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On the Generalized Hypercentralizer of a Lie Ideal in a Prime Ring (*).

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SUMMARY - Let R be an associative ring, $Z(R)$ its center and $H_R(U) = \{a \in R : [a, u^n]_m = 0,\ n = n(a, u) \geq 1,\ m = m(a, u) \geq 1, \text{ all } u \in U\}$, where U is a non-central Lie ideal of R. We prove that if R is a prime ring without nil right ideals, then either $H_R(U) = Z(R)$ or R is an order in a simple algebra of dimension at most 4 over its center.

The aim of this paper is to extend some results about the hypercenter of a ring to the hypercentralizer of a Lie ideal in a prime associative ring.

Let R be a given associative ring and let n be a positive integer. The n-th commutator of $x, y \in R$, denoted by $[x, y]_n$, is defined inductively as follows:

for $n = 1, [x, y]_1 = [x, y] = xy - yx$ is the commutator of x and y

for $n > 1,\ [x, y]_n = [x, y]_{n-1}y - y[x, y]_{n-1}$.

In [2], the n-th hypercenter of R is defined to be the set

$$H_n(R) = \{a \in R : \text{for each } x \in R \text{ there exists an integer } m = m(a, x) \geq 1 \text{ such that } [a, x^m]_n = 0\}$$

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and, moreover, the generalized hypercenter of R is the set

$$H(R) = \{a \in R : \text{for each } x \in R, \text{ there exist integers}$$

$$n = n(a, x) \geqslant 1 \text{ and } m = m(a, x) \geqslant 1, \text{ such that } [a, x^m]_n = 0\}.$$

The classical hypercenter theorem proved by I. N. Herstein [8] asserts that the hypercenter, $H_1(R)$, of a ring without non-zero nil two-sided ideals always coincides with its center.

More recentely Chuang and Lin proved that if R is a ring without non-zero nil two-sided ideals then $H_n(R)$ coincides with the center of R, $Z(R)$.

They also proved that if R is a ring without non-zero nil right ideals then the generalized hypercenter, $H(R)$, coincides with the center (see Theorem 2 and Theorem 4 of [2]).

In this paper we will study a more general situation. More precisely let S be a subset of R, we say that an element $a \in R$ is in the n-th hypercentralizer of S, $H_{n,R}(S)$, if and only if for each $s \in S$ there exists an integer $m = m(a, s) \geqslant 1$ such that $[a, s^m]_n = 0$.

In the same way we define the generalized hypercentralizer of S to be the set

$$H_R(S) = \{a \in R : \text{for each } s \in S \text{ there exist integers}$$

$$n = n(a, s) \geqslant 1 \text{ and } m = m(a, s) \geqslant 1 \text{ such that } [a, s^m]_n = 0\}.$$

Of course if $S = R$ then $H_{n,R}(S)$ is merely the n-th hypercenter of R and $H_R(R)$ coincides with the generalized hypercenter of R. We remark that in [6] Giambruno and Felzenszwalb studied our first hypercentralizer in the case when $S = f(R)$ is the subset of all valutations $f(r_1, \dots, r_d)$ of a multilinear polynomial $f(x_1, \dots, x_d)$ on a prime ring R.

They proved that if R is without non-zero nil right ideals then either $H_{1,R}(f(R)) = Z(R)$ or the polynomial $f(x_1, \dots, x_d)$ is power central valued and R satisfies the standard identity $S_{d+2}(x_1, \dots, x_{d+2})$.

As a consequence of this result it is proved in [1] that if U is a non central Lie ideal of a prime ring R, without non-zero nil right ideals, then either $H_{1,R}(U) = Z(R)$ or R satisfies the standard identity $S_4(x_1, \dots, x_4)$.

Our main result has the same flavour:

THEOREM. *Let R be a prime ring without non-zero nil right ideals and let U be a non central Lie ideal of R, then either $H_R(U) = Z(R)$ or R satisfies the standard identity $S_4(x_1, \dots, x_4)$.*

Preliminar results.

