Rendiconti
del
Seminario Matematico
della
Università di Padova

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Rendiconti del Seminario Matematico della Università di Padova, tome 100 (1998), p. 187-209
<http://www.numdam.org/item?id=RSMUP_1998__100__187_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 100 (1998)

Some Conditions Implying that an Infinite Group is Abelian.

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1. - Introduction.

Following a question of P. Erdös, B. H. Neumann [13] considered groups satisfying the following condition:

A group G is an $\mathbb{A}^{\#}$-group if every infinite set of elements of G contains a 2-element subset $\{x, y\}$ such that $\langle x, y\rangle$ is abelian (or, equivalently $[x, y] = 1$).

Neumann showed that a group is an $\mathbb{A}^{\#}$-group if and only if it is central-by-finite. Corresponding results were obtained by Faber, Laver and McKenzie [2], who considered conditions in which sets of pairwise non-commuting elements had size bounded by infinite cardinals. These #-conditions have also been considered for $\langle x, y\rangle$ being in some other class of groups (see e.g. [1], [3], [9], [12], [14]).

If $\phi(x_1, \ldots, x_n)$ is some word in n variables then one could define G to be a $\mathbb{W}^{\#}$-group if every infinite set of elements in G contains a set of n elements $\{x_1, \ldots, x_n\}$ such that $\phi(x_{\varrho(1)}, \ldots, x_{\varrho(n)}) = 1$ for all permutations ϱ of $\{1, \ldots, n\}$. A related but stronger condition has been considered by Rhemtulla and others. A group is said to be a $\mathbb{W}^*$-group if, given infinite subsets $X_1, \ldots, X_n$ of G, there are elements $x_i \in X_i$, $i = 1, \ldots, n$, such that $\phi(x_{\varrho(1)}, \ldots, x_{\varrho(n)}) = 1$ for all permutations ϱ of $\{1, \ldots, n\}$. It was noted in [11] that if ϕ is the commutator word then every infinite

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$\mathbb{W}^*$-group is abelian, and corresponding results have been obtained for $\phi = x^n$, $\phi = [x_1, x_2, \ldots, x_n]$ in [11], for $\phi = [x, y, y]$ in [16], and for $\phi = [x, y]^2$ in [10]. A related condition in [8] for the word $\phi = [[x_1, x_2], [x_3, x_4]]$ yielded that all such infinite groups were metabelian. These results suggest that if the word ϕ defines a variety $\mathbb{W}$ then an infinite $\mathbb{W}^*$-group ought to be a $\mathbb{W}$-group.

In this paper we consider the conditions $\mathbb{W}^{\#}$ and $\mathbb{W}^*$ for various words ϕ which determine the variety of abelian groups and also consider #- and $*$-conditions for classes related to these words. In [6] we considered the 32 words of the form

$$\phi_n(x, y) = [u, v, w][y, x]$$

with u, v, w being taken from the set $\{x, y, x^{-1}, y^{-1}\}$. It was shown in particular that a group G where ϕ_n is a law is abelian, for any $n = 1, 2, \ldots, 32$ [6; Theorem 5.1]. Some partial results in this direction were already obtained earlier in [4]. For reference we list the 32 possibilities for $[u, v, w]$ as follows:

1) $[x, y, y]$,
2) $[x, y, y^{-1}]$,
3) $[x, y^{-1}, y]$,
4) $[x, y^{-1}, y^{-1}]$,
5) $[x, y, x]$,
6) $[x, y, x^{-1}]$,
7) $[x, y^{-1}, x]$,
8) $[x, y^{-1}, x^{-1}]$,
9) $[x^{-1}, y, y]$,
10) $[x^{-1}, y, y^{-1}]$,
11) $[x^{-1}, y^{-1}, y]$,
12) $[x^{-1}, y^{-1}, y^{-1}]$,
13) $[x^{-1}, y, x]$,
14) $[x^{-1}, y, x^{-1}]$,
15) $[x^{-1}, y^{-1}, x]$,
16) $[x^{-1}, y^{-1}, x^{-1}]$,
17) $[y, x, x]$,
18) $[y, x, x^{-1}]$,
19) $[y, x^{-1}, x]$,
20) $[y, x^{-1}, x^{-1}]$,
21) $[y, x, y]$,
22) $[y, x, y^{-1}]$,
23) $[y, x^{-1}, y]$,
24) $[y, x^{-1}, y^{-1}]$,
25) $[y^{-1}, x, x]$,
26) $[y^{-1}, x, x^{-1}]$,
27) $[y^{-1}, x^{-1}, x]$,
28) $[y^{-1}, x^{-1}, x^{-1}]$,
29) $[y^{-1}, x, y]$,
30) $[y^{-1}, x, y^{-1}]$,
31) $[y^{-1}, x^{-1}, y]$,
32) $[y^{-1}, x^{-1}, y^{-1}]$.

In [6] we considered the following 2 conditions for group elements x, y.

DEFINITION 1.1. *Let* $n \in \{1, \ldots, 32\}$. *The group elements* x, y *satisfy*

(i) $\mathbb{C}n$ *if and only if* $\phi_n(x, y) = 1$,

(ii) $\mathbb{D}n$ *if and only if* $\phi_n(x^\delta, y^\varepsilon) = \phi_n(y^\varepsilon, x^\delta) = 1$, *for all* ε, $\delta = \pm 1$.

It should be noted that the condition $\mathbb{C}n$ on x, y does not imply $[x, y] = 1$ in most of the cases, and even the stronger condition $\mathbb{D}n$ does not imply necessarily $[x, y] = 1$ in some of the cases. A summary of these results from [6] is given in the next section of this paper.

The #- and $*$-conditions for $\mathbb{C}n$ can now be defined as in the general case given above and we can also define similar conditions for $\mathbb{D}n$.

DEFINITION 1.2. *Let $n \in \{1, \dots, 32\}$ and let G be a group.*

(i) *$G \in \mathbb{C}n^{\#}$ if and only if every infinite set of elements in G contains x, y such that $\phi_n(x, y) = \phi_n(y, x) = 1$.*

(ii) *$G \in \mathbb{C}n^{*}$ if and only if, for any two infinite subsets X, Y of G there is $x \in X$ and $y \in Y$ such that $\phi_n(x, y) = \phi_n(y, x) = 1$.*

(iii) *$G \in \mathbb{D}n^{\#}$ if and only if every infinite set of elements in G contains x, y satisfying $\mathbb{D}n$.*

(iv) *$G \in \mathbb{D}n^{*}$ if and only if, for any two infinite subsets X, Y of G there is $x \in X$ and $y \in Y$ such that x, y satisfy $\mathbb{D}n$.*

In all of these definitions the condition on x, y is symmetric in x and y. Longobardi, Maj and Rhemtulla imposed similar symmetry conditions in their investigations in [11]. The main reason for us is our use of Ramsey's Theorem in Sections 4 and 6. Some results can still be obtained if we omit the symmetry restriction from the definition of $\mathbb{C}n^{*}$; in particular, the cases dealt with in Proposition 2.1 and Section 3 do not require symmetric conditions. However, for simplicity of notation we do not pursue the question of how far symmetry is necessary in these investigations and simply give results for the classes defined above.

For the conditions $\mathbb{C}n^{\#}$ and $\mathbb{D}n^{\#}$ one might expect such groups to be central-by-finite. This is clearly the case whenever $\mathbb{C}n$ or $\mathbb{D}n$, respectively, imply $[x, y] = 1$ (see Proposition 2.1 and 2.2), but it is not so in the remaining cases as shown in Examples 2.6 and 2.7.

