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Decreasing diagonal elements in completely positive matrices

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Decreasing Diagonal Elements in Completely Positive Matrices.

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All matrices considered in this paper are real matrices (1).

It is well known that a diagonally dominant symmetric matrix with nonnegative diagonal elements is positive semidefinite; actually, this fact is an immediate consequence of the Gerschgorin circles theorem.

An analogous result, not equally well known, holds for completely positive matrices, i.e. for those matrices A that can be factorized in the form $A = VV^T$, where V is a nonnegative matrix. In fact Kaykobad [4] proved that a nonnegative symmetric diagonally dominant matrix is completely positive.

Hence, if the diagonal elements of a symmetric matrix A have sufficiently large positive values, then A is positive semidefinite and, if A is nonnegative, it is completely positive.

In this paper we consider the question arising by assuming the opposite point of view. Let A be a positive semidefinite or a completely positive matrix; then we will consider the following question: how much a diagonal element of A can be decreased while preserving the semidefinite positivity or, respectively, the complete positivity of A?

We will answer this question concerning positive semidefinite matrices in Section 2; the answer is simple and follows from an inductive test

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for semidefinite positivity (see [7], Cor. 1.3]), which involves the Moore-Penrose pseudo-inverse matrix of a maximal principal submatrix.

As to completely positive matrices the answer arises from the connection of the above question with the property PLSS (positivity of least square solutions) introduced in [7]. More precisely, we will consider in Section 3 *i-minimizable matrices* ($1 \leqslant i \leqslant n$, where n is the order of the considered matrix), which are those completely positive matrices such that the minimal value of the i^{th} diagonal element making the matrix positive semidefinite makes the matrix completely positive too. We will show that 1-minimizability is equivalent to property PLSS, that singular completely positive matrices are i-minimizable for some i, and that completely positive matrices with completely positive associated graph (see [5]) are i-minimizable for all i.

Recent results (see [1]) show that completely positive matrices with cyclic graph of odd lenght are not i-minimizable for any i. Conversely, it follows from results by Berman and Grone [2] that completely positive matrices with cyclic graph of even lenght are i-minimizable for all i. Thus the following problem arises: to find examples of completely positive matrices with more complicate behaviour with respct to minimizability.

In Section 4 we will investigate *excellent* completely positive matrices, already introduced in [1], and defined in terms of their graphs, which are «almost» completely positive. We will give a characterization of excellent 1-minimizable completely positive matrices, which enables us to produce several examples of excellent matrices with different behaviour with the respect to minimizability.

2. – Minimized positive semidefinite matrices.

We recall the following definitions. A symmetric matrix A of order n is *positive semidefinite* if $\underline{x}^T A \underline{x} \geqslant 0$ for all vectors $\underline{x} \in \mathbb{R}^n$. The matrix A is *doubly nonnegative* if it is both semidefinite and (entrywise) nonnegative; it is *completely positive* if there exist a nonnegative (not necessarily square) matrix V such that $A = VV^T$.

It is well known that a completely positive matrix is doubly nonnegative and that the converse is not generally true for matrices of order larger than four.

We will denote by $R(A)$ the column space (range) of the matrix A, by A^+ its Moore-Penrose pseudo-inverse and by $\mathrm{rk}(A)$ its rank. For unexplained notation we refer to [6].

Let A be a symmetric semidefinite matrix of order $n > 1$ in bordered form

(1) $$A = \begin{pmatrix} a & \underline{b}^T \\ \underline{b} & A_1 \end{pmatrix}.$$

In [7] the following result, which provides an inductive test for semidefinite positivity, was proved:

LEMMA 2.1. *The symmetric matrix A in* (1) *is positive semidefinite if and only if A_1 is positive semidefinite, $\underline{b} \in R(A_1)$ and $a \geqslant \underline{b}^T A_1^+ \underline{b}$.*

It follows from Lemma 2.1 that the minimal value of $t \in \mathbb{R}$ such that the matrix

(2) $$A(t) = \begin{pmatrix} t & \underline{b}^T \\ \underline{b} & A_1 \end{pmatrix}$$

is positive semidefinite is $\underline{b}^T A_1^+ \underline{b}$; we will denote it by a_μ. The matrix

(3) $$A_\mu = \begin{pmatrix} a_\mu & \underline{b}^T \\ \underline{b} & A_1 \end{pmatrix}$$

is called the 1-*minimized* of A. We say that A itself is 1-*minimized* if $A = A_\mu$. If $1 \leqslant i \leqslant n$, the *i-minimized* of A is the matrix $(P_i^T A P_i)_\mu$, where P_i is the permutation matrix obtained from the identity matrix by transposing the first and the i^{th} row. The matrix A is said to be *i-minimized* if $P_i^T A P_i = (P_i^T A P_i)_\mu$.