Here we summarize some basic properties of n-th commutators. These simple facts will be used implicitly troughout all the proofs of this paper.

REMARK 1. Let $x, y, z \in R$.

a) If $[x, y]_n = 0$ for some $n \geq 1$ then $[x, y^m]_n = 0$ for any $m \geq 1$ and $[x, y]_q = 0$ for any $q \geq n$.

b) If $[x, y^m]_n = 0$ and $[z, y^t]_n = 0$ then $[x, y^{mt}]_n = [z, y^{mt}]_n = 0$.

c) $[x+y, z]_n = [x, z]_n + [y, z]_n$ and $[xy, z]_n = \sum_{i=0}^{n} \binom{n}{i} [x, z]_{n-i}[y, z]_i$ (here we put $[x, y]_0 = x$).

d) If $[x, y^m]_n = 0$ and $[z, y^m]_q = 0$ then $[xz, y^m]_{n+q-1} = 0$.

As a consequence we also have

REMARK 2.

a) $Z(R) \subseteq H_{n, R}(S) \subseteq H_R(S)$.

b) $H_{n, R}(S)$ is an additive subgroup of R.

c) $H_R(S)$ is a subring of R.

REMARK 3. Let φ be an automorphism of R such that $\varphi(S) \subseteq S$, then $\varphi(H_{n, R}(S)) \subseteq H_{n, R}(S)$ and $\varphi(H_R(S)) \subseteq H_R(S)$.

1. - Some reductions.

We begin the proof of our theorem with the following standard results (see Lemmas 11, 12 of [2]).

LEMMA 1.1. *If R is a prime ring of characteristic $p>0$ then $H_R(S)=H_{1,R}(S)$.*

PROOF. Let $a\in H_R(S)$. Given $s\in S$ there exist positive integers $n=n(a,s)$ and $m=m(a,s)$ such that $[a,s^m]_n=0$. For any $x, y\in R$ we have $[x,y]_n=\sum_{i=0}^{n}\binom{n}{i}(-1)^i y^i x y^{n-i}$.

Hence, for $t\geq 1$ such that $p^t\geq n$, we obtain $0=[a,s^m]_{p^t}=as^{mp^t}-s^{mp^t}a=[a,s^{mp^t}]$ and so $a\in H_{1,R}(S)$. ∎

LEMMA 1.2. *Let R be a domain and let $a, b\in H(S)$ be such that $a+b+ab=a+b+ba=0$. Then $a, b\in H_{1,R}(S)$.*

PROOF. If the charateristic of R is a prime number then by Lemma 1.1 one has $H_R(S)=H_{1,R}(S)$. Hence we may assume that $\mathrm{char}\,R=0$. Given $s\in S$, let h and k be the minimal positive integers such that $[a,s^n]_h=0$ and $[b,s^n]_k=0$ for some $n\geq 1$. Suppose that $h>1$ and $k>1$, then $h+k-2\geq\max(h,k)$. Hence $[a,s^n]_{h+k-2}=[b,s^n]_{h+k-2}=0$. On the other hand

$$0=[a+b+ab,s^n]_{h+k-2}=[ab,s^n]_{h+k-2}=$$
$$=\binom{h+k-2}{h-1}[a,s^n]_{h-1}[b,s^n]_{k-1}.$$

Since R is a domain of characteristic 0, we have $[a,s^n]_{h-1}=0$ or $[b,s^n]_{k-1}=0$. This contradicts with the minimality of h and k. Hence one of h and k must be 1, say $h=1$. Then $0=[a+b+ab,s^n]=[b,s^n]+a[b,s^n]=(1+a)[b,s^n]$. Hence $[b,s^n]=(1+b)(1+a)[b,s^n]=0$ (Note that the use of 1 is purely formal). Thus $h=k=1$. Since this holds for any $s\in S$ we have $a, b\in H_{1,R}(S)$. ∎

LEMMA 1.3. *Let R be a prime ring and U a non-central Lie ideal of R. Then either there exists a non-zero ideal I of R such that $0\neq[I,R]\subseteq U$ or $\mathrm{char}\,R=2$ and R satisfies $S_4(x_1,\dots,x_4)$.*

PROOF. See [7, pp. 4-5], [4, Lemma 2, Proposition 1], [10, Theorem 4]. ∎

2. – The case $R = H_R([I, I])$.

