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FRANCESCO TONIN

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## The Cauchy Problem with Logarithmic Ramifications for $\mathcal{D}$ -Modules.

FRANCESCO TONIN (\*)

**ABSTRACT** - We are concerned with the holomorphic Cauchy problem with logarithmic ramifications for general (overdetermined) systems with simple characteristics. We generalize previous results by Hamada and others. Our situation is related to that of [K-S2] and [D'A-S2], where more general Cauchy data are considered. The same arguments also apply to the study of «swallow's tail» singularities.

### 1. - Introduction.

The Cauchy problem with singular data is a much investigated subject. Logarithmic singularities have been treated by Hamada, Hamada-Leray-Wagschal, Wagschal, Persson, Kashiwara-Schapira and D'Agnolo-Schapira (see [Ha], [H-L-W], [W], [P], [K-S2], [D'A-S1] and [D'A-S2]). In particular Kashiwara-Schapira and D'Agnolo-Schapira have used an algebraic approach to the problem. Roughly speaking, if  $Y$  is an hypersurface in  $X$  given by the equation  $g = 0$ , they considered singularities which can be written as  $f_0(x) \ln(g(x)) + f_1(x)$  with  $f_0 \in \mathcal{O}_X$ ,  $f_1 \in \mathcal{O}_{X \setminus Y}$  (if one denotes by  $\mathcal{O}_\Omega$  the holomorphic functions on  $\Omega$ ). Here we consider a smaller class of singularities, namely those among the previous ones which are tempered (meromorphic). The good notions and theories to deal with this subject are built by use of the functor of moderate cohomology, as developed by Kashiwara ([K1], see also [K-S3]), and have been added microlocal insight with the microlocalization of that functor, which is due to Andronikof (see [A]). In this paper we will

(\*) Indirizzo dell'A.: Dipartimento di Matematica Pura ed Applicata, via Belzoni 7, I-35131 Padova.

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mainly follow the definitions and the outlines of these three papers, as well as those given in [K-S1].

At first we approach the classical case of one operator Cauchy problem. Then we work out a generalization of it to the case of systems. Finally we show how our result can be adapted to another situation, that of tempered singularities ramified along a swallow's tail variety. In the appendix we state a lemma which is helpful for that application.

## 2. - The Cauchy problem with logarithmic ramifications.

Let  $X$  be a complex manifold, let  $Y$  be a hypersurface in  $X$ , let  $Z$  be a hypersurface in  $Y$ . Let  $T^*X$  be the cotangent bundle endowed with its canonical symplectic structure (see [K-S1] for notational conventions). Let us consider the following situation. Take  $P = P(x, D)$  a linear partial differential operator of order  $m$  on  $X$  with holomorphic coefficients for which the hypersurface  $Y$  be non-characteristic. Let  $f: Y \rightarrow X$  be the embedding. Define  ${}^t f'$  and  $f_\pi$  the natural mappings associated to  $f$ :

$$(2.1) \quad T^*Y \xleftarrow{{}^t f'} Y \times_X T^*X \xrightarrow{f_\pi} T^*X.$$

Let us denote by  $\sigma(P)$  the principal symbol of  $P$ . Suppose that  $P$  have simple characteristics transversal to  $Y \times_X T^*X$  at  ${}^t f'^{-1}(T_Z^*Y) \cap \text{char } P$  in the sense of [S1, Prop. III, 2.2.2], that is, if  $\phi = 0$  is a local equation for  $Y$ , then the Poisson bracket  $\{\phi, \sigma(P)\}$  does not vanish at any point of  ${}^t f'^{-1}(T_Z^*Y) \cap \sigma(P)^{-1}(0)$ . Consider the Cauchy problem:

$$(2.2) \quad \begin{cases} P(x, D)u(x) = v(x), \\ D_1^h u(x)|_Y = \omega_h(x'), \quad 0 \leq h < m, \end{cases}$$

where the  $\omega_h$ 's are tempered holomorphic functions on  $Y$  logarithmically ramified along  $Z$  and  $v(x) = \sum \psi_i(x)$  with the  $\psi_i$ 's being tempered holomorphic functions on  $X$  logarithmically ramified along  $Z_i$ .

Let us describe what we mean by holomorphic functions logarithmically ramified. Consider on  $\mathbb{C}$  the functions which can be written as  $f_{-1}(z) + f_0(z) \ln(z) + \sum_{i=1}^k f_i(z) z^{-i}$ , where  $f_i$  is an holomorphic function for  $i = -1, \dots, k$ . Let now  $X$  be an open set in  $\mathbb{C}^n$  containing 0,  $Y$  be an hypersurface in  $X$ , whose local equation we suppose to be  $z_1 = 0$ . Define  $\mathcal{O}_{[Y|X]}^1$  the set of functions which can be written as

$$(2.3) \quad f_{-1}(z) + f_0(z) \ln(z_1) + \sum_{i=1}^k f_i(z) z_1^{-i},$$

where  $f_i$  is an holomorphic function for  $i = -1, \dots, k$ . By  $\mathcal{O}_X$  (respectively  $\mathcal{E}_X$ ) we denote the sheaf of holomorphic finite order differential operators on  $X$  (resp. microdifferential operators on  $X$ , see [S1] for an exposition). We denote by  $D^b(\mathcal{O}_X)$  (resp.  $D_{coh}^b(\mathcal{O}_X)$ ) the derived category of the category of sheaves of left  $\mathcal{O}_X$ -modules with bounded (resp. bounded and coherent) cohomology. We define analogously the categories  $D^b(\mathcal{E}_X)$  and  $D_{coh}^b(\mathcal{E}_X)$ . We will also use the category  $D^b(\mathcal{E}_{X|\mathcal{Q}})$ . The function space  $\mathcal{O}_{[Y|X]}^1$  can be described as

$$(2.4) \quad \mathcal{N} = \frac{\mathcal{O}_X}{\mathcal{O}_X D_1 z_1 D_1}$$

which is a coherent  $\mathcal{O}_X$ -module. In a more general setting, let us consider a hypersurface  $Z = \{g^{-1}(0)\}$  with  $g: X \rightarrow \mathbb{C}$  a holomorphic function with  $dg \neq 0$ . Set  $L_{\{0\}|\mathbb{C}}^1 = R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{O})$ . We write  $\mathcal{O}b_{X_R}$  for the sheaf of distributions on  $X_R$ . We shall make use of Kashiwara's functors (see [K1], [K-S3]):

$$(2.5) \quad \begin{cases} \mathcal{H}om(\cdot, \mathcal{O}b): D_{R-c}^b(C_X)^\circ \rightarrow D(\mathcal{O}_X), \\ \mathcal{H}om(\cdot, \mathcal{O}_X) = R\mathcal{H}om_{\mathcal{O}_{\bar{X}}}(\mathcal{O}_{\bar{X}}, \mathcal{H}om(\cdot, \mathcal{O}b_{X_R})). \end{cases}$$

We define  $\mathcal{H}om(g^{-1}L_{\{0\}|\mathbb{C}}^1, \mathcal{O}_X) = \mathcal{O}_{[Z|X]}^1$ , which is the  $\mathcal{O}_X$ -module of holomorphic functions with logarithmic ramifications along  $Z$ .

Let us reformulate the Cauchy problem in this new setting. We have the following lemma.

