\alpha+2\beta+2)/(\alpha\beta+2\beta+2)} + \\ &\quad + C_8 t \langle \xi \rangle^{(-2\alpha+2\beta+2)/(\alpha\beta+2\beta+2)} + \\ &\quad \left. + \left\{ \begin{array}{ll} C_9 t \langle \xi \rangle^{\alpha\beta/\alpha\beta+2\beta+2} & \text{if } \gamma \geq 0, \\ C_9 \left( \frac{1}{\gamma+1} \right) T^{-\gamma} t^{\gamma+1} \langle \xi \rangle^{\alpha\beta/(\alpha\beta+2\beta+2)} & \text{if } -1 < \gamma < 0 \end{array} \right\} \right\} \\ &\leq E_\delta^{(\varepsilon)}(0) \exp \left\{ (\varrho(t) - \varrho(0)) \langle \xi \rangle_\nu^\kappa + C_{10} \mu(t) \langle \xi \rangle^{(-2\alpha+2\beta+2)/(\alpha\beta+2\beta+2)} \right\}, \end{aligned}$$

where  $C_{10} = C_5 + C_7 + C_8 + C_9 \max \{1, (1/\gamma + 1) T^{-1}\}$ . Here we used

$$\frac{\alpha\beta}{\alpha\beta+2\beta+2} < \frac{-2\alpha+2\beta+2}{\alpha\beta+2\beta+2} \quad \text{for } 0 \leq \alpha \leq 1, 0 \leq \beta \leq \infty.$$

Supposing

$$(20) \quad \kappa > \frac{-2\alpha+2\beta+2}{\alpha\beta+2\beta+2} (\equiv \kappa_0),$$

we get

$$\begin{aligned} (21) \quad E_\delta^{(\varepsilon)}(t) &\leq E_\delta^{(\varepsilon)}(0) \exp \left[ \langle \xi \rangle_\nu^\kappa \left( \varrho(t) - \varrho(0) + C_{10} \mu(t) \frac{\langle \xi \rangle^{\kappa_0}}{\langle \xi \rangle_\nu^\kappa} \right) \right] \leq \\ &\leq E_\delta^{(\varepsilon)}(0) \exp \left\{ \langle \xi \rangle_\nu^\kappa (\varrho(t) - \varrho(0) + C_{10} \mu(t) \nu^{\kappa_0 - \kappa}) \right\}. \end{aligned}$$

Finally if we choose  $\varrho(t) = \varrho(0) - C_{10}\mu(t) \nu^{\kappa_0 - \kappa}$ , we have the energy estimate

$$(22) \quad E_\delta^{(\varepsilon)}(t) \leq E_\delta^{(\varepsilon)}(0).$$

Noting that  $\varrho(t)$  is the decreasing function, by (7), (10), the left side of (22) is changed as follows

$$(23) \quad E_\delta^{(\varepsilon)}(t, \xi) \geq e^{\varrho(T)\langle \xi \rangle^\kappa} \left( C_1 \left( \frac{\delta}{T} \right)^\beta \left( |\xi|^2 |v(t, \xi)|^2 + |v'(t, \xi)|^2 \right) \right)^{1/2} \\ \geq \min \left\{ 1, C_1^{1/2} \left( \frac{\delta}{T} \right)^{\beta/2} \right\} e^{(\varrho(0) - C_{10}\mu(T)\nu^{\kappa_0 - \kappa})\langle \xi \rangle^\kappa} \left( |\xi|^2 |v(t, \xi)|^2 + |v'(t, \xi)|^2 \right)^{1/2}.$$

While by (1) and (7) the right side of (22) is changed as follows

$$(24) \quad E_\delta^{(\varepsilon)}(0, \xi) \leq e^{\varrho(0)\langle \xi \rangle^\kappa} \left( a_\delta^{(\varepsilon)}(0, \xi) |v_0(\xi)|^2 + |v_1(\xi)|^2 \right)^{1/2} = \\ = e^{\varrho(0)\langle \xi \rangle^\kappa} \left( a(0, \xi) |v_0(\xi)|^2 + |v_1(\xi)|^2 \right)^{1/2} \leq \\ \leq C_{11} e^{\varrho(0)\langle \xi \rangle^\kappa} \left( |\xi|^2 |v_0(\xi)|^2 + |v_1(\xi)|^2 \right)^{1/2}.$$