In this section I will be a non-zero two-sided ideal of R. $[I, I]$ will denote the subset $\{[a, b] : a, b \in I\}$. We begin with an immediate consequence of Lemma 1.2.

LEMMA 2.1. *Let R be a division ring. If $R = H_R([R, R])$ then R satisfies $S_4(x_1, \dots, x_4)$.*

PROOF. Let $-1 \neq a \in R$. Let $b = (1+a)^{-1} - 1$, then $a + b + ab = a + b + ba = 0$ and, by Lemma 1.2, $a \in H_{1, R}([R, R])$. On the other hand, by Lemma 1 of [6], we have either $H_1([R, R]) = Z(R)$ or $\dim_{Z(R)} R = N^2$ where $N \leq 2$. In any case R satisfies the standard identity $S_4(x_1, \dots, x_4)$. ∎

LEMMA 2.2. *Let R be a domain with non-zero Jacobson's radical $J(R)$. If $H_R([I, I]) = R$ then R satisfies $S_4(x_1, \dots, x_4)$.*

PROOF. As R is a prime ring $V = I \cap J(R)$ is a non-zero two-sided ideal of R. By Lemma 1.2 $V \subseteq H_{1, R}([I, I])$ and, a fortiori, $V = H_{1, V}([V, V])$. Therefore, by Lemma 7 of [6], V satisfies the standard identity $S_4(x_1, \dots, x_4)$. Since V is a non-zero ideal of R, $S_4(x_1, \dots, x_4)$ is a polynomial identity for R too. ∎

Notice that in the next Lemma we do not assume that $R = H_R([I, I])$.

LEMMA 2.3. *Let R be a primitive ring. If R is not a division ring then either $H_R([I, I]) = Z(R)$ or $R \cong F_2$, the ring of 2×2 matrices over a field F.*

PROOF. Let V be a faithful irreducible right R-module with endomorphisms ring D, a division ring. Since I is a non-zero two-sided ideal of R then R and I are both dense subrings of D-linear transformations on V. Suppose that $\dim_D V \geq 3$. We claim that in this case $H_R([I, I]) = Z(R)$.

In fact, let $a \neq 0$ be in $H_R([I, I])$ and assume that for some $v \in V$ the vectors v and va are linearly independent over D. By our assumption there exists a vector w such that v, va, w are linearly independent over D. As I acts densely on V there exist elements $r, s \in I$ such that

$$vr = 0, \ (va)r = w, \ wr = 0 \quad \text{and} \quad vs = 0, \ (va)s = 0, \ ws = va .$$

Hence $v[r, s] = 0$ and $va[r, s]^m = va$ for any $m \geq 1$. Since $a \in H_R([I, I])$ there exist positive integers n, m such that $[a,[r, s]^m]_n = 0$. Thus we have:

$$0 = v[a,[r, s]^m]_n = v\left(\sum_{i=0}^{n}\binom{n}{i}(-1)^i[r, s]^{mi}a[r, s]^{m(n-i)}\right) =$$

$$= va[r, s]^{mn} = va \neq 0 ,$$

a contradiction.

Hence, given $v \in V$, v and va are linearly dependent over D. It is well known that in this case a must be central (see, for istance, the proof of Lemma 2 in [8]).