A 1-minimized (or, more generally, an *i-minimized*) matrix is singular. This fact is trivial if A_1 is singular, otherwise it follows from the equality $\operatorname{Det}(A) = \operatorname{Det}(A_1)(a - \underline{b}^T A_1^{-1} \underline{b})$ applied to $A = A_\mu$.

The converse is not generally true, as the trivial example

(4) $$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

shows. However, we have the following result:

THEOREM 2.2. *A singular positive semidefinite matrix n-by-n is i-minimized for some $i \leqslant n$.*

In order to prove the preceding theorem, we need two lemmas, whose straightforward proofs are given for sake of completeness.

LEMMA 2.3. *Let A be a positive semidefinite n-by-n matrix. Then its rank coincides with the maximum order of a non-singular principal submatrix.*

PROOF. Let $\mathrm{rk}(A) = k$. Since A is positive semidefinite, there exists a real n-by-k matrix W such that $A = WW^T$. Since $\mathrm{rk}(W) = k$ too, we can find a non-singular k-by-k submatrix of W. Let it W_1. Finally $A' = W_1 W_1^T$ is a non-singular principal submatrix of A of order k. ■

The second result that we need is the following:

LEMMA 2.4. *Let A be a positive semidefinite matrix in bordered form* (1), *with* $\mathrm{rk}(A_1) = k$, *and let M a non-singular principal submatrix of A_1 of order k. Then $a_\mu = \underline{c}^T M^{-1} \underline{c}$, where $\underline{c}$ is the vector obtained by taking the coordinates of $\underline{b}$ corresponding to the rows of M.*

PROOF. The existence of M is ensured by Lemma 2.3. There is no loss of generality if we consider A in the form

$$\begin{pmatrix} a & \underline{c}^T & \underline{d}^T \\ \underline{c} & M & D^T \\ \underline{d} & D & N \end{pmatrix}. \tag{5}$$

There exists a matrix W_1 of rank k such that $A_1 = W_1 W_1^T$ and clearly $A_\mu = WW^T$ for $W^T = (W_1^+ \underline{b}\ \ W_1^T)$, because

$$a_\mu = \underline{b}^T A_1^+ \underline{b} = \underline{b}^T (W_1 W_1^T)^+ \underline{b} = \underline{b}^T (W_1^T)^+ W_1^+ \underline{b} = (W_1^+ \underline{b})^T W_1^+ \underline{b}.$$

Thus $\mathrm{rk}(A_\mu) = k$ and the principal submatrix

$$\begin{pmatrix} a_\mu & \underline{c}^T \\ \underline{c} & M \end{pmatrix} \tag{6}$$

is singular and positive semidefinite, thus $a_\mu = \underline{c}^T M^{-1} \underline{c}$, by Lemma 2.1. ■

We are now able to give the

PROOF OF THEOREM 2.2 Let B the singular positive semidefinite matrix, $k = \mathrm{rk}(B)$. There exists a permutation matrix of the form P_i for some $i \leq n$, such that $A = P_i^T B P_i$, written in bordered form (1), has $\mathrm{rk}(A_1) = k$.

By Lemma 2.4, there exists a non-singular principal submatrix M of order k of A_1, so that A is cogredient to the matrix

$$(7) \qquad A^* = \begin{pmatrix} a & \underline{c}^T & \underline{d}^T \\ \underline{c} & M & D^T \\ \underline{d} & D & N \end{pmatrix}.$$

We will show that this matrix is 1-minimized, hence the matrix B will be i-minimized. The principal submatrix

$$(8) \qquad \begin{pmatrix} a & \underline{c}^T \\ \underline{c} & M \end{pmatrix}$$

is singular, hence $a = \underline{c}^T M^{-1} \underline{c}$ and, by Lemma 2.4, this is the minimal value that makes A^* positive semidefinite. ∎

From the preceding results we get the following

COROLLARY 2.5. *Let A be a singular positive semidefinite matrix in bordered form* (1). *Then A is* 1*-minimized if and only if* $\mathrm{rk}(A) = \mathrm{rk}(A_1)$.