**LEMMA 2.1** [S1, Prop. 2.2.2], [D'A-S1, Prop. 3.1.3]. *Under the previous hypotheses there exist smooth hypersurfaces  $Z_1, \dots, Z_r$  of  $X$  pairwise transversal, transversal to  $Y$ , and such that  $Z_i \cap Y = Z$  for every  $i$ . Moreover, for a neighborhood  $W$  of  $T_Z^*Y$ ,*

$$(2.6) \quad \text{char}(\mathfrak{N}) \cap {}^t f'^{-1}(T_Z^*Y) \subset \bigcup_{i=1}^r T_{Z_i}^*X \cup T_X^*X.$$

Let us assume that, locally,  $Y = \{z_1 = 0\}$ . We consider the following Cauchy problem:

$$(2.7) \quad \begin{cases} Pu = f, \\ D_1^h u|_Y = w_h, \quad h = 0, \dots, m-1, \end{cases}$$

where  $f \in \sum_{j=1}^r f_j$  with  $f_j \in \mathcal{O}_{[z_j|X]}^1$  and  $w_h \in \mathcal{O}_{[Z|Y]}^1$ . We will prove the following theorem.

**THEOREM 2.2.** *The Cauchy problem (2.7) is locally well posed.*

Remark that in general there exist complex analytic functions  $g: Y \rightarrow \mathbb{C}$  with  $dg \neq 0$  and  $g^{-1}(0) = Z$ ,  $g_1, \dots, g_r: X \rightarrow \mathbb{C}$  with  $dg_i \neq 0$  and  $g_i^{-1}(0) = Z_i$ , such that  $g_i \circ f = g$ .

Set  $L := g^{-1}L^1_{\{0\}|\mathbb{C}}$  and  $K_i := g_i^{-1}L^1_{\{0\}|\mathbb{C}}$ . Define  $K$  to be the first term of a distinguished triangle

$$(2.8) \quad K \rightarrow \bigoplus_{i=1}^r K_i \xrightarrow{h} \bigoplus_{i=1}^{r-1} C_X \xrightarrow{+1}$$

where  $h$  is the composite of the natural morphism  $\bigoplus_{i=1}^r \tau_i: \bigoplus_{i=1}^r K_i \rightarrow \bigoplus_{i=1}^r C_X$  and the map  $\bigoplus_{i=1}^r C_X \rightarrow \bigoplus_{i=1}^{r-1} C_X$  given by  $(a_1, \dots, a_r) \mapsto (a_2 - a_1, \dots, a_r - a_{r-1})$ . We also write  $\sum_{i=1}^r \mathcal{O}^1_{[Z_i|X]} = \mathfrak{H}om(K, \mathcal{O}_X)$  the complex of holomorphic tempered functions on  $X$  with ramifications of logarithmic type along the  $Z_i$ 's.

Let us see how one can probe this theorem. We can restrict ourselves to solving separately the two Cauchy problems:

$$(2.9) \quad \begin{cases} Pu = 0, \\ D_1^h u = w_h, \end{cases}$$

and

$$(2.10) \quad Pu = f.$$

The two of them, together, will be proven if we prove the following:

$$(2.11) \quad R \mathcal{H}om_{\mathcal{O}_X} \left( \mathcal{K}, \sum_{i=1}^r \mathcal{O}^1_{[Z_i|X]} \right) \Big|_Z \simeq R \mathcal{H}om_{\mathcal{O}_Y} (\mathcal{K}_Y, \mathcal{O}^1_{[Z|Y]}) \Big|_Z,$$

where  $\mathcal{K}$  denotes the left  $\mathcal{O}_X$ -module  $\mathcal{O}_X/\mathcal{O}_X P$  and  $\mathcal{K}_Y$  its inverse image with respect to the embedding  $f: Y \hookrightarrow X$ . Equation (2.11) rewrites as:

$$(2.12) \quad R \mathcal{H}om_{\mathcal{O}_X} (\mathcal{K}, \mathfrak{H}om(K, \mathcal{O}_X)) \Big|_Z \simeq R \mathcal{H}om_{\mathcal{O}_Y} (\mathcal{K}_Y, \mathfrak{H}om(L, \mathcal{O}_Y)) \Big|_Z.$$

We will prove this last isomorphism.

We introduce Anronikof's functor  $\mathfrak{E}\mu\text{hom}(\cdot, \mathcal{O}b): D_{\mathbb{R}-c}^b(C_X) \rightarrow D^b(\pi^{-1}\mathcal{O}_X)$  and its holomorphic analogue,  $\mathfrak{E}\mu\text{hom}(\cdot, \mathcal{O}_X) = R \mathcal{H}om_{\mathcal{O}_{\bar{X}}}(\mathcal{O}_{\bar{X}}, \mathfrak{E}\mu\text{hom}(\cdot, \mathcal{O}b))$  (see [A]). We will make use of the notion of *perverse sheaves* (see [K-S1, Chapter 10] for an exposition). We remark that in our situation  $\mathfrak{E}\mu\text{hom}(K, \mathcal{O}_X)$  and  $\mathfrak{E}\mu\text{hom}(L, \mathcal{O}_X)$  are concen-

trated in degree zero. In fact,  $L_{\{0\}|C}^1$  is a perverse sheaf on  $C$  (see [K-S1, Chapter 10]) and that, being  $g$  a finite map,  $L = g^{-1}L_{\{0\}|C}^1$  is a perverse sheaf on  $Y$  (see [K-S1, Prop. 10.3.7]); in particular it is  $R$ -constructible. We have the following lemma.

LEMMA 2.3 [A, Cor. 5.6.1]. *If  $F \in \text{Ob}(\mathbf{D}^b(C_X))$  and  $F$  is perverse, then  $\mathfrak{E}\mu\text{hom}(F, \mathcal{O}_X)$  is concentrated in degree zero.*

Applying this lemma we have that  $\mathfrak{E}\mu\text{hom}(L, \mathcal{O}_Y)$  is concentrated in one degree. Moreover, as above, we have that  $K_i := g_i^{-1}L_{\{0\}|C}^1$  is a perverse sheaf for every  $i$ , because  $g_i$  is a finite map. We can then apply Lemma 2.3 (see also [A, Cor. 5.6.1]) to get that  $\mathfrak{E}\mu\text{hom}(K_i, \mathcal{O}_X)$  is concentrated in one degree. We have that, out of the zero section  $T_X^*X$ :

$$(2.13) \quad \mathfrak{E}\mu\text{hom}(K, \mathcal{O}_X) \xrightarrow{\sim} \bigoplus_{i=1}^r \mathfrak{E}\mu\text{hom}(K_i, \mathcal{O}_X)$$

so that, out of the zero section,  $\mathfrak{E}\mu\text{hom}(K, \mathcal{O}_X)$  is concentrated in one degree.

We are dealing with the issue that  $\mathfrak{E}\mu\text{hom}(K, \mathcal{O}_X)$  and other sheaves obtained by applying the functor  $\mathfrak{E}\mu\text{hom}(\cdot, \mathcal{O})$  be concentrated in one degree. This is a sufficient condition in order to make  $\mathfrak{E}\mu\text{hom}(K, \mathcal{O}_X)$  into an object of the category  $\mathbf{D}^b(\mathcal{E}_X)$  (see also [A, 5.6.1]). Alternatively, this condition may be substituted by the requirement that locally out of  $T_X^*X$  (respectively  $T_Y^*Y$ )  $\mathfrak{E}\mu\text{hom}(K, \mathcal{O}_X)$  (resp.  $\mathfrak{E}\mu\text{hom}(L, \mathcal{O}_Y)$ ) be a well defined object in  $\mathbf{D}^b(\mathcal{E}_X)$  (resp.  $\mathbf{D}^b(\mathcal{E}_Y)$ ); for this other condition refer to Section 4 and the Appendix. We remark that it is conjectured that  $\mathfrak{E}\mu\text{hom}(K, \mathcal{O}_X)$  be naturally an object of  $\mathbf{D}^b(\mathcal{E}_X)$ , but we are not taking it for granted in the present paper.