Hence we may assume that $\dim_D V \leq 2$ and so, by our hypotesis, $R \cong D_2$, the ring of 2×2 matrices over the division ring D. Thus R is a simple ring with a non-trivial idempotent and I coincides with R. Moreover, by Remarks 2 and 3, $H_R([R, R])$ is a subring of R which is invariant under all the automorphisms of R. If R is not the ring of 2×2 matrices over $GF(2)$, then, by [7, theorem 1.15], either $H_R([R, R]) = Z(R)$ or $H_R([R, R]) = R$. In the last case we have $H_{D_2}([D_2, D_2]) = D_2$ and we claim that D is commutative. Let e_{ij} be the matrix unit with 1 in (i, j) entry and 0 elsewhere. Let $r = a(e_{12} + e_{22})$ and $s = b(e_{12} + e_{22})$ where $a, b \in D$; hence $[r, s]^m = [a, b]^m(e_{12} + e_{22})$ for any $m \geq 1$. Since $H_{D_2}([D_2, D_2]) = D_2$ there exist positive integers n, m such that $[e_{12},[r, s]^m]_n = 0$. Since $[e_{12},[r, s]^m] = [e_{12},[a, b]^m(e_{12} + e_{22})] = [a, b]^m e_{12}$, we obtain $0 = [e_{12},[r, s]^m]_n = [a, b]^{mn} e_{12}$, and so $[a, b]^{mn} = 0$ in D. Hence $[a, b] = 0$ for all $a, b \in D$, that is D is commutative and we are done. ∎

As an immediate consequence of Lemmas 2.1 and 2.3 we obtain

LEMMA 2.4. *Let R be a primitive ring. If $R = H_R([I, I])$ then R satisfies $S_4(x_1, \ldots, x_4)$.*

PROOF. It is suffices to recall that F_2 satisfies the standard identity $S_4(x_1, \ldots, x_4)$ (see Example 3 page 12 of [9]). ∎

LEMMA 2.5. *Let R be a domain. If $R = H_R([I, I])$ then R satisfies $S_4(x_1, \dots, x_4)$.*

PROOF. If $J(R) \neq (0)$ then the result follows by Lemma 2.2. Now we assume $J(R) = (0)$. So that R is a subdirect product of primitive rings R_γ, $\gamma \in \Gamma$.

Let P_γ be a primitive ideal of R such that $R_\gamma \cong R/P_\gamma$. We consider $\Gamma_1 = \{\gamma \in \Gamma : I \subseteq P_\gamma\}$ and $\Gamma_2 = \{\gamma \in \Gamma : I \nsubseteq P_\gamma\}$, in addition let $I_i = \cap P_\gamma$, for $\gamma \in \Gamma_i$, $i = 1, 2$. Since R is semisimple $I_1 I_2 \subseteq I_1 \cap I_2 = (0)$.

Since R is a domain we must have either $I_1 = (0)$ or $I_2 = (0)$. If $I_1 = (0)$ then $I \subseteq I_1 = (0)$, a contradiction. Hence $I_2 = (0)$ and so R is a subdirect product of primitive rings $R_\gamma \cong R/P_\gamma$, such that $I \nsubseteq P_\gamma$. Of course $I_\gamma = (I + P_\gamma)/P_\gamma$ is a non-zero two-sided ideal of R_γ and we also have $R_\gamma = H_{R\gamma}([I_\gamma, I_\gamma])$.

Therefore, by Lemma 2.4, $S_4(x_1, \dots, x_4)$ is a polinomial identity for R_γ, for each $\gamma \in \Gamma_2$, and so R satisfies $S_4(x_1, \dots, x_4)$. ∎

LEMMA 2.6. *Let R be a prime ring satisfying a polynomial identity. If $R = H_R([R, R])$ then R satisfies $S_4(x_1, \dots, x_4)$.*

PROOF. Since R is a P.I. ring, by Posner's theorem, the ring of central quotients of R, i.e. the ring $Q = \{rz^{-1} : r \in R, 0 \neq z \in Z(R)\}$ is a finite dimensional central simple algebra. Of course $Q = H_Q([Q, Q])$, hence by Lemma 2.4 Q must satisfy $S_4(x_1, \dots, x_4)$ and we are done. ∎

LEMMA 2.7. *Let R be a prime ring without non-zero nil right ideals. If $R = H_R([R, R])$ then R satisfies $S_4(x_1, \dots, x_4)$.*

PROOF. Let ϱ be a non-zero right ideal of R; we claim that if ϱ satisfies a polynomial identity then ϱ satisfies $S_4(x_1, \dots, x_4)\, x_5$.