For this reason we focus our investigations on $*$-conditions. If further conditions like solvable-by-finite or an infinite FC-center are imposed, then $\mathbb{C}n^{*}$-groups are abelian for all n (Theorem 3.6 and 3.7). These lead to a reduction theorem for the general case (Theorem 3.8), enabling us to show in Theorem 4.10 that an infinite $\mathbb{C}14^{*}$-group is abelian. As a corollary we obtain that for all those n which imply $\mathbb{D}14$, an infinite $\mathbb{D}n^{*}$-group is abelian (Corollary 4.11). For the remaining cases, namely $n = 4$ and 30, the corresponding results are obtained in Theorem

5.5. All this can be summarized in the general result that an infinite $\mathbb{D}n^*$-group is abelian for $n \in \{1, \dots, 32\}$ (Corollary 5.6).

We have not obtained a result for $\mathbb{C}n^*$-groups for all values of n. Although $\mathbb{D}14$ seemed to be the weakest of the $\mathbb{D}n$-conditions [6; Proposition 4.8] we do not have results corresponding to Lemmas 4.3 and 4.5 for all values of n. Similar results can be obtained in some cases. For example, we have such results for $n = 1, 20, 23$, and 30, and so have been able to construct similar proofs that $\mathbb{C}n^*$-groups are abelian for those values of n. The proof for $\mathbb{C}1^*$-groups is given in Section 6. There are, of course, fewer implications between the (symmetrized) $\mathbb{C}n$-classes than between the classes $\mathbb{D}n$ and so a result like Corollary 5.6 for $\mathbb{C}n^*$ would require most of the 32 cases to be dealt with individually.

2. – Elementary results and two counterexamples.

In this section we summarize results obtained in [6] for the conditions $\mathbb{C}n$ and $\mathbb{D}n$, $n = 1, \dots, 32$. We show that for those values of n for which $\mathbb{C}n$ or $\mathbb{D}n$ imply $[x, y] = 1$, we obtain results similar to those of B. H. Neumann, namely groups satisfy $\mathbb{C}n^{\#}$ or $\mathbb{D}n^{\#}$ if and only if they are central-by-finite, and $\mathbb{C}n^*$ or $\mathbb{D}n^*$ if and only if they are abelian. We conclude this section with two counterexamples of groups which satisfy $\mathbb{D}n^{\#}$ but are not central-by-finite, and n is a value for which $\mathbb{D}n$ does not imply $[x, y] = 1$. Our first result does not require the symmetrized #- and $*$-conditions.

PROPOSITION 2.1. *Let $S = \{17, 18, 19, 21, 22, 29\}$. If $n \in S$ and the ordered pair (x, y) satisfies $\mathbb{C}n$, then $[x, y] = 1$. Furthermore, any infinite $\mathbb{C}n^{\#}$-group is central-by-finite and any infinite $\mathbb{C}n^*$-group is abelian.*

PROOF. By Proposition 2.1 in [6] it follows that $[x, y] = 1$ if the ordered pair (x, y) satisfies $\mathbb{C}n$, for $n \in S$.

Now let G be an infinite $\mathbb{C}n^{\#}$-group, $n \in S$, and X an infinite subset of G. Then there exist $x, y \in X$ such that $\phi_n(x, y) = 1$. Hence, by the above, $[x, y] = 1$. Thus, by Neumann's result, G is central-by-finite.

Finally, let G be an infinite $\mathbb{C}n^*$-group, $n \in S$, and X, Y infinite subsets of G. Then there are $x \in X$ and $y \in Y$ such that $\phi_n(x, y) = 1$. By the above we have $[x, y] = 1$. Thus G is an $\mathbb{A}^*$-group, and hence abelian by a result from [11]. ■

PROPOSITION 2.2. *Let* $S = \{23, \ldots, 28, 31, 32\}$. *If* $n \in S$ *and the elements* x, y *satisfy* $\mathbb{D}n$, *then* $[x, y] = 1$, *but* $\mathbb{C}n$ *alone does not imply* $[x, y] = 1$. *Furthermore, any infinite* $\mathbb{D}n^{\#}$*-group is central-by-finite, and any infinite* $\mathbb{D}n^{*}$*-group is abelian.*

PROOF. It follows by Propositions 2.2, 2.3, 2.4 and 2.5 of [6], respectively, that if x, y satisfy $\mathbb{D}n$, $n \in S$, then $[x, y] = 1$, but $\mathbb{C}n$ alone does not imply $[x, y] = 1$. The rest of the results therefore follow as in Proposition 2.1. ■

Propositions 3.1 to 3.4 in [6] can be summarized in the following proposition.

PROPOSITION 2.3. *Let* $S = \{1, 2, 3, 5, 6, 7, 10, 12, 13, 15, 16\}$. *For* $n \in S$, *condition* $\mathbb{D}n$ *is equivalent to* $\mathbb{D}1$. *Furthermore,* $\mathbb{D}1$ *does not imply* $[x, y] = 1$.

The next proposition is a consequence of Proposition 3.5 in [6].

PROPOSITION 2.4. Conditions $\mathbb{D}8$, $\mathbb{D}9$ and $\mathbb{D}11$ are equivalent and they do not imply $[x, y] = 1$.

The last proposition of this section summarizes results from [6] which can be found there in Corollaries 3.6, 4.2, 4.4, 4.6, 4.7 and 4.9.

PROPOSITION 2.5. *Between the nonabelian* $\mathbb{D}$*-conditions* $\mathbb{D}1$, $\mathbb{D}4$, $\mathbb{D}8$, $\mathbb{D}14$, $\mathbb{D}20$ *and* $\mathbb{D}30$ *we have exactly the following implications*:

$$
\begin{tikzcd}
& & & & \mathbb{D}14 & \\
& \mathbb{D}4 & & \mathbb{D}8 \arrow[ur] & & \mathbb{D}20 . \arrow[ul] \\
\mathbb{D}30 \arrow[ur] & & \mathbb{D}1 \arrow[ul] \arrow[ur] & & &
\end{tikzcd}
$$

Our next two examples show that a $\mathbb{D}n^{\#}$-group need not be central-by-finite if $\mathbb{D}n$ is a condition which does not imply $[x, y] = 1$, i.e. $n = 1, 4, 8, 14, 20, 30$.

EXAMPLE 2.6. *There exists an infinite* $\mathbb{D}n^{\#}$*-group* G, $n = 1, 4, 8, 14$, *which is not central-by-finite.*

PROOF. Let A be an elementary abelian 3-group of infinite rank and $G = A \rtimes \langle g \rangle$, the split extension of A by $\langle g \rangle$, where $\langle g \rangle$ is cyclic of order 2 such that $gag = a^{-1}$ for all $a \in A$. Obviously $Z(G) = 1$, hence G is not central-by-finite. By Proposition 2.5, $\mathbb{D}1$ implies $\mathbb{D}4$, $\mathbb{D}8$, and $\mathbb{D}14$. Thus it suffices to show that G is a $\mathbb{D}1^{\#}$-group. Let X be an infinite set in G. Since $G = A \cup Ag$, then either $A \cap X$ or $Ag \cap X$ is infinite. If $A \cap X$ is infinite, then clearly there are $x, y \in X$ satisfying $\mathbb{D}1$ since A is abelian. If $Ag \cap X$ is infinite, it suffices to show that $\phi_1(bg, cg) = 1$ for all $b, c \in A$. Now $\phi_1(bg, cg) = [[bg, cg], cg][cg, bg] = [b^{-1}c, cg] \cdot bc^{-1} = b^3c^{-3} = 1$, proving our claim. Hence G is a $\mathbb{D}1^{\#}$-group. ■

To facilitate the calculations in our next example and in some of the upcoming lemmas, we write $a = [x, y]$ and consider the near-ring $\mathcal{N}_a$ of mappings of a into G' generated by the elements of G acting by conjugation on a. Frequently we will suppress the a and write $f(x, y) = 0$ instead of $a^{f(x, y)} = 1$. For further details see [6] and [15; 1.5].