LEMMA 2.4. *a) There is a canonical isomorphism:*

$$(2.14) \quad \begin{aligned} R\pi_X * R\mathcal{H}om_{\omega_X}(\mathcal{N}, \mathfrak{E}\mu\text{hom}(K, \mathcal{O}_X)) &= \\ &= R\mathcal{H}om_{\omega_X}(\mathcal{N}, \mathfrak{E}\text{hom}(K, \mathcal{O}_X)). \end{aligned}$$

*b) (See [D'A-S1, Proof Th. 2.1.1]) Suppose  $P$  has simple characteristics and is non characteristic with respect to  $Y$ , then:*

$$(2.15) \quad \begin{aligned} R\pi_Y * R^t f'_! f_\pi^{-1} R\mathcal{H}om_{\omega_X}(\mathcal{N}, \mathfrak{E}\mu\text{hom}(K, \mathcal{O}_X)) &= \\ &= f^{-1} R\mathcal{H}om_{\omega_X}(\mathcal{N}, \mathfrak{E}\text{hom}(K, \mathcal{O}_X)). \end{aligned}$$

PROOF. *a)* It is obvious.

*b)* By our hypothesis  $f$  is non-characteristic for  $\mathcal{N}$ . We consider

the following commutative diagram:

$$(2.16) \quad \begin{array}{ccccc} T^*Y & \xleftarrow{t'f} & Y \times_X T^*X & \xrightarrow{f_\pi} & T^*X \\ \pi_Y \downarrow & & \pi \downarrow & & \pi_X \downarrow \\ Y & \xleftarrow{\text{id}_Y} & Y & \xrightarrow{f} & X \end{array}$$

We have the chain of isomorphisms:

$$(2.17) \quad \begin{aligned} R\pi_Y * R^t f'_! f_\pi^{-1} R \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{E}\mu\text{hom}(K, \mathcal{O}_X)) &\simeq \\ &\simeq R\pi_Y * R^t f'_* f_\pi^{-1} R \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{E}\mu\text{hom}(K, \mathcal{O}_X)) \simeq \\ &\simeq R\pi * f_\pi^{-1} R \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{E}\mu\text{hom}(K, \mathcal{O}_X)) \simeq \\ &\simeq f^{-1} R\pi_X * R \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{E}\mu\text{hom}(K, \mathcal{O}_X)) \simeq \\ &\simeq f^{-1} R \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{E}\mu\text{hom}(K, \mathcal{O}_X)), \end{aligned}$$

where the first isomorphism is due to the fact (being  $f$  non-characteristic for  $\mathcal{N}$ ) that  $t'f'$  is pro-perfect on  $f_\pi^{-1} \text{char}(\mathcal{N})$  and hence on  $f_\pi^{-1} \text{supp}(R \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{E}\mu\text{hom}(K, \mathcal{O}_X)))$  and the third to the fact that, being  $R \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{E}\mu\text{hom}(K, \mathcal{O}_X))$  conic,

$$(2.18) \quad \begin{aligned} R\pi_X * R \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{E}\mu\text{hom}(K, \mathcal{O}_X)) &\simeq \\ &\simeq i^{-1} R \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{E}\mu\text{hom}(K, \mathcal{O}_X)), \end{aligned}$$

$i$  denoting the immersion of the zero-section  $X$  in  $T^*X$ . The last isomorphism follows from (a). ■

Then our Cauchy problem is reduced to proving the following:

$$(2.19) \quad \begin{aligned} R\pi_Y * R^t f'_! f_\pi^{-1} R \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{E}\mu\text{hom}(K, \mathcal{O}_X)) &\simeq \\ &\simeq R\pi_Y * R \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{N}_Y, \mathcal{E}\mu\text{hom}(L, \mathcal{O}_Y)). \end{aligned}$$

This isomorphism is indeed, by use of Sato's distinguished triangle, embedded in a morphism of distinguished triangles:

$$(2.20) \quad \begin{array}{ccccccc} R\pi_{Y!} A & \longrightarrow & R\pi_Y * A & \longrightarrow & R\dot{\pi}_Y * A & \xrightarrow{+1} \\ a \downarrow & & \beta \downarrow & & \gamma \downarrow & \\ R\pi_{Y!} B & \longrightarrow & R\pi_Y * B & \longrightarrow & R\dot{\pi}_Y * B & \xrightarrow{+1} \end{array}$$

with

$$(2.21) \quad \begin{aligned} A &= R^t f'_! f_\pi^{-1} R \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathfrak{E}\mu\text{hom}(K, \mathcal{O}_X)), \\ B &= R \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{N}_Y, \mathfrak{E}\mu\text{hom}(L, \mathcal{O}_Y)). \end{aligned}$$

Taking Lemma 2.4(a) into account, our theorem will be proven if we get to prove that  $\beta$  is an isomorphism. In order to achieve that we will prove that  $\alpha$  and  $\gamma$  are isomorphisms.

We have to prove that:

$$(2.22) \quad R\pi_{Y!} A \simeq R\pi_{Y!} B.$$

We have:

$$(2.23) \quad \begin{aligned} R\pi_{Y!} A &\simeq R\pi_{Y!} R^t f'_! f_\pi^{-1} R \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathfrak{E}\mu\text{hom}(K, \mathcal{O}_X)) \simeq \\ &\simeq R\pi_{Y!} f_\pi^{-1} R \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathfrak{E}\mu\text{hom}(K, \mathcal{O}_X)) \simeq \\ &\simeq f^{-1} \pi_{X!} R \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathfrak{E}\mu\text{hom}(K, \mathcal{O}_X)) \simeq \\ &\simeq f^{-1} \pi_{X!} R \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mu\text{hom}(K, \mathcal{O}_X)), \end{aligned}$$

where the last isomorphism follows from [A, Prop. 3.1.4]. On the other hand, thanks again to [A, Prop. 3.1.4], we have:

$$(2.24) \quad \begin{aligned} R\pi_{Y!} B &\simeq R\pi_{Y!} R \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{N}_Y, \mathfrak{E}\mu\text{hom}(L, \mathcal{O}_Y)) \simeq \\ &\simeq R\pi_{Y!} R \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{N}_Y, \mu\text{hom}(L, \mathcal{O}_Y)). \end{aligned}$$

But  $\mathcal{N}$  is a coherent module non-characteristic with respect to  $f$ , so also  $\mathcal{N}_Y$  is a coherent  $\mathcal{O}_Y$ -module, so by standard arguments it suffices to prove:

$$(2.25) \quad \begin{aligned} f^{-1} R\pi_{X!} \mu\text{hom}(K, R \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}\mathcal{O}_X)) &\simeq \\ &\simeq R\pi_{Y!} \mu\text{hom}(L, R \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{N}_Y, \mathcal{O}_Y)), \end{aligned}$$

that is (by the Cauchy-Kowalevski theorem):

$$(2.26) \quad \begin{aligned} f^{-1} R\pi_{X!} \mu\text{hom}(K, R \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}\mathcal{O}_X)) &\simeq \\ &\simeq R\pi_{Y!} \mu\text{hom}(L, f^{-1} R \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{O}_X)). \end{aligned}$$

One has a technical lemma.