In fact let $l(\varrho) = \{x \in R : x\varrho = 0\}$ the left annihilator of ϱ. Then the quotient ring $\bar{\varrho} = \varrho/(l(\varrho) \cap \varrho)$ is also a prime P.I. ring such that $H_{\bar{\varrho}}([\bar{\varrho}, \bar{\varrho}]) = \bar{\varrho}$. Hence, by Lemma 2.6, $S_4(x_1, \dots, x_4)$ is a polynomial identity of $\bar{\varrho}$, that is $S_4(r_1, \dots, r_4) \in l(\varrho)$, for all $r_i \in \varrho$, and so ϱ satisfies $S_4(x_1, \dots, x_4)\, x_5$, as required.

Now if R is a domain then our result follows by Lemma 2.5. Suppose that R is not a domain and let $ab = 0$ for some non-zero elements $a, b \in R$. Then $bRa \neq (0)$ and so there exists $r \in R$ such that $c = bra$ is a non-zero square-zero element of R. Let $\varrho = cR$, and, as above let $l(\varrho)$ its left annihilator. We consider the prime ring $\overline{\varrho} = \varrho/(l(\varrho) \cap \varrho)$ which is without non-zero nil right ideals.

Let $r_1, r_2 \in R$, since $c \in H_R([R, R])$ there exist positive integers n, m such that $0 = [c, [cr_1, cr_2]^n]_m$. Hence $[cr_1, cr_2]^{nm} c = 0$, since $c^2 = 0$. In other words the polynomial $[x_1, x_2]$ is nil on $\overline{\varrho}$, and so, by Theorem 1 of [3], $\overline{\varrho}$ is commutative, that is $[r_1, r_2]r_3 = 0$, for all $r_i \in \varrho$. Therefore R contains non-zero right ideals satisfying a polynomial identity; thus, by first part of the proof, they must satisfy $S_4(x_1, \dots, x_4)\, x_5$.

Hence, by Zorn's Lemma, there exists a non-zero right ideal ϱ' which is maximal with respect to the property that it satisfy $S_4(x_1, \dots, x_4) x_5$. Now let $r \in R$ and $s_1, s_2, s_3, s_4, s_5 \in \varrho'$, then, since each $s_i r \in \varrho'$, we have

$$S_4(rs_1, rs_2, rs_3, rs_4)\, rs_5 = rS_4(s_1 r, s_2 r, s_3 r, s_4 r)\, s_5 = 0\,.$$

This says that the right ideal $r\varrho'$ satisfies the identity $S_4(x_1, \dots, x_4)\, x_5$. Since both ϱ' and $r\varrho'$ satisfy a polynomial identity then, by a theorem of Rowen [11], $\varrho' + r\varrho'$ also satisfies some identity.

Therefore, as we showed above, $\varrho' + r\varrho'$ satisfies $S_4(x_1, \dots, x_4)\, x_5$. By the maximality of ϱ' we have $r\varrho' \subseteq \varrho'$, for all $r \in R$, that is ϱ' is a non-zero two-sided ideal of R. Hence, by Lemma 1 of [4], R is a P.I. ring and by Lemma 2.6 it satisfies $S_4(x_1, \dots, x_4)$. ■

3. – Some results on invariant subrings.