By (22), (23), (24) we have

$$\min \left\{ 1, C_1^{1/2} \left( \frac{\delta}{T} \right)^{\beta/2} \right\} e^{(\varrho(0) - C_{10}\mu(T)\nu^{\kappa_0 - \kappa})\langle \xi \rangle^\kappa} \left( |\xi|^2 |v(t, \xi)|^2 + |v'(t, \xi)|^2 \right)^{1/2} \leq \\ \leq C_{11} e^{\varrho(0)\langle \xi \rangle^\kappa} \left( |\xi|^2 |v_0(\xi)|^2 + |v_1(\xi)|^2 \right)^{1/2}.$$

Therefore we obtain

$$e^{\varrho_1\langle \xi \rangle^\kappa} \left( |\xi|^2 |v(t, \xi)|^2 + |v'(t, \xi)|^2 \right)^{1/2} \leq \\ \leq \max \left\{ 1, C_1^{-1/2} \left( \frac{\delta}{T} \right)^{-\beta/2} \right\} C_{11} e^{(\varrho_1 + C_{10}\mu(T)\nu^{\kappa_0 - \kappa})\langle \xi \rangle^\kappa} \left( |\xi|^2 |v_0(\xi)|^2 + |v_1(\xi)|^2 \right)^{1/2} \leq \\ \leq C_{12} e^{(\varrho_1 - \varrho_0 + C_{10}\mu(T)\nu^{\kappa_0 - \kappa})\langle \xi \rangle^\kappa + \varrho_0(\langle \xi \rangle^\kappa - \langle \xi \rangle_{\nu_0}^\kappa)} \left( \frac{\delta}{T} \right)^{-\beta/2} \\ \cdot e^{\varrho_0\langle \xi \rangle^\kappa} \left( |\xi|^2 |v_0(\xi)|^2 + |v_1(\xi)|^2 \right)^{1/2},$$

where  $0 < \varrho_1 < \varrho_0$ ,  $C_{12} = \max \{ 1, C_1^{-1/2} (\delta/T)^{-\beta/2} \} C_{11}$ .

Moreover noting that for  $\nu_1 \geq \nu_2$

$$\begin{aligned} \langle \xi \rangle_{\nu_1}^\kappa - \langle \xi \rangle_{\nu_2}^\kappa &= (\nu_1 - \nu_2) \int_0^1 \partial_\nu \langle \xi \rangle_\nu^\kappa \Big|_{\nu=\nu_2+\theta(\nu_1-\nu_2)} d\theta = \\ &= (\nu_1 - \nu_2) \int_0^1 \kappa(\nu_2 + \theta(\nu_1 - \nu_2)) \langle \xi \rangle_{\nu_2+\theta(\nu_1-\nu_2)}^{\kappa-2} d\theta = \kappa(\nu_1 - \nu_2) \nu_2^{\kappa-2} \leq \kappa \nu_1^2 \nu_2^{\kappa-2}, \end{aligned}$$

and taking

$$\nu = \max \left\{ \nu_0, \left( \frac{\varrho_0 - \varrho_1}{2C_{10}\mu(T)} \right)^{1/(\kappa_0 - \kappa)} \right\},$$

by (19) we get

$$\begin{aligned} e^{(\varrho_1 - \varrho_0 + C_{10}\mu(T)\nu^{\kappa_0 - \kappa})\langle \xi \rangle_\nu^\kappa + \varrho_0(\langle \xi \rangle_\nu^\kappa - \langle \xi \rangle_{\nu_0}^\kappa)} \left( \frac{\delta}{T} \right)^{-\beta/2} &\leq \\ &\leq e^{(\varrho_1 - \varrho_0 + C_{10}\mu(T)((\varrho_0 - \varrho_1)/(2C_{10}\mu(T)))\langle \xi \rangle_\nu^\kappa + \varrho_0\kappa\nu^2\nu_0^{\kappa-2})\langle \xi \rangle^{\alpha\beta/(\alpha\beta + 2\beta + 2)}} \\ &\leq e^{\varrho_0\kappa\nu^2\nu_0^{\kappa-2}} (e^{((\varrho_1 - \varrho_0)/2)\langle \xi \rangle_\nu^\kappa} \langle \xi \rangle^{\alpha\beta/(\alpha\beta + 2\beta + 2)}) \leq C_{13} e^{\varrho_0\kappa\nu^2\nu_0^{\kappa-2}}. \end{aligned}$$

At last we have

$$\begin{aligned} (25) \quad e^{\varrho_1\langle \xi \rangle_\nu^\kappa} (|\xi|^2 |v(t, \xi)|^2 + |v'(t, \xi)|^2)^{1/2} &\leq \\ &\leq \text{const } e^{\varrho_0\langle \xi \rangle_{\nu_0}^\kappa} (|\xi|^2 |v_0(\xi)|^2 + |v_1(\xi)|^2)^{1/2} \\ &\quad \text{for } \kappa_0 < \nu \kappa < 1, \quad 0 < \nu \varrho_1 < \varrho_0. \end{aligned}$$