LEMMA 2.5 [D'A-S1, Lemma 3.2.1]. *There exists a natural map  $\tau: L \rightarrow C_C$ . Moreover, applying the functor  $R\Gamma_{\{0\}}(\cdot)$  to this map we get*



an isomorphism:

$$(2.27) \quad \mathbf{R}\Gamma_{\{0\}} L_{\{0\}|\mathbb{C}}^1 \xrightarrow{\sim} \mathbf{R}\Gamma_{\{0\}} \mathbb{C}_{\mathbb{C}}.$$

This lemma implies:

LEMMA 2.6. *A morphism  $\psi: L \rightarrow f^{-1}K$  is induced by (2.27). The isomorphism induced by  $\psi$ ,  $\mathbf{R}\Gamma_{\{y\}}(L \otimes \omega_Y) \rightarrow \mathbf{R}\Gamma_{\{y\}}(K \otimes \omega_X)$ , is an isomorphism for every  $y \in Z$ .*

Then it is possible to argue as in [D'A-S1, § 2A] to conclude that  $\alpha$  is an isomorphism. So what is left to do is to prove that  $\gamma$  is an isomorphism. Our thesis is now that the natural morphism

$$(2.28) \quad \mathbf{R}\dot{\pi}_{Y*} A \rightarrow \mathbf{R}\dot{\pi}_{Y*} B$$

is an isomorphism. In other words that, for every  $p_Y \in \dot{\pi}_Y^{-1}(Z)$ ,

$$(2.29) \quad \begin{aligned} \mathbf{R}^t f'_! f_\pi^{-1} \mathbf{R} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{E} \mu \text{hom}(K, \mathcal{O}_X))_{p_Y} \rightarrow \\ \rightarrow \mathbf{R} \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{N}_Y, \mathcal{E} \mu \text{hom}(L, \mathcal{O}_Y))_{p_Y} \end{aligned}$$

is an isomorphism. Thanks to  $\text{char}(P)$  being transversal to  $Y \times_X T^*X$  and the hypothesis of simple characteristics, there exist  $p_1, \dots, p_r \in {}^t f'^{-1}(p_Y)$  such that:

$$(2.30) \quad {}^t f'^{-1}(p_Y) \cap f_\pi^{-1}(\text{char}(P)) \subset \{p_1, \dots, p_r\}$$

we then have that  $f$  is non-characteristic for  $K$  and  $p_1, \dots, p_r$  are isolated in  ${}^t f'^{-1}(p_Y) \cap f_\pi^{-1}(\text{SS}(K))$ . Then (see [D'A-S1, Lemma 2.1.3]) we can find sheaves  $K_1, \dots, K_r$  on  $X$  such that:

- 1) there exists a distinguished triangle  $\bigoplus_{i=1}^r K_i \rightarrow K \rightarrow K_0 \xrightarrow{+1}$ ;
- 2)  $f_\pi^{-1} \text{SS}(K_i) \cap {}^t f'^{-1}(p_Y) \subset \{p_i\}$  and  $p_i \notin \text{SS}(K_0)$  for  $i = 1, \dots, r$ ;
- 3)  $f_{\mu, p_i}^{-1} K_i = f^{-1}K$  at  $p_Y$  for  $i = 1, \dots, r$  (for the notation  $f_{\mu, p_i}^{-1}$  see Lemma (2.7)).

Also (see [P-S, Lemma 4.2], [S-K-K, Chaper II, Th. 2.2.2]) we can find  $\mathcal{E}_X$ -modules  $\mathcal{N}_1, \dots, \mathcal{N}_r$  such that:

- 4)  $\mathcal{E}_X \otimes_{\mathcal{O}_X} \mathcal{N} \simeq \bigoplus_{i=1}^r \mathcal{N}_i$  in  $\mathbf{D}^b(\mathcal{E}_X|_{f_\pi {}^t f'^{-1}(p_Y)})$ ;
- 5)  $f_\pi^{-1} \text{SS}(\mathcal{N}_i) \cap {}^t f'^{-1}(p_Y) \subset \{p_i\}$ ;
- 6)  $f^{-1}(\mathcal{E}_X \otimes_{\mathcal{O}_X} \mathcal{N}) \simeq \bigoplus_{i=1}^r f^{-1} \mathcal{N}_i$  in  $\mathbf{D}^b(\mathcal{E}_Y|_{p_Y})$ .

Let us now briefly clarify condition (3)

LEMMA 2.7 [K-S1, Prop. 6.9.1], [D'A-S1, Lemma 1.3.4]. *Let  $F \in \text{Ob}(\mathbf{D}^b(C_X; p))$ . Suppose that:*

$$(2.31) \quad {}^t f'^{-1}({}^t f'(p)) \cap f_\pi^{-1}(\text{SS}(F)) \subset \{p\} \text{ in a neighborhood of } p.$$

*Then:*

a) *There exists  $F_{Y,p} \in \text{Ob}(\mathbf{D}^b(C_X))$  and a morphism in  $\mathbf{D}^b(C_X)$ ,  $F_{Y,p} \rightarrow F$  such that:*

$$(2.32) \quad {}^t f'^{-1}({}^t f'(p)) \cap f_\pi^{-1}(\text{SS}(F_{Y,p})) \subset \{p\},$$

*and  $f$  is noncharacteristic for  $F_{Y,p}$ ; moreover  $F_{Y,p} \rightarrow F$  is an isomorphism at  $p$ .*

b) *For  $F \in \text{Ob}(\mathbf{D}^b(C_X; p))$  satisfying (2.31), the object  $f^{-1}F_{Y,p}$  (resp.  $f^!F_{Y,p}$ ) of  $\mathbf{D}^b((C_Y; {}^t f'(p)))$  does not depend (up to isomorphism) on the choice of  $F_{Y,p}$ .*

In the situation of Lemma 2.7 we define the *microlocal inverse image* of  $Y$  by  $f_{\mu,p}^{-1}F = f^{-1}F_{Y,p}$ . Hence  $f_{\mu,p}^{-1}$  is a functor from the full subcategory of  $\mathbf{D}^b(C_X; p)$  the objects of which verify (2.31) to  $\mathbf{D}^b((C_Y; {}^t f'(p)))$ .

Remark that the  $K_i$ 's are R-constructible. In fact in [D'A-S1, Lemma 2.1.3] the  $K_i$ 's are obtained by a «microlocal cut-off». See [A, § A.1] for the «microlocal cut-off lemma» [K-S1, Prop. 6.1.4] in the framework of R-constructible sheaves.

Due to  $p_i \notin \text{SS}(K_0)$  and the distinguished triangle  $\bigoplus_{i=1}^r K_i \rightarrow K \rightarrow K_0 \xrightarrow{+1}$ , the  $\mathfrak{E}\mu\text{hom}(K_i, \mathcal{O}_X)$ 's are concentrated in one degree. We then get the chain of isomorphisms:

$$\begin{aligned} (2.33) \quad R^t f'_! f_\pi^{-1} R \mathcal{H}om_{\omega_X}(\mathcal{N}, \mathfrak{E}\mu\text{hom}(K, \mathcal{O}_X))_{p_Y} &\simeq \\ &\simeq R^t f'_! f_\pi^{-1} R \mathcal{H}om_{\varepsilon_X} \left( \varepsilon_X \overset{L}{\otimes}_{\omega_X} \mathcal{N}, \mathfrak{E}\mu\text{hom}(K, \mathcal{O}_X) \right)_{p_Y} \simeq \\ &\simeq R^t f'_! f_\pi^{-1} \bigoplus_{i=1}^r R \mathcal{H}om_{\varepsilon_X} \left( \varepsilon_X \overset{L}{\otimes}_{\omega_X} \mathcal{N}, \mathfrak{E}\mu\text{hom}(K_i, \mathcal{O}_X) \right)_{p_i} \simeq \\ &\simeq \bigoplus_{i=1}^r R^t f'_! f_\pi^{-1} R \mathcal{H}om_{\varepsilon_X} \left( \varepsilon_X \overset{L}{\otimes}_{\omega_X} \mathcal{N}, \mathfrak{E}\mu\text{hom}(K_i, \mathcal{O}_X) \right)_{p_i} \simeq \\ &\simeq \bigoplus_{i=1}^r R^t f'_! f_\pi^{-1} R \mathcal{H}om_{\varepsilon_X}(\mathcal{N}_i, \mathfrak{E}\mu\text{hom}(K_i, \mathcal{O}_X))_{p_i}. \end{aligned}$$

Remark now that, thanks to the assumption that  $\text{char}(P)$  be transversal to  $Y \times_X T^*X$ , we can apply [S1, Cor. A.4.5] with  $W = Y \times_X T^*X$  and  $V = \text{char}(P)$ . Let us give a definition.