EXAMPLE 2.7. *There exists an infinite* $\mathbb{D}n^{\#}$*-group* H, $n = 20, 30$, *which is not central-by-finite.*

PROOF. Let $A = Dr(\langle u_i \rangle \times \langle v_i \rangle)$ be a direct product of infinitely many four-groups so that A is an elementary abelian 2-group. Let $H = A \rtimes \langle h \rangle$ be the split extension of A by $\langle h \rangle$, where $\langle h \rangle$ is a cyclic group of order 3 such that $h^{-1}u_i h = v_i$ and $h^{-1}v_i h = u_i v_i$. Obviously $Z(H) = 1$, hence H is not central-by-finite. Let X be an infinite set in H. Since $H = A \cup Ah \cup Ah^{-1}$, then at least one of $A \cap X, Ah \cap X, Ah^{-1} \cap X$ is infinite. If $A \cap X$ is infinite, then clearly there are $x, y \in X$ satisfying $\mathbb{D}20$ or $\mathbb{D}30$, respectively, since A is abelian.

Now suppose $Ah \cap X$ is infinite. It suffices to show that uh, vh for all $u, v \in A$ satisfy $\mathbb{D}n$, $n \in \{20, 30\}$. Set $x = uh$, $y = vh$. Then $y = wx$ with $w = vu^{-1}$ and $a = [x, y] = [x, w]^x = w^{x^2 + x}$, since w has order 2. Furthermore, since x has order 3, $a^x = w^{1 + x^2}$ and $a^{x^2} = w^{x+1}$. Similarly, $a^y = w^{1 + x^2}$ and $a^{y^2} = w^{x+1}$. Observing that H is metabelian and H' has exponent 2, the 8 conditions for $\mathbb{D}20$, namely (4.1.1)-(4.1.4) and (4.1.1a)-(4.1.4a) of Proposition 4.1, and the 4 conditions for $\mathbb{D}30$, namely (4.3.1)-(4.3.4) of Proposition 4.3 in [6], can be verified in a straightforward but lengthy manner. As an example we give here the verification of (4.1.1) which can be stated as $x + x^2 = 1$ in $\mathcal{N}_a$. By the above we have $a^{x+x^2} = w^{1+x^2} \cdot w^{x+1} = w^{x^2+x} = a$. Thus (4.1.1) holds. It follows that all $x, y \in Ah$ satisfy $\mathbb{D}n$, $n \in \{20, 30\}$. A similar argument shows that all $x, y \in Ah^{-1}$ satisfy $\mathbb{D}n$, $n \in \{20, 30\}$. It follows that H is a $\mathbb{D}20^{\#}$-group as well as a $\mathbb{D}30^{\#}$-group. ■

3. – A reduction theorem.

In this section we show that an infinite $\mathbb{C}n^*$-group is abelian if further conditions are imposed on the structure of the group. In particular, a $\mathbb{C}n^*$-group, $n \in \{1, 2, \ldots, 32\}$, which is solvable-by-finite or has an infinite FC-center is abelian (Theorem 3.6 and 3.7). These results are proved in the following sequence of lemmas which also provide us with a reduction theorem for the general case (Theorem 3.8). As we will see in the proofs these results do not require the symmetrized $*$-condition.

The proof of the first lemma is straightforward and thus omitted here.

LEMMA 3.1. *Any infinite section of a $\mathbb{C}n^*$-group is also a $\mathbb{C}n^*$-group.*

LEMMA 3.2. *If N is an infinite normal subgroup of the $\mathbb{C}n^*$-group G, then G/N is abelian.*

PROOF. Let $x, y \in G$. Consider the infinite sets xN and yN. Since G is a $\mathbb{C}n^*$-group, there exist elements $n_1, n_2 \in N$ such that $\phi_n(n_1 x, n_2 y) = 1$. Hence $\phi_n(Nx, Ny) = 1$ in G/N. Thus ϕ_n is a law in G/N, and so G/N is abelian by Theorem 5.1 in [6]. ■

LEMMA 3.3. *An infinite FC-group in the class $\mathbb{C}n^*$ is abelian.*

PROOF. Let $x, y \in G$ and let $C = C_G(x) \cap C_G(y)$. Then $[G: C] < \infty$, so that C is an infinite FC-group and thus by Lemma 8.4 in [17] it contains an infinite abelian subgroup A. Now consider the infinite sets Ax and Ay. Since C is a $\mathbb{C}n^*$-group, there are elements $a, b \in A$ such that $\phi_n(ax, by) = 1$. But $[ax, by] = [x, y]$. If ϕ_n is the word $[u, v, w][y, x]$

with $[u, v, w]$ a word in x, y, and if $[u', v', w']$ is the corresponding word in ax, by, then $[u', v', w'] = [u, v, w]$. Hence $\phi_n(x, y) = 1$, and so ϕ_n is a law in G. By Theorem 5.1 of [6] it follows that G is abelian. ■

LEMMA 3.4. *If G is a $\mathbb{C}n^*$-group with infinite center Z then G is abelian.*

PROOF. By Lemma 3.2, G/Z is abelian and so G is nilpotent of class at most two. Let $x, y \in G$ and consider the infinite sets Zx and Zy. Since G is a $\mathbb{C}n^*$-group, there are elements $z_1, z_2 \in Z$ such that $\phi_n(z_1 x, z_2 y) = 1$. Hence $[x, y] = [z_1 x, z_2 y] = [u, v, w]$. But $[u, v, w] = 1$ since G has class two and so $[x, y] = 1$ for all $x, y \in G$. ■

LEMMA 3.5. *Let A be an infinite abelian normal subgroup of the $\mathbb{C}n^*$-group G. Then $A \leqslant Z(G)$, and hence G is abelian.*

PROOF. Suppose $g \notin C_G(A)$ so that $C_A(g) \neq A$ and $A \setminus C_A(g)$ is an infinite set. Let ϕ_n be the word $[u, v, w][y, x]$. If $w = x$ or x^{-1}, let $X = A \setminus C_A(g)$ and $Y = Ag$; if $w = y$ or y^{-1}, let $X = Ag$ and $Y = A \setminus C_A(g)$. Then, for any $x \in X$ and $y \in Y$, we have $[x, y] \neq 1$. But $[u, v] \in A$ (since one of u, v is in $A \setminus C_A(g)$) and $w \in A$ so that $[u, v, w] = 1$. Therefore, for any $x \in X$ and $y \in Y$, $\phi_n(x, y) \neq 1$, contrary to $G \in \mathbb{C}n^*$. ■

THEOREM 3.6. *Let G be an infinite solvable-by-finite $\mathbb{C}n^*$-group, $n \in \{1, \ldots, 32\}$. Then G is abelian.*

PROOF. There is an infinite solvable normal subgroup S in G. We prove the theorem by induction on the derived length d of S.

Let $A = S^{(d-1)}$. If A is infinite then it follows from Lemma 3.5 that G is abelian. If A is finite, then, by induction, G/A is abelian. Thus G is finite-by-abelian and so is an FC-group and the result follows from Lemma 3.3. ■

THEOREM 3.7. *Let G be an infinite $\mathbb{C}n^*$-group, $n \in \{1, 2, \ldots, 32\}$. If $FC(G)$ is infinite, then G is abelian.*

PROOF. By Lemma 3.3, $FC(G)$ is abelian and so, by Lemma 3.5, G is abelian. ■

For the general situation in which we consider infinite $\mathbb{C}n^*$-groups without further restrictions these results are most usefully combined in the following reduction theorem.