As we said in remark 3, $H_R(S)$ is a subring of R which is invariant under any automorphism φ of R such that $\varphi(S) \subseteq S$. This fact is enough to focus our attention on invariant subrings A of R. In this section we will consider the following situation:

R will be a prime ring with non-zero Jacobson's radical $J(R)$ and A will be a subring of R which is invariant under the automorphisms of R which are induced by all the elements of $J(R)$. More precisely, let a be a quasi-regular element of R with quasi-inverse a', that is $a + a' + aa' = a'a + a' + a = 0$.

Notice that if R has a unit element 1 then $1 + a$ is invertible and $(1 + a)^{-1} = 1 + a'$.

Let $\varphi_a : R \to R$ be the map defined by

$$\varphi_a(r) = r + ar + ra' + ara'\,.$$

φ_a is an automorphism of R, we write $\varphi_a(r) = (1 + a)\, r(1 + a)^{-1}$ and we say that a is formally invertible. As in the proof of Lemma 1.2 , we also write $r(1+a)$ for $r+ra$ and $(1+a)r$ for $r+ar$.

Some of the following results are implicitly contained in [6]. We include these statement in this form for the sake of clearness and completeness.

Let R, A be as described in the beginning of this section; we have

LEMMA 3.1. *Let I be a non-zero two-sided ideal of R then either $A \subseteq Z(R)$ or $A \cap I \neq (0)$.*

PROOF. Since R is a prime ring $V = I \cap J(R)$ is a non-zero two-sided ideal of R. Since the centralizer of a non-zero two-sided ideal in a prime ring is equal to the center of the ring then either $A \subseteq Z(R)$ or there exist $a \in A$, $r \in V$ such that $(1+r)a(1+r)^{-1} \neq a$, that is $a + ra + ar' + rar' \neq a$.

Since $a + ra + ar' + rar'$ is an element of A then $0 \neq ra + ar' + rar' \in A \cap I$. ■

LEMMA 3.2. *If A has no non-zero nilpotent elements then any non-zero element of A is regular in R and $Z(A) \subseteq Z(R)$.*

PROOF. See [6] page 423, rows 10-30. ■

We remark that the same argument used in the previous Lemma (of course «$x \in D - \{-1\}$» instead of «$x \in J(R)$») shows the following result

LEMMA. 3.3. *Let R be a division ring. If A is a subring of R which is invariant under all inner automorphism of R then $Z(A) \subseteq Z(R)$.*

In the next Lemma we will use the following definition:

Let R be a prime ring with non-zero Jacobson's radical $J(R)$, then we put

$\varrho_a = \{x \in J(R) : ax = 0\}$. Clearly ϱ_a is a right ideal of R which is the right annihilator of a in $J(R)$.

LEMMA 3.4. *If A does not contain a non-zero two-sided ideal of R, then the set $\{\varrho_a : a \in A\}$ is linearly ordered, that is: for all $a, b \in A$ either $\varrho_a \subseteq \varrho_b$ or $\varrho_b \subseteq \varrho_a$.*

PROOF. See [6] page 424, rows 10-24. ■

We conclude this section by proving the following result:

LEMMA 3.5. *Let A be a domain such that $Z(R) \subseteq A$; if A satisfies a polynomial identity then either $A = Z(R)$ or Q, the ring of central quotient of R, is a simple ring with* 1.

PROOF. Since A is a P.I. domain then by Posner's theorem its center $Z(A)$ is non-zero and any non-zero element of A is invertible in $Q(A) = \{az^{-1}: a \in A, 0 \neq z \in Z(A)\}$. Moreover, by Lemma 3.2, $Z(A) = Z(R)$, hence $Q(A)$ is a subdivision ring of Q, the ring of central quotients of R, and it has the same unit element of R. Assume now that $A \not\subseteq Z(R)$ and let V be a non-zero two-sided ideal of Q. Then $V \cap R$ is a non-zero two-sided ideal of R and so, by Lemma 3.1, $A \cap (V \cap R) \neq (0)$.

Therefore V contains an invertible element of A and so $V = Q$. ■

4. – The general case.

We begin with the case when R is a division ring.