**DEFINITION 2.8.** Let  $V$  and  $T$  be two conic smooth involutive manifolds in a neighborhood of  $p \in \dot{T}^*X$ ,  $V$  being regular involutive. We say that  $V$  is *non glancing* with respect to  $T$  if for any function  $\phi$  defined in a neighborhood of  $p$ , such that  $\phi|_T = 0$  and  $d\phi \neq 0$ , the vector  $H_\phi$  is not tangent to  $V$ . If  $Y$  is a submanifold of  $X$  we say that  $V$  is *non glancing with respect to  $Y$*  if  $V$  is non glancing with respect to  $Y \times_X T^*X$ .

**LEMMA 2.9** [S1, Cor. A.4.5], [S1, Cor. I.6.2.3]. *Let  $V$  and  $W$  be two conic involutive manifolds in a neighborhood of  $p \in \dot{T}^*X$ ,  $V$  being regular. Assume that  $V$  is non glancing with respect to  $W$ . Then there exists a system of local homogeneous symplectic coordinate  $(x; \xi)$  such that:*

$$(2.34) \quad \begin{cases} p = (0; dx_n), \\ V = \{(x; \xi); \xi_1 = \dots = \xi_r = 0\}, & r < n, \\ W = \{(x; \xi); x_1 = \dots = x_d = 0\}, & d \leq r. \end{cases}$$

We may quantize the contact transformation using the theory of contact transforms for the functor  $\mathfrak{E}\mu\text{hom}(\cdot, \mathcal{O}_X)$  as developed in [A, Chaper 5]. Now, as  $\mathfrak{N}_i$  is determined up to isomorphism by  $\text{char}(P)$  (see [S1, Cor. I.6.2.3.]) we may suppose that in a local chart:

$$(2.35) \quad \mathfrak{N}_i = \frac{\varepsilon_X}{\varepsilon_X D_1 + \dots + \varepsilon_X D_r}$$

where  $r := \text{codim}(\text{char}(P))$ . We just treat the case  $\text{codim}\text{char}(P) = 1$ , the other cases being more general but equally treatable by this argument. So suppose  $\mathfrak{N}_i = \varepsilon_X / (\varepsilon_X D_1)$ . We remark that in the present situation  $\varepsilon_X / (\varepsilon_X D_1) = \varepsilon_{X \rightarrow Y}$  and  $\mathfrak{N}_{Y, p_i} \simeq \varepsilon_Y$ , where  $\mathfrak{N}_{Y, p_i}$  denotes the  $\varepsilon_X$ -module inverse image of  $\mathfrak{N}$  at  $p_i$ . We remark that locally  $X = Y \times Z$ , with  $Y$  and  $Z$  being local charts, with a projection  $p: X \rightarrow Y$  and  $p \circ f = \text{id}_Y$ . Thanks to [K-S1, Prop. 6.6.2] we have  $K_i = p^{-1}L' \exists L' \in D^b(C_X)$ . But  $p \circ f = 1_Y$ , so that  $K_i = p^{-1}L$ .

So we have:

$$\begin{aligned} R^t f'_! f_\pi^{-1} R \mathfrak{I}Com_{\varepsilon_X}(\mathfrak{N}_i, \mathfrak{E}\mu\text{hom}(K_i, \mathcal{O}_X))_{p_i} &\simeq \\ &\simeq R^t f'_! f_\pi^{-1} R \mathfrak{I}Com_{\varepsilon_X}(\varepsilon_{X \rightarrow Y}, \mathfrak{E}\mu\text{hom}(p^{-1}L, \mathcal{O}_X))_{p_i}. \end{aligned}$$

PROPOSITION 2.10. *Let  $p: X \rightarrow Y$  be a smooth morphism of complex analytic manifolds and let  $L \in \mathbf{D}_{\mathbb{R}-c}^b(C_Y)$ . Then one has a canonical isomorphism:*

$$(2.36) \quad \mathbf{R}^t p'_* p_\pi^{-1} \mathfrak{E} \mu \text{hom}(L, \mathcal{O}_Y) \xrightarrow{\sim} \\ \xrightarrow{\sim} R \mathfrak{H} \text{Com}_{\omega_X}(\omega_{X \rightarrow Y}, \mathfrak{E} \mu \text{hom}(p^{-1}L, \mathcal{O}_X)).$$

PROOF. We apply the isomorphism (see [K-S3, (5.5)]):

$$(2.37) \quad R \mathfrak{H} \text{Com}_{p^{-1}\omega_Y}(p^{-1}\mathcal{O}_{\bar{Y}}, R \mathfrak{H} \text{Com}_{\omega_{\bar{X}}}(\omega_{X_{\mathbb{R}} \rightarrow Y_{\mathbb{R}}}, N)) \simeq \\ \simeq R \mathfrak{H} \text{Com}_{\omega_X}(\omega_{X \rightarrow Y}, R \mathfrak{H} \text{Com}_{\omega_{\bar{X}}}(\mathcal{O}_{\bar{X}}, N)),$$

with  $N$  replaced by  $\mathfrak{E} \mu \text{hom}(p^{-1}L, \omega_{b_X})$  and then we apply the following proposition. ■

PROPOSITION 2.11. *Let  $p: X \rightarrow Y$  be a smooth morphism of real analytic manifolds and let  $L \in \mathbf{D}_{\mathbb{R}-c}^b(C_Y)$ . Then one has a canonical isomorphism:*

$$(2.38) \quad \mathbf{R}^t p'_* p_\pi^{-1} \mathfrak{E} \mu \text{hom}(L, \omega_{b_Y}) \xrightarrow{\sim} \\ \xrightarrow{\sim} R \mathfrak{H} \text{Com}_{\omega_X}(\omega_{X \rightarrow Y}, \mathfrak{E} \mu \text{hom}(p^{-1}L, \omega_{b_X})).$$

When restricting (2.38) to the zero section, one recovers the isomorphism (see [K2], [A, Prop. 1.1.3], [K-S3, Th. 4.5(i)]):

$$(2.39) \quad R \mathfrak{H} \text{Com}_{\omega_X}(\omega_{X \rightarrow Y}, \mathfrak{E} \text{hom}(p^{-1}L, \omega_{b_X})) \xrightarrow{\sim} p^{-1} \mathfrak{E} \text{hom}(L, \omega_{b_Y}).$$

PROOF. The morphism in (2.38) is obtained by adjunction from the morphism

$$(2.40) \quad \mathbf{R}^t p'_! \left( \omega_{X \rightarrow Y} \otimes_{p^{-1}\omega_Y}^L p_\pi^{-1} \mathfrak{E} \mu \text{hom}(L, \omega_{b_Y}) \right) \rightarrow \\ \rightarrow \mathfrak{E} \mu \text{hom}(p^!L, \omega_{b_X})[\dim(X) - \dim(Y)].$$

This last morphism is deduced from (2.39) by the same method as Andronikof proves Theorem [A, Th. 3.3.6]. We have to show that the morphism in (2.38) is an isomorphism in  $\mathbf{D}^b(C_X)$ . When restricted to the zero section of  $T^*Y$ , (2.38) is nothing but the inverse of

the isomorphism (2.39). Let then  $\xi \in T^*Y$ . We have the embedding  ${}^t p': T^*Y \hookrightarrow T^*X$ . We will write  $\xi$  for  ${}^t p'(\xi)$ .