Since  $u_0(x)$  and  $u_1(x)$  belong to  $G_0^s$ , there exist  $\varrho_0, \nu_0 > 0$  such that

$$(26) \quad e^{\varrho_0\langle \xi \rangle_{\nu_0}^{1/s}} (|\xi|^2 |v_0(\xi)|^2 + |v_1(\xi)|^2)^{1/2} < \infty.$$

Thus by (P), (20), (25) we find that the solution  $u \in C^2([0, T], G^s)$ , where  $s = \kappa^{-1}$  satisfies

$$1 \leq s < 1 + \frac{\alpha(\beta + 2)}{2(\beta - \alpha + 1)}.$$

This implies (3).

We shall next consider the case of  $k = 1, 2, \dots$ . We also define the approximate energy

$$(27) \quad E_\delta(t, \xi) = e^{\varrho(t)\langle \xi \rangle^k} (a_\delta(t, \xi) |v|^2 + |v'|^2)^{1/2},$$

where  $a_\delta(t, \xi) \equiv (A_\delta(t) \xi, \xi)$ .

Similarly we get (see (8))

$$(28) \quad E_\delta(t) \leq E_\delta(0) \exp \left\{ (\varrho(t) - \varrho(0)) \langle \xi \rangle_v^k + \right. \\ \left. + \frac{1}{2} \int_0^t \frac{|da_\delta/dt|}{a_\delta} ds + \frac{1}{2} \int_0^t \frac{|a_\delta - a|}{a_\delta^{1/2}} ds + \frac{1}{2} \int_0^t \frac{|b|}{a_\delta^{1/2}} ds \right\} \quad \text{for } t \in [0, T].$$

In order to estimate the coefficients which belong to  $C^1$  at least, we need the following lemma.

LEMMA (Colombini, Jannelli, Spagnolo). *Let  $f(t)$  be a real function of class  $C^{k+\alpha}$  on some compact interval  $I \subset \mathbf{R}$ , with  $k$  integer  $\geq 1$  and  $0 \leq \alpha \leq 1$ , and assume that  $f(t) \geq 0$  on  $I$ . Then the function  $f(t)^{1/(k+\alpha)}$  is absolutely continuous on  $I$ . Moreover*

$$\|(f^{1/(k+\alpha)})'\|_{L^1(I)}^k \leq C(k, \alpha, I) \|f\|_{C^{k+\alpha}(I)}.$$

For the proof, refer to [3].

Hence by (1) we get

$$(29) \quad \frac{1}{2} \int_0^t \frac{|da_\delta/dt|}{a_\delta} ds = \frac{1}{2} \int_0^t \frac{|da_\delta/dt|}{a_\delta^{1/(k+\alpha)} a_\delta^{1-1/(k+\alpha)}} ds \\ \leq \frac{1}{2} \left( C_1 \left( \frac{\delta}{T} \right)^\beta |\xi| \right)^{-1/(k+\alpha)} \int_0^t \frac{|da_\delta/dt|}{a_\delta^{1-1/(k+\alpha)}} ds \leq \psi(t) T^{\beta/(k+\alpha)} \delta^{-\beta/(k+\alpha)},$$

where  $\psi(t)$  is the increasing function satisfying  $\psi(0) = 0$ .

In the same way with the second term of (15), we get

$$(30) \quad \frac{1}{2} \int_0^t \frac{|a_\delta - a|}{a_\delta^{1/2}} ds \leq C_8 T^{-1-\beta/2} t \delta^{1+\beta/2} |\xi|.$$

When  $1 \leq k + \alpha \leq 2$ , in the same way with (18), we get

$$(31) \quad \frac{1}{2} \int_0^t \frac{|b|}{(a_\delta)^{1/2}} ds \leq \begin{cases} C_9 T^{\beta/2} t \delta^{-\beta/2} & \text{if } \gamma \geq 0, \\ C_9 \left( \frac{1}{\gamma + 1} \right) T^{\beta/2-\gamma} t^{\gamma+1} \delta^{-\beta/2} & \text{if } -1 < \gamma < 0, \end{cases}$$

$$\leq \begin{cases} C_9 T^{\beta/2} t \delta^{-\beta/(k+\alpha)} & \text{if } \gamma \geq 0, \\ C_9 \left( \frac{1}{\gamma + 1} \right) T^{\beta/(2-\gamma)} t^{\gamma+1} \delta^{-\beta/(k+\alpha)} & \text{if } -1 < \gamma < 0. \end{cases}$$