LEMMA 4.1. *Let R be a division ring then either $H_R([R, R]) = Z(R)$ or R satisfies the standard identity $S_4(x_1, \dots, x_4)$.*

PROOF. Let $A = H_R([R, R])$, as we said above A is invariant under all the automorphisms of R. Hence $Z(A) \subseteq Z(R)$ by Lemma 3.3 and so $Z(A) = Z(R)$.

Since $H_A([A, A]) = A$, by Lemma 2.5, we obtain that A satisfies the standard identity $S_4(x_1, \dots, x_4)$. By Posner's theorem the ring $B = \{az^{-1}: a \in A, 0 \neq z \in Z(A)\}$ of central quotients of A is a finite dimensional central simple algebra which satisfies $S_4(x_1, \dots, x_4)$. Of course B is a subdivision ring of R, moreover it is invariant under all automorphism of R. Therefore, by Brauer-Cartan-Hua theorem, either $B = R$ or $B \subseteq Z(R)$.

In the latter case $H_R([R, R]) = A = B = Z(R)$, while in the first case R satisfies the standard identity $S_4(x_1, \dots, x_4)$. ■

LEMMA 4.2. *Let R be a prime ring with no non-zero nil right ideals and $J(R) \neq 0$. Let I be a non-zero two-sided ideal of R. If*

$H_R([I, R]) = A$ does not contain a non-zero two-sided ideal of R then $H_R([I, R]) = Z(R)$.

PROOF. Since A does not contain a non-zero two-sided ideal of R, then, by Lemma 3.4, for all $a, b \in A$ we must have either $\varrho_a \subseteq \varrho_b$ or $\varrho_b \subseteq \varrho_a$.

We claim that A does not contain non-zero nilpotent elements. Let $a \in A$ be such that $a^2 = 0$ and $a \neq 0$.

If a annihilates on the left every square-zero element of A then

$$a(1+x)\,a(1+x)^{-1} = 0 \qquad \text{for all } x \in J(R).$$

Hence, since $a^2 = 0$, we have $aJ(R)a = (0)$ and so $a = 0$, a contradiction.

Thus there exists $b \in A$ with $b^2 = 0$ and $ab \neq 0$. Then $0 \neq abJ(R) \subseteq \subseteq a\varrho_b$, so $\varrho_b \nsubseteq \varrho_a$. Hence $\varrho_a \subseteq \varrho_b$, in particular we have:

$$baJ(R) \subseteq b\varrho_a \subseteq b\varrho_b = (0) \qquad \text{and so } ba = 0\,.$$

Since $a \in H_R([I, R])$, for any $r \in I$, there exist positive integers n, m such that $0 = [a, [r, ab]^m]_n$. And so, since $a^2 = b^2 = ba = 0$, we obtain $0 = [a, [r, ab]^m]_n br = (-1)^n (abr)^{nm}$, that is abI is a nil right ideal of R. Hence $abI = (0)$ and so $ab = 0$, a contradiction again.

Therefore $A = H_R([I, R])$ does not contain non-zero nilpotent elements and, by Lemma 3.2, we obtain that any non-zero element of A is regular in R and $Z(A) = Z(R)$. In particular A is a domain, moreover if $A \nsubseteq Z(R)$ then $A \cap I \neq (0)$, by Lemma 3.1.

Therefore $A \cap I$ is a non-zero two-sided ideal of A and we also have $A = H_A([I \cap A, A])$. Hence, by Lemma 2.5, A satisfies $S_4(x_1, \dots, x_4)$. Since $A \nsubseteq Z(R)$, Q, the ring of central quotients of R, is a simple ring with 1 (see Lemma 3.5), and so it is a primitive ring.

Of course $A = H_R([I, R]) \subseteq H_Q([Q, Q])$. But, by Lemmas 2.3 and 4.1, either $H_Q([Q, Q]) = Z(Q)$ or Q satisfies $S_4(x_1, \dots, x_4)$.