As we have to examine the morphism induced on the stalks, the claim is of local nature. Then we can suppose, until the end of the proof,  $X := Y \times Z$  with  $Y$  and  $Z$  finite dimensional real vector spaces,  $\dim(X) = n$ ,  $\dim(Y) = m$ . Let  $p$  and  $q$  be the projections  $p: X \rightarrow Y$ ,  $q: X \rightarrow Z$ , and  $\pi_Y(\xi) = 0$ . Also recall the maps  $q_1: X \times X \rightarrow X$ ;  $(x, x') \mapsto x$  and  $q_2: X \times X \rightarrow X$ ;  $(x, x') \mapsto x'$ . Let us begin by stating two lemmas.

**LEMMA 2.12.** *Let  $\gamma$  be a closed cone in  $X$  with  $p$  proper on  $\gamma$ . Take  $W \subset W'$  open neighborhoods of 0 in  $Z$ ,  $V \subset V'$  open neighborhoods of 0 in  $Y$ . Suppose that:*

$$(2.41) \quad q(\gamma \cap p^{-1}(V' + V^a)) + W \subset W',$$

where  $V^a := -V$ . Then:

$$(2.42) \quad \begin{aligned} Z(\gamma) \cap q_2^{-1}(V' \times W') \cap q_1^{-1}(V \times W) = \\ = Z(\gamma) \cap q_2^{-1}(V' \times Z) \cap q_1^{-1}(V \times W). \end{aligned}$$

**PROOF.** Consider the following diagram:

$$(2.43) \quad \begin{array}{ccccccc} Y & \xleftarrow{p} & Y \times Z & \xleftarrow{q_1} & X \times X & \xrightarrow{q_2} & Y \times Z & \xrightarrow{p} & Y \\ & & \downarrow q & & & & \downarrow q & & \\ & & Z & & & & Z & & \end{array}$$

Let  $(x, x') \in Z(\gamma)$  with  $x \in V \times W$  and  $x' \in V' \times Z$ . Then  $p(x' - x) \in V' - V$  implies  $x' - x \in p^{-1}(V' - V)$ . Also,  $x' - x \in \gamma$ , so:

$$(2.44) \quad x' - x \in \gamma \cap p^{-1}(V' - V)$$

hence

$$(2.45) \quad q(x' - x) \in q(\gamma \cap p^{-1}(V' - V)).$$

This together with  $q(x) \in W$  give us:

$$(2.46) \quad q(x') = q(x' - x) + q(x) \in q(\gamma \cap p^{-1}(V' - V)) + W$$

and, by (2.41),  $q(x') \in W'$ ; in other words  $(x, x') \in Z(\gamma) \cap q_2^{-1}(V' \times W') \cap q_1^{-1}(V \times W)$ . ■

LEMMA 2.13. *Let  $\gamma$  be a closed cone in  $X$ . Let  $p|_\gamma$  be proper with convex fibers. Let  $\gamma' := p(\gamma)$  and let  $G \in \mathbf{D}^b(C_Y)$ . Then:*

$$(2.47) \quad p^{-1}(C_{Z(\gamma')} \circ G) \simeq C_{Z(\gamma)} \circ p^{-1}G.$$

PROOF. Let  $\Delta_p := \{(x, y) \in X \times Y; y = p(x)\}$ . Then one has:

$$(2.48) \quad C_{Z(\gamma)} \circ C_{\Delta_p} \simeq q_{13}!(C_{(Z(\gamma) \times Y) \cap (X \times \Delta_p)}) \simeq C_{\{(x, y) \in X \times Y; (x + \gamma) \cap p^{-1}(y) \neq \emptyset\}},$$

where  $q_{13}: X \times X \times Y \rightarrow X \times Y$  is the projection on the first and third factors. Analogously:

$$(2.49) \quad C_{\Delta_p} \circ C_{Z(\gamma')} \simeq q_{13}!(C_{(\Delta_p \times Y) \cap (X \times Z(\gamma'))}) \simeq C_{\{(x, y) \in X \times Y; y \in p(x) + \gamma'\}}.$$

So we have  $C_{\Delta_p} \circ C_{Z(\gamma')} \simeq C_{Z(\gamma)} \circ C_{\Delta_p}$ . ■

END OF PROOF OF PROPOSITION (2.11). We make use of the stalk formula for  $\mathfrak{E}\mu\text{hom}(\cdot, \mathcal{O}b_Y)$  at  $\xi$  (see [A, Prop. 2.3.3] and also [K-S1, Prop. 3.5.4] and proof of [K-S1, Prop. 4.4.4]). We get:  $\forall j$

$$(2.50) \quad H^j(\mathfrak{E}\mu\text{hom}(L, \mathcal{O}b_Y))_\xi = \lim_{\substack{\rightarrow \\ V, \gamma'}} H^j R\Gamma(V; \mathfrak{E}\text{hom}(C_{Z(\gamma')} \circ L_V, \mathcal{O}b_Y)),$$

where  $\gamma'$  is a convex proper closed subanalytic cone in  $\mathbb{R}^m$  such that  $\gamma' \subset \text{Int}(\{\xi\}^{0a}) \cup \{0\}$ , and  $V$  is a subanalytic open neighborhood of  $\pi(p'(\xi)) = 0$  in  $Y$ .

Then we use the stalk formula for  $\mathfrak{E}\mu\text{hom}(\cdot, \mathcal{O}b_X)$ . Let  $\gamma$  be a convex proper closed subanalytic cone in  $\mathbb{R}^n$  such that  $\gamma \subset \text{Int}(\{\xi\}^{0a}) \cup \{0\}$ , and let  $U, U'$  be subanalytic open neighborhoods of  $\pi(\xi) = 0$  in  $X$ , with  $U \subset U'$ . It is not restrictive to take  $U = V \times W, U' = V' \times W'$  with  $V, V'$  subanalytic open sets in  $Y, W, W'$  subanalytic open sets in  $Z$ . Also, take  $U, U'$  as to satisfy (2.41). The stalk formula reads:  $\forall j$

$$(2.51) \quad H^j R \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_{X \rightarrow Y}, \mathfrak{E}\mu\text{hom}(p^{-1}L, \mathcal{O}b_X))_\xi = \\ = \lim_{\substack{\rightarrow \\ U', U, \gamma}} H^j R\Gamma(R \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_{X \rightarrow Y}, \mathfrak{E}\text{hom}_X(C_{Z(\gamma)} \circ ((p^{-1}L)_{U'}), \mathcal{O}b_X))) \simeq \\ \simeq R\Gamma(U; R \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_{X \rightarrow Y}, \mathfrak{E}\text{hom}_X((C_{Z(\gamma)} \circ ((p^{-1}L)_{U'}))_U, \mathcal{O}b_X))) \simeq \\ \simeq R\Gamma(U; R \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_{X \rightarrow Y}, \mathfrak{E}\text{hom}_X(\mathbf{R}q_{1!}(q_2^{-1}p^{-1}L \otimes \\ \otimes C_{Z(\gamma) \cap q_2^{-1}U' \cap q_1^{-1}U}), \mathcal{O}b_X))).$$