By (28),(29),(30),(31) we have the estimate

$$E_\delta(t) \leq E_\delta(0) \exp \{ (\varrho(t) - \varrho(0)) \langle \xi \rangle_\nu^k + \psi(t) T^{\beta/(k+\alpha)} \delta^{-\beta/(k+\alpha)} +$$

$$+ C_8 T^{-1-\beta/2} t \delta^{1+\beta/2} |\xi| +$$

$$+ \begin{cases} C_9 T^{\beta/2} t \delta^{-\beta/(k+\alpha)} \} & \text{if } \gamma \geq 0, \\ C_9 \left( \frac{1}{\gamma + 1} \right) T^{\beta/2-\gamma} t^{\gamma+1} \delta^{-\beta/(k+\alpha)} \} & \text{if } -1 < \gamma < 0. \end{cases}$$

Here, if we take

$$\delta = T \langle \xi \rangle^{-1/(1+\beta/2+\beta/(k+\alpha))},$$

and put

$$\tilde{\mu}(t) = t + t^\lambda + \psi(t)$$

with the parameter  $\lambda = \min \{ 1, \gamma + 1 \}$ , we have

$$E_\delta(t) \leq E_\delta(0) \exp \{ (\varrho(t) - \varrho(0)) \langle \xi \rangle_\nu^k + C_{14} \tilde{\mu}(t) \langle \xi \rangle^{2\beta/(2\beta+(k+\alpha)(\beta+2))} \},$$

where  $C_{14} = 1 + C_8 + C_9 \max \{ 1, (1/(\gamma + 1)) T^{-1} \}$ .

Supposing

$$(32) \quad \kappa > \frac{2\beta}{2\beta + (k + \alpha)(\beta + 2)} (\equiv \kappa_0),$$

we get

$$(33) \quad E_\delta(t) \leq E_\delta(0) \exp \left\{ (\varrho(t) - \varrho(0)) \langle \xi \rangle_\nu^\kappa + C_{14} \tilde{\mu}(t) \nu^{\kappa_0 - \kappa} \right\}.$$

When  $2 \leq k + \alpha \leq \infty$ , we remark that if  $\beta/2 - \gamma - 1 < 0$ , the following holds

$$\begin{aligned} \frac{1}{2} \int_0^t \frac{|b|}{a_\delta^{1/2}} ds &\leq \frac{1}{2} \int_0^t \frac{|b|}{a^{1/2}} ds \leq \frac{1}{2} \int_0^t \frac{C_2(1-s/T)^\gamma |\xi|}{\{C_1(1-s/T)^\beta |\xi|^2\}^{1/2}} ds = \\ &= \frac{1}{2} C_1^{-1/2} C_2 \int_0^t \left(1 - \frac{s}{T}\right)^{\gamma - \beta/2} ds \leq \text{const}. \end{aligned}$$

This implies that the lower term doesn't influence the well-posedness of the problem. Hence it is sufficient to investigate the case  $\beta/2 - \gamma - 1 \geq 0$ . With the parameter  $\tau < 1$ , we get

$$\begin{aligned} \frac{1}{2} \int_0^t \frac{|b|}{a_\delta^{1/2}} ds &\leq \frac{1}{2} \int_0^t \frac{|b|}{a_\delta^{1/2 - (\gamma + \tau)/\beta} a_\delta^{(\gamma + \tau)/\beta}} ds \leq \\ &\leq \frac{1}{2} \left\{ C_1 \left( \frac{\delta}{T} \right)^\beta |\xi|^2 \right\}^{-1/2 + (\gamma + \tau)/\beta} \int_0^t \frac{|b|}{a_\delta^{(\gamma + \tau)/\beta}} ds \leq \\ &\leq \frac{1}{2} \left\{ C_1 \left( \frac{\delta}{T} \right)^\beta |\xi|^2 \right\}^{-1/2 + (\gamma + \tau)/\beta} \int_0^t \frac{C_2(1-s/T)^\gamma |\xi|}{\{C_1(1-s/T)^\beta |\xi|^2\}^{(\gamma + \tau)/\beta}} ds = \\ &= \frac{1}{2} C_1^{-1/2} C_2 T^{\beta/2 - (\gamma + \tau)} \delta^{\gamma + \tau - \beta/2} \int_0^t \left(1 - \frac{s}{T}\right)^{-\tau} ds = \\ &= \frac{1}{2} C_1^{-1/2} C_2 \frac{1}{1 - \tau} T^{\beta/2 - (\gamma + \tau) + 1} \delta^{\gamma + \tau - \beta/2} \left\{ 1 - \left(1 - \frac{t}{T}\right)^{1 - \tau} \right\}. \end{aligned}$$