THEOREM 3.8. *Suppose that there exists an infinite nonabelian $\mathbb{C}n^*$-group. Then there exists an infinite nonabelian $\mathbb{C}n^*$-group G with $FC(G) = 1$ and G having no abelian normal subgroup.*

PROOF. Let H be an infinite nonabelian $\mathbb{C}n^*$-group. By Theorem 3.7, $FC(H)$ is finite. Hence $G = H/FC(H)$ is an infinite $\mathbb{C}n^*$-group. Also G is nonabelian, for otherwise H would be finite-by-abelian and hence $FC(H) = H$. Also $FC(G) = 1$, since $FC(H)$ is finite. If A is an abelian normal subgroup of G then, by Lemma 3.5, A must be finite, but then $A \leq FC(G) = 1$. ■

4. – The class of $\mathbb{C}14^*$-groups.

In this section we will show that an infinite $\mathbb{C}14^*$-group is abelian (Theorem 4.10). As corollaries we obtain that infinite $\mathbb{D}14^*$-groups, and with it all $\mathbb{D}n^*$-groups for those values of n, where $\mathbb{D}n$ implies $\mathbb{D}14$, are abelian. Our results in the remaining sections will require Ramsey's Theorem which we will quote as Theorem 4.1 in the form used here. (see [17; Theorem 7.1]).

THEOREM 4.1. *Let S be an infinite set and suppose that the family $[S]^2$ of 2-element subsets of S is expressed as a union of n subfamilies $[S]^2 = \Delta_1 \cup \ldots \cup \Delta_n$, where n is finite. Then there is an infinite subset T of S and an integer k, $1 \leq k \leq n$, such that $[T]^2 \subseteq \Delta_k$.*

We also require a version of Ramsey's Theorem for ordered pairs of elements of S. Again we state this only in the form required here.

COROLLARY 4.2. *Let S be an infinite set and let $P(x, y)$ be some statement about the ordered pair (x, y) such that whenever X and Y are infinite subsets of S, there are elements $x \in X$ and $y \in Y$ such that $P(x, y)$ is true. Then S contains an infinite subset $\{x_1, x_2, \ldots\}$ such* *that one of the following is true:*

(i) $P(x_i, x_j)$ *is true whenever* $i<j$;

(ii) $P(x_j, x_i)$ *is true, whenever* $i<j$.

PROOF. Take an infinite countable subset $C=\{s_1, s_2, \ldots\}$ of S. Define the three subsets $\Delta_1, \Delta_2, \Delta_3$ of $[C]^2$ as follows:

$\{s_i, s_j\}\in\Delta_1$ if $i<j$ and $P(s_i, s_j)$ is true,

$\{s_i, s_j\}\in\Delta_2$ if $i<j$ and $P(s_j, s_i)$ is true,

$\{s_i, s_j\}\in\Delta_3$ if $P(s_i, s_j)$ and $P(s_j, s_i)$ are both false.

By Theorem 4.1, there is an infinite subset $T=\{x_1, x_2, \ldots\}=$ $=\{s_{i_1}, s_{i_2}, \ldots\}\subseteq C$ with $[T]^2\subseteq\Delta_k$ for some $k=1, 2$ or 3. Considering $X=$ $=\{x_1, x_3, x_5, \ldots\}$ and $Y=\{x_2, x_4, x_6, \ldots\}$ we see that there are i, j with $P(x_i, x_j)$ true and so $\{x_i, x_j\}\notin\Delta_3$. So $[T]^2\subseteq\Delta_1$ or $[T]^2\subseteq\Delta_2$. If $[T]^2\subseteq\Delta_1$, then $\{x_i, x_j\}\in\Delta_1$ for all i, j and so $P(x_i, x_j)$ is true, whenever $i<j$. If $[T]^2\subseteq\Delta_2$, then $P(x_j, x_i)$ is true, whenever $i<j$. ■

One application of this result will use the condition C14 applied to the pair (x^2, y) simultaneously with the pair (x, y). We begin by considering the implications of this pair of conditions.

LEMMA 4.3. *Let* $x, y\in G$ *satisfy the following two conditions*

$$[x^{-1}, y, x^{-1}]=[x, y], \tag{4.3.1}$$

$$[x^{-2}, y, x^{-2}]=[x^2, y]. \tag{4.3.2}$$

Then

(i) $[x, y]^{12}=1$,

(ii) $[x^6, y]=1$,

(iii) *if* $[x^3, y]=1$, *then* $[x, y]^4=1$.

PROOF. Write $a=[x, y]$ and consider the near ring $\mathcal{N}_a$. First we show (i). In $\mathcal{N}_a$ (4.3.1) can be written as

$$x=x^2+1\,. \tag{4.3.3}$$

Similarly, (4.3.2) becomes $x+1=x^{-1}+x^{-2}-x^{-4}-x^{-3}$, and, multiplying by x^4 from the right, we obtain

$$x^5+x^4+x+1=x^3+x^2 . \tag{4.3.4}$$

We observe that (4.3.3) yields the following reduction formulas for powers of x:

$$x^n=x^{n-1}-x^{n-2}, \qquad n=2,3,\dots . \tag{4.3.5}$$

Repeated applications of (4.3.5) and collection of terms lead to $x^5+x^4=$ $=x^3-2x^2$. Substituting into (4.3.4) yields $-2x^2+x+1=x^2$. By using (4.3.3) and collecting terms we arrive at

$$2=2x^2 , \tag{4.3.6}$$

and subsequently

$$x-1+x=3 . \tag{4.3.7}$$

By (4.3.5) and (4.3.7) we obtain $x^3=3-2x$. Therefore $x^4=3x-2x^2=$ $=3x-2$. But also $x^4=x^2 \cdot x^2=x^3-x^2$. By the above and using (4.3.7), this leads to $x^4=x-4$. Hence $3x-2=x-4$ and so

$$2x=-2 , \tag{4.3.8}$$

and subsequently $x^3=3-2x=5$. Using (4.3.5) and the above leads to $x^5=x-9$, but also $x^5=x^4 \cdot x=x^2-4x=x+3$ so that $x+3=x-9$, giving $12=0$ in $\mathcal{N}_a$. That is $[x,y]^{12}=1$ in G.

Next we turn to (ii). The usual commutator expansion gives $[x^6,y]=$ $=a^{x^5+x^4+x^3+x^2+x+1}$. In $\mathcal{N}_a$ we obtain $\sum_{n=0}^{5} x^{5-n}=x+3+x-4+5+x-$ $-1+x+1$ by replacing x^n, $n=2,\dots,5$, by their linear expressions obtained in the proof of (i). Repeated use of (4.3.7) and (4.3.8) leads to $\sum_{n=0}^{5} x^{5-n}=0$ in $\mathcal{N}_a$. That is $[x^6,y]=1$ in G.

Finally we prove (iii). If $[x^3,y]=1$, then $x^2+x+1=0$ in $\mathcal{N}_a$. Combining this with (4.3.3) gives $2x^2+2=0$. Using (4.3.6) yields $4=0$ in $\mathcal{N}_a$. That is $[x,y]^4=1$ in G. ∎

The results given in part (i) and (ii) of Lemma 4.3 are best possible as the following example shows.

EXAMPLE 4.4. *There exists a group* $G = \langle x, y \rangle$ *of order* 72 *satisfying* (4.3.1) *and* (4.3.2) *with* x *having order* 6 *and* $[x, y]$ *having order* 12.