By Lemma 2.12 this is equal to:

$$\begin{aligned}
 (2.52) \quad & \mathbf{R}\Gamma(U; R \mathcal{H}om_{\mathcal{O}_X} \cdot \\
 & \cdot (\mathcal{O}_{X \rightarrow Y}, \mathcal{H}om_X(Rq_{1!}(\overrightarrow{q_2^{-1} p^{-1} L} \otimes C_{Z(\gamma') \cap q_2^{-1} p^{-1} V' \cap q_1^{-1} U}), \mathcal{O}b_X))) \simeq \\
 & \simeq \mathbf{R}\Gamma(U; R \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_{X \rightarrow Y}, \mathcal{H}om_X((C_{Z(\gamma')} \circ p^{-1} L_{V'})_U, \mathcal{O}b_X))) \simeq \\
 & \simeq \mathbf{R}\Gamma(U; R \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_{X \rightarrow Y}, \mathcal{H}om_X(C_{Z(\gamma')} \circ p^{-1} L_{V'}, \mathcal{O}b_X))).
 \end{aligned}$$

By Lemma 2.13 this is equals:

$$(2.53) \quad \mathbf{R}\Gamma(U; R \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_{X \rightarrow Y}, \mathcal{H}om_X(p^{-1}(C_{Z(\gamma')}) \circ L_{V'}), \mathcal{O}b_X))).$$

As  $p$  is smooth, by (2.39) we get:

$$\begin{aligned}
 (2.54) \quad & \mathbf{R}\Gamma((U; p^{-1} \mathcal{H}om_Y(C_{Z(\gamma')} \circ L_{V'}, \mathcal{O}b_Y))) \simeq \\
 & \simeq \mathbf{R}\Gamma(V \times Z; p^{-1} \mathcal{H}om_Y(C_{Z(\gamma')} \circ L_{V'}, \mathcal{O}b_Y)) \simeq \\
 & \simeq \mathbf{R}\Gamma(V; \mathbf{R}p_* p^{-1} \mathcal{H}om_Y(C_{Z(\gamma')} \circ L_{V'}, \mathcal{O}b_Y)).
 \end{aligned}$$

But  $p$  has contractible fibers, so  $\mathbf{R}p_* p^{-1}(\cdot) = \text{id}_Y$  and we have:

$$(2.55) \quad \mathbf{R}\Gamma(V; \mathcal{H}om_Y(C_{Z(\gamma')} \circ L_{V'}, \mathcal{O}b_Y)).$$

By this and (2.50) we get:  $\forall j$

$$\begin{aligned}
 (2.56) \quad & H^j R \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_{X \rightarrow Y}, \mathcal{H}om(p^{-1} L, \mathcal{O}b_X))_\xi = \\
 & = \lim_{\substack{\rightarrow \\ V', \bar{V}, \gamma'}} H^j \mathbf{R}\Gamma(V; \mathcal{H}om_Y(C_{Z(\gamma')} \circ L_{V'}, \mathcal{O}b_Y)) = \\
 & = H^j R^t p'_* p_\pi^{-1} \mathcal{E}muhom(L, \mathcal{O}b_Y)_\xi,
 \end{aligned}$$

which proves (2.38). *This completes the proof of Proposition 2.11.*

Then for  $\xi = p_i$  we have:

$$\begin{aligned}
 (2.57) \quad & R^t p'_* p_\pi^{-1} \mathcal{E}muhom(L, \mathcal{O}_Y)_\xi \xrightarrow{\sim} \\
 & \xrightarrow{\sim} R \mathcal{H}om_{\mathcal{E}_X}(\mathcal{E}_{X \rightarrow Y}, \mathcal{E}muhom(p^{-1} L, \mathcal{O}_X))_\xi.
 \end{aligned}$$

Taking into account that  $\mathcal{E}muhom(p^{-1} L, \mathcal{O}_X) \in \mathbf{D}^b(\mathcal{E}_X)$  (thanks to  $\mathcal{E}muhom(K, \mathcal{O}_X)$  being concentrated in degree 0) and that  $p$  is smooth, we

have  $\varepsilon_{X \rightarrow Y} \simeq \varepsilon_X \bigotimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{O}_{X \rightarrow Y}$  so we can interchange  $\varepsilon_{X \rightarrow Y}$  and  $\mathcal{O}_{X \rightarrow Y}$  thus obtaining the following formulation:

$$(2.58) \quad R \mathcal{H}om_{\varepsilon_X}(\varepsilon_{X \rightarrow Y}, \mathfrak{E}hom(p^{-1}L, \mathcal{O}_X))_{p_i} \simeq \mathfrak{E}hom(L, \mathcal{O}_Y)_{p_i}.$$

By Lemma 2.9 we may also write (2.58) as:

$$(2.59) \quad R \mathcal{H}om_{\varepsilon_X}(\mathcal{M}_i, \mathfrak{E}hom(p^{-1}L, \mathcal{O}_X))_{p_i} \simeq R \mathcal{H}om_{\varepsilon_Y}(\mathcal{M}_{Y, p_i}, \mathfrak{E}hom(L, \mathcal{O}_Y))_{p_i},$$

where  $\mathcal{M}_{Y, p_i}$  denotes the  $\varepsilon_X$ -module inverse image of  $\mathcal{M}$  at  $p_i$ . One has the chain of isomorphisms:

$$(2.60) \quad \bigoplus_{i=1}^r R^t f'_i f_\pi^{-1} R \mathcal{H}om_{\varepsilon_X} \left( \varepsilon_X \bigotimes_{\mathcal{O}_X}^L \mathcal{M}_i, \mathfrak{E}hom(p^{-1}L, \mathcal{O}_X) \right)_{p_i} \simeq \bigoplus_{i=1}^r R^t f'_i f_\pi^{-1} R^t p'_* p_\pi^{-1} \mathfrak{E}hom(L, \mathcal{O}_Y)_{p_i} \simeq \bigoplus_{i=1}^r R^t f'_i f_\pi^{-1} R^t p'_i p_\pi^{-1} \mathfrak{E}hom(L, \mathcal{O}_Y)_{p_i} \simeq \bigoplus_{i=1}^r \mathfrak{E}hom(L, \mathcal{O}_Y)_{p_i},$$

where the first isomorphism comes from Proposition 2.11, while the second is due to the fact that  ${}^t p'$  is injective and the last isomorphism was obtained taking into account that (since  $p \circ f = \text{id}_Y$ )  $R^t f'_i f_\pi^{-1} R^t p'_i p_\pi^{-1} = {}_1 \tau_{*Y}$ . So what we get in the end is nothing but

$$(2.61) \quad \bigoplus_{i=1}^r R \mathcal{H}om_{\varepsilon_Y}(\mathcal{M}_Y, \mathfrak{E}hom(L, \mathcal{O}_Y))_{p_i},$$

and as  $\mathcal{M} = \bigoplus_{i=1}^r \mathcal{M}_i$  at  $p_Y$  this is nothing but  $R \mathcal{H}om_{p_Y}(\mathcal{M}_Y, \mathfrak{E}hom(L, \mathcal{O}_X))_{p_Y}$ , so  $\gamma$  is an isomorphism.

Now proof of Theorem 2.2 is complete.

### 3. - The abstract result.

The argument used to prove Theorem 2.2 is sufficiently general to suit other applications.