In the first case we obtain $A \subseteq Z(R)$ which contradicts with our last assumption. In the last case Q is a simple algebra which is at most 4-dimensional over its center.

Therefore Q must satisfy all the polynomial identities of 2×2 matrices over its center (see [9]), hence $[x_1, [x_2, x_3]^2]$ is a polynomial identity for $R \subseteq Q$.

In other words $A = H_R([I, R]) = R$, that is A contains a non-zero two-sided ideal of R, and this is a contradiction again.

Hence $A \subseteq Z(R)$ and we are done. ■

A special case of our final result is the following

PROPOSITION 4.1. *Let R be a prime ring without non-zero nil right ideals. Let I be a non-zero two-sided ideal of R and let $A = H_R([I, R])$. Then either $A = Z(R)$ or R satisfies $S_4(x_1, \dots, x_4)$ and $A = R$.*

PROOF. Suppose R is semisimple. Then, as in the proof of Lemma 2.5, R is a subdirect product of primitive rings $R_\gamma = R/P_\gamma$, such that $I \not\subseteq P_\gamma$, for each γ in the set Γ of indeces.

For each $\gamma \in \Gamma$, let A_γ and I_γ be the images in R_γ of A and I respectively. Then, since $A_\gamma \subset H_{R_\gamma}([I_\gamma, R_\gamma])$, by Lemmas 2.3 and 4.1, either $A_\gamma \subseteq Z(R_\gamma)$ or R_γ satisfies $S_4(x_1, \dots, x_4)$.

Now, let $\Gamma_1 = \{\gamma \in \Gamma : A_\gamma \subseteq Z(R_\gamma)\}$ and $\Gamma_2 = \{\gamma \in \Gamma : A_\gamma \not\subseteq Z(R_\gamma)\}$.

Then $\Gamma = \Gamma_1 \cup \Gamma_2$. Let $I_1 = \cap P_\gamma$, $\gamma \in \Gamma_1$ and $I_2 = \cap P_\gamma$, $\gamma \in \Gamma_2$.

So $(0) = J(R) = I_1 \cap I_2$, moreover if $\gamma \in \Gamma_2$ then R_γ satisfies $S_4(x_1, \dots, x_4)$.

Since R is prime and $I_1 I_2 \subseteq I_1 \cap I_2 = (0)$ we must have either $I_1 = 0$ or $I_2 = 0$. If $I_1 = 0$ then we conclude that $A \subseteq Z(R)$. Hence if $A \not\subseteq Z(R)$ then $I_2 = (0)$ and consequentely R satisfies $S_4(x_1, \dots, x_4)$. In this case, by Posner's theorem, R is an order in a simple algebra at most 4-dimensional over its center. Hence R satisfies the polynomial identity $[[x_1, x_2]^2, x_3]$ and so $A = H_R([I, R]) = R$.

Therefore we must assume that $J(R) \neq 0$.

If A does not contain a non-zero ideal of R then, by Lemma 4.2, $H_R([I, R]) = Z(R)$.

Hence we may assume that A contains a non-zero ideal V of R. Since R is prime, $R_1 = V \cap I$ is a prime ring without non-zero nil right ideals, moreover, as $V \subseteq A$, we have that $I \cap V = H_{I \cap V}([I \cap V, I \cap V])$. By Lemma 2.7, $I \cap V$ satisfies $S_4(x_1, \dots, x_4)$, and so R too. As above this implies $H_R([I, R]) = R$ and we are done. ■

THOREM 4.1. *Let R be a prime ring without non-zero nil right ideals, U a noncentral Lie ideal of R. Then either $H_R(U) = Z(R)$ or R satisfies $S_4(x_1, \dots, x_4)$.*

PROOF. By Lemma 1.3 if R does not satisfy $S_4(x_1, \dots, x_4)$ then there exists a non-zero two-sided ideal I of R such that $[I, R] \subseteq U$. Since $H_R(U) \subseteq H_R([I, R])$, we conclude by previous proposition. ■

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