PROOF. Let $H = Q \times C_3$, where Q is the quaternion group $Q = \langle a, b \mid a^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1} \rangle$ and $C_3 = \langle c \mid c^3 = 1 \rangle$. Let G be the split extension of H by a cyclic group $\langle x \rangle$ of order 6, where x induces the following automorphism of order 6 on H:

$$a^x = b, \qquad b^x = ab, \qquad c^x = c^{-1}.$$

In G, let $y = x(b^{-1}, c^{-1})$. Then $[x, y] = (a, c)$ and $[x^2, y] = (ab^{-1}, 1)$. Also $[x^{-1}, y] = (ab^{-1}, c)$ so that $[x^{-1}, y, x^{-1}] = (a, c) = [x, y]$, and $[x^{-2}, y] = (a, 1)$ so that $[x^{-2}, y, x^{-2}] = (ab^{-1}, 1) = [x^2, y]$. Thus (x, y) satisfies (4.3.1) and (4.3.2) with x having order 6 and $[x, y]$ having order 12. ■

LEMMA 4.5. *Let* G *be a* C14**-group and let* A *be an abelian subgroup of* G. *Then either* $A^6 = \{a^6 \mid a \in A\}$ *is finite or* $A^6 \leqslant Z(G)$.

PROOF. Suppose A^6 is infinite; then certainly $A^2 = \{a^2 \mid a \in A\}$ is infinite. Let $g \in G$; we show that $A^6 \leqslant C_G(g)$. For each element $x \in A^2$, choose an element $b \in A$ such that $b^2 = x$. This gives an infinite subset B of A such that the elements b^2, $b \in B$, are distinct. Let S be an infinite subset of B. Consider the infinite sets $X = S$ and $Y = Sg = \{sg \mid s \in S\}$. The C14*-condition shows that there are elements $b, c \in S$ such that $[b^{-1}, cg, b^{-1}] = [b, cg]$. But A abelian implies

(4.5.1) $$[b^{-1}, g, b^{-1}] = [b, g].$$

Since every infinite subset S of B contains an element b satisfying (4.5.1), it follows that if $C = \{b \in B \mid [b^{-1}, g, b^{-1}] = [b, g]\}$ then $B \backslash C$ is finite. Now let T be any infinite subset of $C^2 = \{b^2 \mid b \in C\}$. By considering the infinite sets T and Tg we see that there are elements $d^2, e^2 \in T$ such that $[d^{-2}, e^2 g, d^{-2}] = [d^2, e^2 g]$, and hence

(4.5.1) $$[d^{-2}, g, d^{-2}] = [d^2, g].$$

It follows that if $D = \{b \in B \mid [b^{-1}, g, b^{-1}] = [b, g] \text{ and } [b^{-2}, g, b^{-2}] = [b^2, g]\}$ then $B \backslash D$ is finite. For each $d \in D$, we have $[d^6, g] = 1$, by Lemma 4.3, and since $A^6 = B^6 = \{b^6 \mid b \in B\}$ and $B \backslash D$ is finite, it follows that $A^6 \backslash A^6 \cap C_G(g)$ is finite. Now A^6 infinite implies that $A^6 \cap C_G(g) = A^6$ and so $A^6 \leqslant C_G(g)$, for any $g \in G$. ■

COROLLARY 4.6. *Suppose that G is an infinite nonabelian* C14*-*group. If A is an abelian subgroup of G, then A^6 is finite.*

PROOF. If A^6 were infinite, then it would be central contrary to Lemma 3.4. ■

COROLLARY 4.7. *Suppose that G is an infinite nonabelian* C14*-*group. Then G is periodic and has no infinite abelian $\{2, 3\}'$-subgroup.*

PROOF. If G contained an element g of infinite order then $\langle g \rangle^6$ would be infinite, contrary to Corollary 4.6. If G contained an infinite abelian $\{2, 3\}'$-subgroup A then $A^6 = A$ would be infinite, again contradicting Corollary 4.6. ■

The next stage in our proof is to reduce to the consideration of a $\{2, 3\}$-group.

LEMMA 4.8. *Suppose that G is an infinite nonabelian* C14*-*group with $FC(G) = 1$. Then G is a $\{2, 3\}$-group.*

PROOF. If G is not a $\{2, 3\}$-group then it contains an element u of prime order $p > 3$ and u has infinitely many conjugates. It follows from the C14*-condition and Theorem 4.1 that this set of conjugates contains an infinite subset S such that $[x^{-1}, y, x^{-1}] = [x, y]$, for all $x, y \in S$. This is the first occasion where we have required the symmetry in x, y in the definitions of Cn*. Let X and Y be two infinite subsets of S. Since the elements have order $p \neq 2$, the squares of the elements are distinct. So $X^2 = \{x^2 \mid x \in X\}$ is infinite and applying the C14*-condition to X^2 and Y gives elements $x \in X$ and $y \in Y$ such that

$$[x^{-2}, y, x^{-2}] = [x^2, y]. \tag{4.8.1}$$

Therefore (4.8.1) satisfies the hypothesis on $P(x, y)$ in Corollary 4.2 and so S contains an infinite set of elements $\{x_1, x_2, \ldots\}$ such that either (1) $[x_i^{-2}, x_j, x_i^{-2}] = [x_i^2, x_j]$, whenever $i < j$, or (2) $[x_i^{-2}, x_j, x_i^{-2}] = [x_i^2, x_j]$, whenever $i > j$. Lemma 4.3 shows that either (1) $[x_i^6, x_j] = 1$, whenever $i < j$, or (2) $[x_i^6, x_j] = 1$, whenever $i > j$. Since x_i has prime order $p > 3$, it follows that $[x_i, x_j] = 1$ for all i, j and so $\langle x_1, x_2, \ldots \rangle$ is an infinite elementary abelian p-group, contrary to Corollary 4.7. Therefore G is a $\{2, 3\}$-group. ■

The main step which remains is to show that G is locally finite, which we achieve by showing that G has exponent 6.

LEMMA 4.9. *Suppose that G is an infinite nonabelian* $\mathbb{C}14^*$*-group with* $FC(G) = 1$. *Then G has exponent* 6.

PROOF. By Lemma 4.8, G is a $\{2, 3\}$-group and so we have only to show that G has no elements of order 4 or 9.

Suppose first that G has an element u of order 4. Then u^2 has order 2 and u^2 has infinitely many conjugates. Therefore there is an infinite set C of conjugates of u whose squares are distinct. If C contained an infinite commuting subset, then the elements of this set would generate an abelian subgroup A of exponent 4 with $A^6 = A^2$ being infinite. This is contrary to Corollary 4.6. By Theorem 4.1, C must contain an infinite subset S of pairwise noncommuting elements. The $\mathbb{C}14^*$-condition and Theorem 4.1 show that S contains an infinite subset T such that $[x^{-1}, y, x^{-1}] = [x, y] \neq 1$, for all $x, y \in T$. If X and Y are infinite subsets of T then the set $X^2 = \{x^2 \mid x \in X\}$ is also infinite and by the $\mathbb{C}14^*$-condition there are elements $x \in X$ and $y \in Y$ such that

$$(4.9.1)\ [x^{-2}, y, x^{-2}] = [x^2, y] \quad \text{and} \quad [y^{-1}, x^2, y^{-1}] = [y, x^2].$$

The equations of (4.9.1) therefore satisfy the hypothesis on $P(x, y)$ in Corollary 4.2 and so T contains an infinite subset $\{x_1, x_2, \dots\}$ such that either

(1) $[x_i^{-2}, x_j, x_i^{-2}] = [x_i^2, x_j]$, whenever $i < j$, or

(2) $[x_i^{-2}, x_j, x_i^{-2}] = [x_i^2, x_j]$, whenever $i > j$.