PROOF. Assume A 1-minimized. The proof of Lemma 2.4 shows that W has the same number of columns as W_1, hence $\mathrm{rk}(A) = \mathrm{rk}(A_1)$. Conversely, if $\mathrm{rk}(A) = \mathrm{rk}(A_1)$, in the proof of Theorem 2.2, M can be choosen as a submatrix of A_1 without using the permutation matrix P_i, and one can conclude that A itself is 1-minimized. ∎

3. – Minimizable completely positive matrices.

Let A be a completely positive matrix of order n in bordered form (1). An immediate consequence of the fact that the class of the completely positive matrices of order n is closed in the class of the symmetric matrices of the same order (see [3]) is that there exists a minimal value $a_e \in \mathbb{R}$

such that the matrix

$$A_e = \begin{pmatrix} a_e & \underline{b}^T \\ \underline{b} & A_1 \end{pmatrix} \tag{9}$$

is completely positive; obviously $a_e \geqslant a_\mu$. The matrix A_e is called the 1-*extremized* of A. A itself is called an 1-*extreme* matrix if $A = A_e$. If $i \leqslant n$, the *i-extremized* of A is the 1-extremized of $P_i^T A P_i$ and A itself is an *i-extreme* matrix if $P_i^T A P_i$ is 1-extreme.

DEFINITION. We say that a completely positive matrix A in bordered form (1) is *i-minimizable* if the i-minimized matrix of A is also completely positive, i.e. if $(P_i^T A P_i)_\mu = (P_i^T A P_i)_e$.

DEFINITION. We say that a completely positive matrix A in bordered form (1) is *totally minimizable* if it is i-minimizable for all $i \leqslant n$; equivalently, if all the matrices cogredient to A are 1-minimizable.

The apparently new notion of 1-minimizable completely positive matrix is equivalent to the notion of «property PLSS» already introduced in [7], and simply called «property (P)» in [6]. Recall that the symmetric non-negative matrix A in bordered form (1) satisfies property PLSS if there exists a nonnegative matrix V_1 such that $A_1 = V_1 V_1^T$ and $V_1^+ \underline{b} \geqslant \underline{0}$.

PROPOSITION 3.1. *A completely positive matrix A in bordered form (1) is 1-minimizable if and only if it satisfies the property* PLSS.

PROOF. Let A satisfy the property PLSS. Then A_μ also satisfies this property; hence, by ([7], Theorem 2.1), A_μ is completely positive.

Conversely, let A be 1-minimizable. By ([7], Proposition 1.2), there exists a nonnegative matrix V_1 and a nonnegative vector $\underline{v}$ such that:

$$A_1 = V_1 V_1^T \qquad \underline{b} = V_1 \underline{v} \qquad \underline{b}^T A_1^+ \underline{b} = \underline{v}^T \underline{v}.$$

From the equalities

$$\|\underline{v}\|_2^2 = \underline{v}^T \underline{v} = \underline{b}^T A_1^+ \underline{b} = \underline{b}^T (V_1 V_1^T)^+ \underline{b} =$$
$$= \underline{b}^T (V_1^T)^+ V_1^+ \underline{b} = \underline{b}^T (V_1^+)^T V_1^+ \underline{b} = \|V_1^+ \underline{b}\|_2^2$$

and from the uniqueness of the least square solution of the linear system

$V_1\underline{x} = \underline{b}$, there follows that $\underline{v} = V_1^+ \underline{b}$. Thus $V_1^+ \underline{b} \geqslant \underline{0}$ and property PLSS holds. ■

The class of totally minimizable matrices is quite large, as the following result shows. Recall that a graph Γ is called *completely positive* if every doubly nonnegative matrix A whose associated graph $\Gamma(A)$ equals Γ is completely positive. Kogan and Berman [5] proved that a graph is completely positive if and only if it does not contain a cycle of odd length larger than three.