By (17) we get

$$(34) \quad \frac{1}{2} \int_0^t \frac{|b|}{a_\delta^{1/2}} ds \leq \begin{cases} \frac{1}{2} C_1^{-1/2} C_2 T^{\beta/2 - (\gamma + \tau)} t \delta^{\gamma + \tau - \beta/2} & \text{if } \tau \leq 0, \\ \frac{1}{2} C_1^{-1/2} C_2 \frac{1}{1 - \tau} T^{\beta/2 - \gamma} t^{1 - \tau} \delta^{\gamma + \tau - \beta/2} & \text{if } 0 < \tau < 1. \end{cases}$$

By (28), (29), (30), (34) we have the estimate

$$\begin{aligned}
 E_\delta(t) \leq E_\delta(0) \exp \{ (\varrho(t) - \varrho(0)) \langle \xi \rangle_\nu^\kappa + \psi(t) T^{\beta/(k+\alpha)} \delta^{-\beta/(k+\alpha)} + \\
 + C_8 T^{-1-\beta/2} t \delta^{1+\beta/2} |\xi| + \\
 + \begin{cases} \frac{1}{2} C_1^{-1/2} C_2 T^{\beta/2-(\gamma+\tau)} t \delta^{\gamma+\tau-\beta/2} & \text{if } \tau \leq 0, \\ \frac{1}{2} C_1^{-1/2} C_2 \frac{1}{1-\tau} T^{\beta/2-\gamma} t^{1-\tau} \delta^{\gamma+\tau-\beta/2} & \text{if } 0 < \tau < 1. \end{cases}
 \end{aligned}$$

Here, if we take

$$\delta = T \langle \xi \rangle^{\max \left\{ \frac{-1}{1+(\beta/2)+\beta/(k+\alpha)}, \frac{-1}{1-\tau+\beta-\gamma} \right\}},$$

and put

$$\tilde{\mu}(t) = t + t^\gamma + \psi(t)$$

with the parameter  $\lambda = \min \{1, 1-\tau\}$ , we have

$$\begin{aligned}
 E_\delta(t) \leq E_\delta(0) \exp \{ (\varrho(t) - \varrho(0)) \langle \xi \rangle_\nu^\kappa + \\
 + C_{15} \tilde{\mu}(t) \langle \xi \rangle^{\max \{ 2\beta/(2\beta+(k+\alpha)(\beta+2)), (\beta/2-\gamma-\tau)/(1+\beta-\gamma-\tau) \}} \},
 \end{aligned}$$

where  $C_{15} = 1 + C_8 + (1/2) C_1^{-1/2} C_2 \max \{1, 1/(1-\tau) T^\tau\}$ .

We remark that for any fixed

$$(35) \quad \kappa > \max \left\{ \frac{2\beta}{2\beta+(k+\alpha)(\beta+2)}, \frac{\beta/2-\gamma-1}{\beta-\gamma} \right\} (\equiv \kappa_0),$$

there exists  $\tau < 1$  such that

$$\kappa > \max \left\{ \frac{2\beta}{2\beta+(k+\alpha)(\beta+2)}, \frac{\beta/2-\gamma-\tau}{1+\beta-\gamma-\tau} \right\} (\equiv \kappa_1).$$

Then we get

$$(36) \quad E_\delta(t) \leq E_\delta(0) \exp \{ \langle \xi \rangle_\nu^\kappa (\varrho(t) - \varrho(0) + C_{15} \tilde{\mu}(t) \nu^{\kappa_1-\kappa}) \}.$$

Taking place of (21) by (33),(36), and noting that if  $k = 1, 2, \dots, \beta - 2\gamma - 2 < 0$ , by (32) it holds that

$$s (= \kappa^{-1}) < \frac{1}{\kappa_0} = 1 + \frac{(k+\alpha)(\beta+2)}{2\beta}$$

and if  $k = 2, 3, \dots, \beta - 2\gamma - 2 \geq 0$ , by (35) it holds that

$$s (= \kappa^{-1}) < \frac{1}{\kappa_0} = \frac{1}{\max\{2\beta/(2\beta + (k + \alpha)(\beta + 2)), (\beta/2 - \gamma - 1)/(\beta - \gamma)\}} = \\ = 1 + \min\left\{\frac{(k + \alpha)(\beta + 2)}{2\beta}, \frac{\beta + 2}{\beta - 2\gamma - 2}\right\},$$

we can conclude the theorem.

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