Lemma 4.3 together with the fact that each x_i has order 4 shows that either (1) $[x_i^2, x_j] = 1$, whenever $i < j$, or (2) $[x_i^2, x_j] = 1$, whenever $i > j$. In both cases $\langle x_1^2, x_2^2, \dots\rangle$ is an infinite elementary abelian 2-group.

In case (2), consider the group $H = \langle x_1, x_2, x_3^2, x_4^2, \dots\rangle$. This is a $\mathbb{C}14^*$-subgroup with infinite central subgroup $\langle x_3^2, x_4^2, \dots\rangle$. By Lemma 3.4, H is abelian and so $[x_1, x_2] = 1$, contrary to the choice of the set S.

In case (1), fix $g = x_k$; then for each i we have $[x_i^{-1}, g, x_i^{-1}] = [x_i, g]$. For each integer $l > k$, consider the infinite sets $X_l = \{x_i^2 \mid i > l\}$ and $Y_l = X_l g$. By the $\mathbb{C}14^*$-condition there are integers $i, j > l$ such that $[x_i^{-2}, x_j^2 g, x_i^{-2}] = [x_i^2, x_j^2 g]$. But since $[x_i^2, x_j^2] = 1$, this reduces to $[x_i^{-2}, g, x_i^{-2}] = [x_i^2, g]$. By Lemma 4.3, $[x_i^6, g] = 1$ and so $[x_i^2, g] = 1$. It follows that there are infinitely many integers $i > k$ such that $[x_i^2, x_k] = 1$. We can therefore obtain an infinite subset $\{y_1, y_2, \dots\}$ of $\{x_1, x_2, \dots\}$ such that $[y_i^2, y_j] = 1$, whenever $i > j$. Thus we have case (2) again and we obtain a contradiction. Therefore G has no elements of order 4.

Now suppose that G has an element x of order 9. Then x^3 has order 3 and x^3 has infinitely many conjugates. Using the same argument as in the previous case we obtain an infinite set T of pairwise noncommuting conjugates of x and then an infinite subset $\{x_1, x_2, \dots\}$ such that $[x_i^{-1}, x_j, x_i^{-1}] = [x_i, x_j] \neq 1$ for all i, j and either (1) $[x_i^{-2}, x_j, x_i^{-2}] = [x_i^2, x_j]$, whenever $i < j$, or (2) $[x_i^{-2}, x_j, x_i^{-2}] = [x_i^2, x_j]$, whenever $i > j$.

If $[x_i^{-1}, x_j, x_i^{-1}] = [x_i, x_j]$ and $[x_i^{-2}, x_j, x_i^{-2}] = [x_i^2, x_j]$, then, by Lemma 4.3, $[x_i, x_j]^{12} = 1$ and $[x_i^6, x_j] = 1$. Since x_i has order 9, we have $[x_i^3, x_j] = 1$ and, by (iii) of Lemma 4.3, $[x_i, x_j]^4 = 1$. But G has no elements of order 4 so $[x_i, x_j]^2 = 1$. The equation $[x_j^{-1}, x_i, x_j^{-1}] = [x_j, x_i]$ is equivalent to $[x_i, x_j]^{x_j} = [x_i, x_j]^{1+x_j^2}$ and so $[x_i, x_j^3] = [x_i, x_j]^{1+x_j+x_j^2} = [x_i, x_j]^{2+2x_j^2} = 1$. Therefore $[x_i, x_j^3] = 1$ for all i, j and so $\langle x_1^3, x_2^3, \dots\rangle$ is an infinite central subgroup of the $\mathbb{C}14^*$-group $\langle x_1, x_2, \dots\rangle$. By Lemma 3.4, $\langle x_1, x_2, \dots\rangle$ is abelian contrary to the choice of T. Hence G has no elements of order 9 either. ■

Now we are ready to prove the main result of this section.

THEOREM 4.10. *An infinite* $\mathbb{C}14^*$-*group is abelian.*

PROOF. If there is an infinite nonabelian $\mathbb{C}14^*$-group then, by Theorem 3.8, we can choose a counterexample G with $FC(G) = 1$. By Lemma 4.9, G has exponent 6 and so is locally finite [5; 18.4.8], and by Burnside's Theorem [5; 9.3.2] it is also locally solvable.

Let H be any countably infinite subgroup of G; then $H = PQ = QP$, where P is a 2-group and Q is a 3-group [7; p. 22]. Since P has exponent 2 it is abelian. If Q is finite then H is abelian-by-finite and so is abelian by Lemma 3.5, and hence Q is abelian. If Q is not abelian then it is infinite and so is not central-by-finite. Hence Q contains an infinite set S of pairwise noncommuting elements. Since Q is a $\mathbb{C}14^*$-group, Theorem 4.1 shows that S has an infinite subset S_1 such that $[x^{-1}, y, x^{-1}] = [x, y] \neq 1$, for all $x, y \in S_1$. We can write S_1 as the disjoint union of two infinite sets X and Y and consider the infinite sets $X^2 = \{x^2 \mid x \in X\}$ and Y. By the $\mathbb{C}14^*$-condition, there are elements $x \in X$ and $y \in Y$ such that $[x^{-2}, y, x^{-2}] = [x^2, y]$. Since Q has exponent 3, $[x^3, y] = 1$ and so, by

(iii) of Lemma 4.3, $[x, y]^4 = 1$, and hence $[x, y] = 1$, a contradiction. Therefore Q is abelian. Thus, by Ito's Theorem [15; 8.5.3], H is metabelian, and so by Theorem 3.6, H is abelian. Since H was an arbitrary countably infinite subgroup of G it follows that G is abelian. ■

We have now immediately the following corollary.

COROLLARY 4.11. *An infinite* $\mathbb{D}n$*-*group is abelian, for* $n = 1, 8, 14, 20$.

PROOF. By definition, any $\mathbb{D}14$*-group is a $\mathbb{C}14$*-group, and hence abelian by Theorem 4.10. Proposition 2.5 implies that every infinite $\mathbb{D}n$*-group, $n = 1, 8, 20$, is a $\mathbb{D}14$*-group, and hence abelian by the above. ■

5. – The class of $\mathbb{D}4$*-groups.

In this section we show that in the remaining cases, i.e. $n = 4, 30$, infinite $\mathbb{D}n$*-groups are abelian. As stated in Proposition 2.4, neither $\mathbb{D}4$ nor $\mathbb{D}30$ imply $\mathbb{D}14$. Thus we cannot use Theorem 4.10 to prove our claim. However, by Proposition 2.4, $\mathbb{D}30$ implies $\mathbb{D}4$. Thus it suffices to prove our claim for $n = 4$. First we recall some relevant results from Proposition 4.5 in [6].

LEMMA 5.1. *If* $\{x, y\}$ *satisfies* $\mathbb{D}4$, *then*

(i) $[x^6, y] = [x, y^6] = 1$,

(ii) $[x^3, y] = [x, y]^{2x}, [x, y^3] = [x, y]^{2y}$.

Our proof that an infinite $\mathbb{D}4$*-group is abelian is similar in structure to that for $\mathbb{C}14$*-groups but is slightly easier because of the additional information contained in the above result.