PROPOSITION 3.2. *The class of totally minimizable matrices contains all the completely positive matrices with completely positive graphs.*

PROOF. Let A be a completely positive matrix such that its graph $\Gamma(A)$ is completely positive. Let B the i-minimized matrix of A for some $i \leqslant n$. Obviously $\Gamma(B)$ is completely positive, thus B, which is obviously doubly nonnegative, is completely positive. ■

The connection between singularity and minimizability is illustrated in the following

PROPOSITION 3.3. *Let A be a singular completely positive matrix in bordered form* (1). *Then A is i-minimizable for some $i \leqslant n$ and is* 1-*minimizable if* $\mathrm{rk}(A) = \mathrm{rk}(A_1)$.

PROOF. By Theorem 2.2, A is i-minimized for a certain i; let $A^* = P_i^T A P_i$. A^* is completely positive and 1-minimized, so it is obviously 1-minimizable, hence A is i-minimizable. Furthermore, if $\mathrm{rk}(A) = \mathrm{rk}(A_1)$, by Corollary 2.5 it follows that A is 1-minimizable. ■

We conclude this section with a result providing the construction of a completely positive singular matrix (hence i-minimizable for some $i \leqslant n$) which fails to be 1-minimizable; this matrix is obtained by bordering in a suitable way a non-singular not 1-minimizable completely positive matrix. Such a matrix does exists (see Example 4.6).

PROPOSITION 3.4. *Let C be a non-singular completely positive matrix of order $n > 1$ in bordered form*

$$C = \begin{pmatrix} a & \underline{c}^T \\ \underline{c} & C_1 \end{pmatrix} \tag{10}$$

which is not 1-minimizable. Then there exists a completely positive matrix A obtained by suitably bordering the matrix C

$$A = \begin{pmatrix} a & \underline{c}^T & d \\ \underline{c} & C_1 & \underline{d} \\ d & \underline{d}^T & e \end{pmatrix} \tag{11}$$

which is not 1-minimizable, but is i-minimizable for some $1 < i \leqslant n$.

PROOF. There exists a matrix $V \geqslant 0$ such that $C = VV^T$. Let $V^T = (\underline{v}_1 \ \ V_1^T)$. Then $\mathrm{rk}(V) = \mathrm{rk}(C) = n$ and $\mathrm{rk}(V_1) = \mathrm{rk}(C_1) = n-1$. Let $W^T = (\underline{v}_1 \ \ V_1^T \ \ \underline{v})$, where $\underline{v}$ is a nonnegative vector of the column space $R(V_1^T)$. Obviously $\mathrm{rk}(W^T) = n$. Let now $A = WW^T$. Then A is completely positive of order $n+1$ and rank n, and has the form (11) for $d = \underline{v}_1^T \underline{v}$, $\underline{d} = V_1 \underline{v}$ and $e = \underline{v}^T \underline{v}$. By Proposition 3.3, A is i-minimizable for some i. Assume, by way of contraddiction, that A is 1-minimizable; then the 1-minimized matrix A_μ of A is completely positive, hence the its submatrix

$$\begin{pmatrix} a_\mu & \underline{c}^T \\ \underline{c} & C_1 \end{pmatrix} \tag{12}$$

is completely positive too. By Lemma 2.4, $a_\mu = \underline{c}^T C_1^{-1} \underline{c}$, hence we reach the contraddiction, since C is not 1-minimizable. ■

4. – Minimizability of excellent completely positive matrices.

A cycle of even length is a bipartite graph, hence it is a completely positive graph; therefore, in view of Proposition 3.2, a cyclic completely positive matrix of even order is totally minimizable.

Cyclic doubly nonnegative matrices of odd order which are not completely positive can be easily produced (see [3]). In [1] it is proved that a doubly nonnegative cyclic matrix A of odd order is completely positive if and only if $\mathrm{Det}(A) \geq 4h_1 h_2 \dots h_n$, where the h_i's are the non-zero elements over the main diagonal. It follows that such a matrix cannot be singular and consequently, given any $i \leq n$, it is not i-minimizable.

In this section we study a family of completely positive matrices, containing the cyclic ones, which provides more interesting examples with respect to minimizability.