LEMMA 5.2. *Suppose that* G *is an infinite nonabelian* $\mathbb{D}4$**-group. If* A *is an abelian subgroup of* G *then* A^6 *is finite.*

PROOF. Suppose that A^6 is infinite and for each $x \in A^6$ choose $b \in A$ such that $b^6 = x$. This gives an infinite subset B of A such that the elements b^6, $b \in B$, are distinct. Given any element $g \in G$ and let X be any infinite subset of B. Consider X and $Y = Xg = \{xg \mid x \in X\}$. By the $\mathbb{D}4$*-

condition, there are elements $b, c \in X$ such that $\{b, cg\}$ satisfies $\mathbb{D}4$. In particular $[b^6, cg] = 1$ and hence $[b^6, g] = 1$. Thus every infinite subset of A^6 contains an element of $C_G(g)$. Therefore $A^6 \backslash A^6 \cap C_G(g)$ is finite and hence $A^6 \leqslant C_G(g)$. It follows that $A^6 \leqslant Z(G)$, contrary to Lemma 3.5. ■

LEMMA 5.3. *Suppose that G is an infinite nonabelian $\mathbb{D}4^*$-group with $FC(G) = 1$. Then G is a $\{2, 3\}$-group.*

PROOF. By Lemma 5.2, G has no elements of infinite order. If G is not a $\{2, 3\}$-group it contains an element x of prime order $p > 3$ and x has infinitely many conjugates. By Theorem 4.1 and the $\mathbb{D}4^*$-condition there is an infinite set S of conjugates of x such that each $\{u, v\} \in S^{[2]}$ satisfies $\mathbb{D}4$. By Lemma 5.1 we have $[u^6, v] = 1$, but since u has order p this implies that $[u, v] = 1$. Therefore $\langle u \mid u \in S\rangle$ is an infinite elementary abelian p-subgroup, contrary to Lemma 5.2. ■

LEMMA 5.4. *Suppose that G is an infinite nonabelian $\mathbb{D}4^*$-group with $FC(G) = 1$. Then G has exponent 6.*

PROOF. Suppose first that G has an element x of order 4. Then x^2 has order 2 and has infinitely many conjugates. Therefore there is an infinite set C of conjugates of x whose squares are distinct. If C contained an infinite commuting subset then this subset would generate an abelian subgroup A with $A^6 = A^2$ being infinite, contrary to Lemma 5.2. By Theorem 4.1, C contains an infinite subset S of pairwise noncommuting elements.

Applying Theorem 4.1 again with the $\mathbb{D}4^*$-condition, we obtain an infinite subset T of S such that $\{u, v\}$ satisfies $\mathbb{D}4$, for all $u, v \in T$. In particular $\langle u^2 \mid u \in T\rangle$ is an infinite central subgroup of $\langle u \mid u \in T\rangle$. It follows from Lemma 3.5 that $\langle u \mid u \in T\rangle$ is abelian, contrary to the elements of T being pairwise noncommuting. Therefore G has no elements of order 4.

Now suppose G has an element x of order 9. As above, we obtain an infinite set W of pairwise noncommuting conjugates of x such that $[u^6, v] = 1$, for all $u, v \in W$. Since u has order 9 it follows that $[u^3, v] = 1$, for all $u, v \in W$. Now $\langle u^3 \mid u \in W\rangle$ is an infinite central subgroup of $\langle u \mid u \in W\rangle$ and we again obtain a contradiction to the elements of W being noncommuting. ■

We are now ready to prove the main result of this section.

THEOREM 5.5. *An infinite* $\mathbb{D}n^*$*-group,* $n = 4, 30$, *is abelian.*

PROOF. Let $n = 4$. If there is an infinite nonabelian $\mathbb{D}4^*$-group then, by Theorem 3.8, we can choose it to have $FC(G) = 1$. By Lemma 5.4, G has exponent 6 and so is locally finite. If H is any countably infinite subgroup of G then $H = PQ = QP$ where P is an elementary abelian 2-subgroup and Q has exponent 3. As in the proof of Theorem 4.10, we have only to show that if Q is infinite then it is abelian.

If Q is infinite and nonabelian then it contains an infinite set S of pairwise noncommuting elements. By the $\mathbb{D}4^*$-condition there are elements $x, y \in S$ such that $\{x, y\}$ satisfies $\mathbb{D}4$. Since $x^3 = 1$, it follows from (ii) of Lemma 5.1 that $[x, y]^{2x} = 1$ and, since $[x, y] \in Q$ has order dividing 3, this implies that $[x, y] = 1$. This contradiction shows that Q is abelian. Hence $H = PQ$ is metabelian and hence abelian, by Theorem 3.6.

Finally, let $n = 30$. By Proposition 2.5, condition $\mathbb{D}30$ implies $\mathbb{D}4$. Therefore, by the above, it follows that an infinite $\mathbb{D}30^*$-group is abelian. ■

The following corollary is an immediate consequence of Theorem 5.5 and Corollary 4.11.

COROLLARY 5.6. *For any* $n \in \{1, \ldots, 32\}$, *an infinite* $\mathbb{D}n^*$*-group is abelian.*

6. – The class of $\mathbb{C}1^*$-groups.

In this section we will show that an infinite $\mathbb{C}1^*$-group is abelian. The proof follows along the same lines of the one given in Section 4 for $\mathbb{C}14^*$-groups, but it is much simpler, since we can reduce it to a 2-group. By Proposition 2.3 there are 10 other $\mathbb{D}n$-conditions equivalent to $\mathbb{D}1$. However, there are only 5 of them, namely $n = 2, 3, 5, 6, 13$, where the symmetrized $\mathbb{C}n$-condition remains equivalent to the symmetrized $\mathbb{C}1$-condition. In all of these cases we are able to show that infinite $\mathbb{C}n^*$-groups are abelian.

We start with a lemma analogous to Lemma 4.3, where we will use the $\mathbb{C}1$ condition applied to the pair (x, y^2) simultaneously with the pair (x, y).

LEMMA 6.1. *Let x, $y \in G$ satisfy the following two conditions*

(6.1.1) $$[x, y, y] = [x, y],$$

(6.1.2) $$[x, y^2, y^2] = [x, y^2].$$

Then

(i) $[x, y]^3 = 1$,

(ii) $[x, y^2] = 1$.

PROOF. Write $a = [x, y]$ and consider the near ring $\mathcal{N}_a$. First we show (i). In $\mathcal{N}_a$ (6.1.1) can be written as

(6.1.3) $$y = 2.$$

Similarly, (6.1.2) becomes

(6.1.4) $$3y^2 = 6.$$

Now (6.1.3) implies $3y = 6$. Substitution into (6.1.4) leads to $3y^2 = 3y$. Multiplying by y^{-1} from the right yields $3y = 3$. Using (6.1.3), we obtain $3 = 0$, or $a^3 = 1$, proving (i). The second claim follows from observing $[x, y^2] = a^3$. ■

LEMMA 6.2. *Let G be a* $\mathbb{C}1^*$*-group and let A be an abelian subgroup of G. Then either $A^2 = \{a^2 \mid a \in A\}$ is finite or $A^2 \leqslant Z(G)$.*

PROOF. Suppose A^2 is infinite. Let $g \in G$; we show that $A^2 \leqslant C_G(g)$. For each element $x \in A^2$, choose an element $b \in A$ such that $b^2 = x$. This gives an infinite subset B of A such that the elements b^2, $b \in B$, are distinct. Let S be an infinite subset of B. Consider the infinite sets $X = S$ and $Y = Sg = \{sg \mid s \in S\}$. The $\mathbb{C}1^*$-condition shows that there are elements b, $c \in S$ such that $[cg, b, b] = [cg, b]$. But A abelian implies

(6.2.1) $$[g, b, b] = [g, b].$$

Since every infinite subset S of B contains an element b satisfying (6.2.1), it follows that if $C = \{b \in B \mid [g, b, b] = [g, b]\}$ then $B \backslash C$ is finite. Now let T be any infinite subset of $C^2 = \{b^2 \mid b \in C\}$. By considering the infinite sets T and Tg we see that there are elements d^2, $e^2 \in T$ such that $[e^2 g, d^2, d^2] = [e^2 g, d^2]$, and hence

(6.2.2) $$[g, d^2, d^2] = [g, d^2].$$

It follows that if $D = \{b \in B \mid [g, b, b] = [g, b]$ and $[g, b^2, b^2] = [g, b^2]\}$ then $B \setminus D$ is finite. For each $d \in D$, we have $[d^2, g] = 1$, by Lemma 6.1, and since $A^2 = B^2 = \{b^2 \mid b \in B\}$ and $B \setminus D$ is finite, it follows that $A^2 \setminus A^2 \cap C_G(g)$ is finite. Now A^2 infinite implies that $A^2 \cap C_G(g) = A^2$ and so $A^2 \leqslant C_G(g)$, for any $g \in G$. ■

COROLLARY 6.3. *Suppose that G is an infinite nonabelian* $\mathbb{C}1^*$-*group. If A is an abelian subgroup of G, then A^2 is finite.*

PROOF. If A^2 were infinite, then it would be central contrary to Lemma 3.4. ■

COROLLARY 6.4. *Suppose that G is an infinite nonabelian* $\mathbb{C}1^*$-*group. Then G is periodic and has no infinite abelian $2'$-subgroup.*

PROOF. If G contained an element g of infinite order then $\langle g \rangle^2$ would be infinite, contrary to Corollary 6.3. If G contained an infinite abelian $2'$-subgroup A then $A^2 = A$ would be infinite, again contradicting Corollary 6.3. ■

The next stage in our proof is to reduce to the consideration of a 2-group.

LEMMA 6.5. *Suppose that G is an infinite nonabelian* $\mathbb{C}1^*$-*group with $FC(G) = 1$. Then G is a 2-group.*

PROOF. If G is not a 2-group then it contains an element u of prime order $p \neq 2$ and u has infinitely many conjugates. It follows from the $\mathbb{C}1^*$-condition and Theorem 4.1 that this set of conjugates contains an infinite subset S such that $[x, y, y] = [x, y]$, for all $x, y \in S$. Let X and Y be two infinite subsets of S. Since the elements have order $p \neq 2$, the squares of the elements are distinct. So $Y^2 = \{y^2 \mid y \in Y\}$ is infinite and applying the $\mathbb{C}1^*$-condition to X and Y^2 gives elements $x \in X$ and $y \in Y$ such that

$$[x, y^2, y^2] = [x, y^2] \quad \text{and} \quad [y^2, x, x] = [y^2, x]. \tag{6.5.1}$$

The equations of (6.5.1) therefore satisfy the hypothesis on $P(x, y)$ in Corollary 4.2 and so S contains an infinite set of elements $\{x_1, x_2, \ldots\}$ such that either (1) $[x_j, x_i^2, x_i^2] = [x_j, x_i^2]$, whenever $i < j$, or (2)

$[x_j, x_i^2, x_i^2] = [x_j, x_i^2]$, whenever $i > j$. Lemma 5.1 shows that either (1) $[x_i^2, x_j] = 1$, whenever $i < j$, or (2) $[x_i^2, x_j] = 1$, whenever $i > j$. Since x_i has prime order $p \neq 2$, it follows that $[x_i, x_j] = 1$ for all i, j and so $\langle x_1, x_2, \ldots \rangle$ is an infinite elementary abelian p-group, contrary to Corollary 6.4. Therefore G is a 2-group. ■

The next step is to show that G has no elements of order 4.

LEMMA 6.6. *Suppose that G is an infinite nonabelian* $\mathbb{C}1^*$*-group with $FC(G) = 1$. Then G has exponent* 2.

PROOF. By Lemma 6.5, G is a 2-group and so we have only to show that G has no elements of order 4.

Suppose first that G has an element u of order 4. Then u^2 has order 2 and u^2 has infinitely many conjugates. Therefore there is an infinite set C of conjugates of u whose squares are distinct. If C contained an infinite commuting subset, then the elements of this set would generate an abelian subgroup A of exponent 4 with A^2 being infinite. This is contrary to Corollary 6.3. By Theorem 4.1, C must contain an infinite subset S of pairwise noncommuting elements. The $\mathbb{C}1^*$-condition and Theorem 4.1 show that S contains an infinite subset T such that $[x, y, y] = [x, y] \neq 1$, for all $x, y \in T$. If X and Y are infinite subsets of T then the set $Y^2 = \{y^2 \mid y \in Y\}$ is also infinite and by the $\mathbb{C}1^*$-condition there are elements $x \in X$ and $y \in Y$ such that

$$(6.6.1) \qquad [x, y^2, y^2] = [x, y^2], \qquad \text{and } [y^2, x, x] = [y^2, x].$$

The equations of (6.6.1) therefore satisfy the hypothesis on $P(x, y)$ in Corollary 4.2 and so T contains an infinite subset $\{x_1, x_2, \ldots\}$ such that either

(1) $[x_j, x_i^2, x_i^2] = [x_j, x_i^2]$, whenever $i < j$, or

(2) $[x_j, x_i^2, x_i^2] = [x_j, x_i^2]$, whenever $i > j$.

But Lemma 6.1 implies that $[x, y]^3 = 1$ and, since G is a 2-group, $[x, y] = 1$ contrary to the choice of S. ■

Now we are ready to prove our final result.

THEOREM 6.7. *An infinite* $\mathbb{C}n^*$*-group,* $n = 1, 2, 3, 5, 6, 13$, *is abelian.*

PROOF. First let $n = 1$. If there is an infinite nonabelian $\mathbb{C}1^*$-group then, by Theorem 3.8, we can choose a counterexample G with $FC(G) = 1$. By Lemma 6.5 and 6.6, G is a 2-group without elements of order 4, hence is abelian, a contradiction. We conclude that there are no non-abelian infinite $\mathbb{C}1^*$-groups.

Turning to the other values of n, we observe that conditions $\mathbb{C}2$ and $\mathbb{C}3$ can be rewritten as $[x, y^{-1}, y^{-1}] = [x, y^{-1}]$. Thus (x, y^{-1}) satisfies $\mathbb{C}1$ iff (x, y) satisfies $\mathbb{C}2$ or $\mathbb{C}3$, respectively. Similarly, $\mathbb{C}5$ can be rewritten as $[y, x, x] = [y, x]$. Thus (y, x) satisfies $\mathbb{C}1$ iff (x, y) satisfies $\mathbb{C}5$. Finally, $\mathbb{C}6$ and $\mathbb{C}13$ can be rewritten as $[y, x^{-1}, x^{-1}] = [y, x^{-1}]$. Thus (y, x^{-1}) satisfies $\mathbb{C}1$ iff (x, y) satisfies $\mathbb{C}6$ or $\mathbb{C}13$, respectively.

Suppose now that G is a $\mathbb{C}2^*$-group. We have to show that for any infinite subsets X and Y of G there exist $x \in X$, $y \in Y$ such that $\phi_1(x, y) = \phi_1(y, x) = 1$. Let X, V be any infinite subsets of G. There exist $x \in X$, $v \in V$ such that $\phi_2(x, v) = \phi_2(v, x) = 1$, or equivalently $\phi_1(x, v^{-1}) = \phi_1(v^{-1}, x) = 1$. Now for any subset Y of G there exists a subset V such that $Y = V^{-1} = \{w^{-1};\ w \in V\}$. Thus if $v \in V$, then $v^{-1} \in Y$. Set $y = v^{-1}$ and we conclude (x, y) satisfies $\mathbb{C}1$. Hence $\mathbb{C}2^*$ implies $\mathbb{C}1^*$. Similar arguments show that $\mathbb{C}n^*$, $n = 3, 5, 6, 13$, implies $\mathbb{C}1^*$. ■

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Manoscritto pervenuto in redazione il 14 febbraio, 1997.