In [1] doubly nonnegative matrices $A = (a_{ij})$ in bordered form (1) have been considered satisfying the following properties:

1) $a_{2,n} = a_{n,2} = 0$;

2) $\underline{b} = (\beta_2\ 0\ 0\ \dots\ 0\ \beta_n)^T$ with $\beta_2 \neq 0 \neq \beta_n$;

3) The graph $\Gamma(A_1)$ associated to A_1 is completely positive.

The graph $\Gamma(A)$ associated to the matrix A is obtained from the graph $\Gamma(A_1)$ by adding one more vertex, connected with only two vertices of $\Gamma(A_1)$, which are not connected each other. We will say that such a graph is an *excellent graph*, and that a doubly nonnegative matrix A satisfying the preceding properties is an *excellent matrix*.

In the preceding notation, let us denote by A^- the matrix obtained from A by substituting β_n by $-\beta_n$, or, equivalently, substituting the vector $\underline{b}$ by the vector $\underline{b}_1 - \underline{b}_2$, where $\underline{b}_1 = (\beta_2\ 0\ 0\ \dots\ 0)^T$ and $\underline{b}_2 = (0\ 0\ \dots\ 0\ \beta_n)^T$. In [1], part of the following result was proved.

THEOREM 4.1. *Let A be an excellent doubly nonnegative n-by-n matrix with $n > 2$ in bordered form* (1), *with $\underline{b} = (\beta_2\ 0\ 0\ \dots\ 0\ \beta_n)^T$. The following facts are equivalent:*

1) *A is completely positive*;

2) *A^- is positive semidefinite*;

3) *The vector $(1\ 0\ \dots\ 0\ 0)^T$ of $\mathbb{R}^{n-1}$ belongs to the column space $R(A_1)$ of A_1 and $a \geq a_\mu - 4\beta_2\beta_n\alpha$, where α is the element in the first row and last column of A_1^+.*

PROOF. For 1) $\Leftrightarrow$ 2) see [1]. For 2) $\Leftrightarrow$ 3), observe that, by Lemma 2.1, A^- is positive semidefinite if and only if $\underline{b}_1 - \underline{b}_2 \in R(A_1)$ and $a \geq (\underline{b}_1^T - \underline{b}_2^T) A_1^+ (\underline{b}_1 - \underline{b}_2)$. But since $\underline{b} = \underline{b}_1 + \underline{b}_2 \in R(A_1)$, $\underline{b}_1 - \underline{b}_2 \in R(A_1)$ if and only if $\underline{b}_1 \in R(A_1)$ (or equivalently $\underline{b}_2 \in R(A_1)$), if and only if

$\underline{b} = (1\ 0\ \dots\ 0\ 0)^T \in R(A_1)$. Moreover, we have:

$$(\underline{b}_1^T - \underline{b}_2^T)\, A_1^+ (\underline{b}_1 - \underline{b}_2) = \underline{b}_1^T A_1^+ \underline{b}_1 + \underline{b}_2^T A_1^+ \underline{b}_2 - 2\, \underline{b}_2^T A_1^+ \underline{b}_1 =$$

$$= \underline{b}_1^T A_1^+ \underline{b}_1 + \underline{b}_2^T A_1^+ \underline{b}_2 + 2\, \underline{b}_2^T A_1^+ \underline{b}_1 - 4\, \underline{b}_2^T A_1^+ \underline{b}_1 = a_\mu - 4\beta_2\beta_n \alpha\,. \qquad \blacksquare$$

We must remark that it is not possible to eliminate in Theorem 4.1 the hypothesis that A is doubly nonnegative, since there are symmetric nonnegative matrices A with A_1 positive semidefinite and $\underline{b} \in R(A_1)$, which fail to be positive semidefinite, and such that A^- is positive semidefinite. This happens (by using the preceding notation and with A in bordered form (2.1)) when $\alpha > 0$ and $\underline{b}^T A_1^+ \underline{b} > a > \underline{b}^T A_1^+ \underline{b} - 4\beta_2\beta_n \alpha$. On the other hand, in the hypothsis of Theorema 4.1, if $\alpha \geqslant 0$, the condition $a \geqslant a_\mu - 4\beta_2\beta_n\alpha$ is automatically verified, since $a \geqslant a_\mu$. An immediate consequence of this remark is the following

COROLLARY 4.2. *Let A be an excellent completely positive matrix od order $n > 1$ in bordered form* (1), *with $\underline{b} = (\beta_2\ 0\ 0\ \dots\ 0\ \beta_n)^T$, and let α be the element in the first row and last column of A_1^+. Then*

1) $a_e = \max(a_\mu,\ a_\mu - 4\beta_2\beta_n\alpha)$;

2) *A is 1-minimizable if and only if $\alpha \geqslant 0$.*

PROOF. 1) is an immediate consequence of Theorem 4.1.

2) If A is 1-minimizable, then $a_e = a_\mu$, hence $\alpha \geqslant 0$ by point 1). Conversely $\alpha \geqslant 0$ and point 1) imply $a_e = a_\mu$. $\blacksquare$

We give now some examples of excellent matrices with different behaviour with respect to 1-minimizability.

EXAMPLE 4.3. Consider the following symmetric matrix with excellent associated graph

$$A(a) = \begin{bmatrix} a & 1 & 0 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 \\ 0 & 2 & 5 & 3 & 1 \\ 0 & 1 & 3 & 2 & 2 \\ 1 & 0 & 1 & 2 & 11 \end{bmatrix}. \tag{13}$$

The principal submatrix A_1 (obtained from the last four rows and columns) is singular and completely positive, and the vector $\underline{b} = (1\ 0\ 0\ 1)^T \in R(A_1)$, hence $A(a)$ is doubly nonnegative for $a \geqslant \underline{b}^T A_1^+ \underline{b}$. Howewer, the matrix $A(a)$ is not completely positive for any value of a, since, according to Theorem 4 the vector $(1\ 0\ 0\ 0)^T$ does not belong to $R(A_1)$.

EXAMPLE 4.4. Consider the following symmetric matrix with excellent associated graph

$$A(a) = \begin{bmatrix} a & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 & 2 \end{bmatrix}. \tag{14}$$

In this case the principal submatrix A_1 is non-singular and completely positive, so the vectors $\underline{b} = (1\ 0\ 0\ 1)^T$ and $\underline{b}_1 = (1\ 0\ 0\ 0)^T$ belong to $R(A_1)$, hence $A(a)$ is doubly nonnegative for $a \geqslant \underline{b}^T A_1^+ \underline{b} = 7$. Moreover, the element a in the first row and last column of A_1^+ equals 1, hence $A(a)$ is completely positive and 1-minimizable for all $a \geqslant 7$. If $7 > a \geqslant 3 = 7 - 4\beta_2\beta_n a$, A is not doubly nonnegative and A^- is positive semidefinite.

EXAMPLE 4.5. Consider the following symmetric matrix with excellent associated graph

$$A(a) = \begin{bmatrix} a & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 3 & 6 \end{bmatrix}. \tag{15}$$

In this case A_1 is non-singular and completely positive, so $\underline{b} =$

$= (1\ 0\ 0\ 1)^T$ and $\underline{b}_1 = (1\ 0\ 0\ 0)^T$ belong to $R(A_1)$, hence $A(a)$ is doubly nonnegative for $a \geqslant \underline{b}^T A_1^+ \underline{b} = 3$. Moreover $\alpha = -1$, hence $A(a)$ is completely positive if and only if $a \geqslant a_\mu - 4\alpha = 7$ and one can conclude that A is not 1-minimizable.

EXAMPLE 4.6. Consider the following completely positive cyclic matrix of order 5

$$C = \begin{bmatrix} 2 & 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 1 & 2 \end{bmatrix} \tag{16}$$

which is not i-minimizable for any i. A factorization of $C = VV^T$ is obtained for

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}. \tag{17}$$

Using Proposition 3.4, one can see that, setting $W^T = (V^T \underline{b})$ where $\underline{b} = (0\ 0\ 0\ 1\ 1)^T$, the following completely positive matrix

$$A = WW^T = \begin{bmatrix} 2 & 1 & 0 & 0 & 1 & 1 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 1 & 0 & 0 & 1 & 2 & 2 \\ 1 & 0 & 0 & 1 & 2 & 2 \end{bmatrix} \tag{18}$$

is singular, hence i-minimizable for some i, but it is not 1-minimizable.

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