Let us first generalize our ingredients. Instead of  $Y$  being an hypersurface let us just take  $Y$  a submanifold of  $X$ ,  $f: Y \hookrightarrow X$  the embedding, Let  $Z$  be a subset of  $Y$ . Let  $M$  be a left coherent  $\mathcal{O}_X$ -module. In place of  $\text{char}(P)$  we will consider  $V$ , a smooth conic, involutive, regular subma-



nifold of  $T^*X$ . Instead of  $K$  being the first term of the distinguished triangle

$$(3.1) \quad K \rightarrow \bigoplus_{i=1}^r K_i \xrightarrow{h} \bigoplus_{i=1}^{r-1} C_X \xrightarrow{+1}$$

we just consider  $K$  to be an object of  $D_{R-c}^b(C_X)$ ,  $L$  an object of  $D_{R-c}^b(C_X)$ . Instead of the morphism  $\psi$  of Lemma 2.6, we will just suppose a morphism  $\psi: L \rightarrow f^{-1}K$  to be given. Our new hypotheses we will be:

- i) let  $\mathcal{N}$  have simple characteristics along  $V$ ;
- ii) let  $V$  be non glancing with respect to  $Y$  in a neighborhood of  $\tilde{\pi}_X^{-1}(Z)$ ;
- iii) let  $\text{SS}(K) \subset V$ ;
- iv) let  $\mathfrak{E}\mu\text{hom}(K, \mathcal{O}_X)$  (respectively  $\mathfrak{E}\mu\text{hom}(F, \mathcal{O}_Y)$ ) be locally concentrated in one degree outside of the zero section  $T_X^*X$  (resp.  $T_Y^*Y$ );
- v) let the morphism induced by  $\psi$ ,  $L \rightarrow f_{\mu,p}^{-1}K$  be an isomorphism in  $D^b(C_Y; p_Y) \forall p \in {}^t f'^{-1}(p_Y) \cap f_\pi^{-1}(\text{char}(\mathcal{N}))$ ;
- vi) let the morphism induced by  $\psi$ ,  $R\Gamma_{\{y\}}(L \otimes \omega_Y) \rightarrow R\Gamma_{\{y\}}(K \otimes \omega_X)$ , be an isomorphism for every  $y \in Z$ .

**THEOREM 3.1.** *In the above hypotheses we have that the natural morphism induced by  $\psi$ :*

$$(3.2) \quad f^{-1}R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathfrak{E}hom(K, \mathcal{O}_X))|_Z \rightarrow R\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{N}_Y, \mathfrak{E}hom(L, \mathcal{O}_Y))|_Z,$$

*is an isomorphism.*

**REMARK 3.1.** Identify  $Y \times_X T^*X$  to a subset of  $T^*X$ . Thanks to assumptions (i), (ii)  $\forall p_Y \in \tilde{\pi}_Y^{-1}(Z)$  there exists  $p_1, \dots, p_r \in {}^t f'^{-1}(p_Y)$  with  ${}^t f'^{-1}(p_Y) \cap f_\pi^{-1}(\text{char}(\mathcal{N})) \subset \{p_1, \dots, p_r\}$ ; in fact, (ii) implies:

$$(3.3) \quad \forall p \in (Y \times_X T^*X) \cap V, \quad T_p(Y \times_X T^*X)^\perp \cap T_p V = \{0\}.$$

Since  $T_p(Y \times_X T^*X)^\perp = T_p({}^t f'^{-1}{}^t f'(p))$ , this implies:

$$(3.4) \quad T_p({}^t f'^{-1}{}^t f'(p)) \cap T_p(\text{char}(\mathcal{N})) = \{0\}.$$

Then  $f$  is non-characteristic for  $\mathcal{N}$ , that is  ${}^t f'$  is finite on  $f_\pi^{-1}(\text{char}(\mathcal{N}))$ .

REMARK 3.2. Theorem 3.1 extends Theorem 2.2 to general (over-determined) systems.

This more general theorem can be read as a Cauchy-Kowalevski theorem, as we have seen in the case of logarithmic ramifications, and as we will see in the next section.

#### 4. - Application to a different type of ramifications.

We will consider the Cauchy problem with data ramified along a particular type of variety, called «swallow's tail». Of course, the ramification will be of tempered type, but the approach is meant to parallel that of non tempered ramifications which can be found in [D'A-S2]. We will see how the general statement of Theorem 3.1 can be applied to this second geometrical situation. This case of the swallow's tail appeared first in [Le2]. Let  $X$  be an open subset of  $\mathbb{C}^{n+1}$  containing 0 and endowed with coordinates  $x = (x_0, x')$ . Set  $Y = \{x \in X; x_0 = 0\}$ . Let

$T = \{x' \in Y; \text{the polynomial in } z,$

$$A(z, x') = z^{n+1} - x_n z^{n-1} - \dots - x_2 z - x_1 \text{ has at least one double root}\}.$$

Here we exploit a particular feature, that is the conormals to  $TA = T_{T, \text{reg}}^* \bar{Y}$  give a Lagrangean manifold in  $T^*X$ , featuring just one direction above 0. Define  $\bar{Y} \subset \mathbb{C}_z \times Y$  the variety given by the equation  $A(z, x') = 0$ . If we consider the projection  $\eta: \bar{Y} \rightarrow Y$ , we find that the «swallow's tail»  $T$  is nothing but the image by  $\eta$  of the point in  $\bar{Y}$  where  $\eta$  is not smooth.  $T$  is thus a singular variety. Let  $L = \eta_! \mathcal{O}_{\bar{Y}}$ ; there is a canonically induced morphism  $\tau: L \rightarrow \mathcal{O}_Y$ . Define  $\mathcal{O}_{[T|Y]}^{\text{ram}} := \mathcal{E}hom(L, \mathcal{O}_Y)$ .

We will need another assumption. Namely, suppose there exist  $T_1, \dots, T_r$  «swallow's tails» in  $X$  such that the  $T_i$  are mutually transversal and transversal to  $Y$  (i.e.  $A_i \cap A_j \subset T_X^* X$  for  $i \neq j$  and  $A_i \cap T_Y^* X \subset T_X^* X$  for every  $i$ ) with  $T_i \cap Y = T$ . Define  $K_i := \eta_{i!} \mathcal{O}_{\bar{X}_i}$ ,  $i = 1, \dots, r$  where  $\tau_i: K_i \rightarrow \mathcal{O}_X$ . Let as above  $K$  be the complex defined by (2.8) with this choice of the  $K_i$ 's. Define  $\sum_i \mathcal{O}_{[T_i|Y]}^{\text{ram}} := \mathcal{E}hom(K, \mathcal{O}_X)$ .

LEMMA 4.1. *The canonically induced morphism:  $R\Gamma_{\{0\}} \eta_! \mathcal{O}_{\bar{Y}} \rightarrow R\Gamma_{\{0\}} \mathcal{O}_Y$  is an isomorphism.*

PROOF. Remark that  $\eta^{-1}(0) = \{0\}$ , and that  $\bar{Y}$  is a complex manifold of the same dimension as  $Y$ . ■

We will now introduce the differential system. Let  $\mathcal{M}$  be a left coherent  $\mathcal{O}_X$ -module. Let  $V := \text{char}(\mathcal{M})$ . Let  $\mathcal{M}$  have simple characteristics

transversal to  $Y \times_X T^* X$ ; that is,  $\mathcal{N}$  has simple characteristics and  $V$  is non glancing with respect to  $Y \times_X T^* X$ . We denote by  $f$  the immersion of  $Y$  into  $X$ . Suppose that for a neighborhood  $W$  of  $T^*_T Y$   $f$  is non-characteristic for  $\mathcal{N}$ . Suppose also that:

$$(4.1) \quad \text{char}(\mathcal{N}) \cap {}^t f'^{-1}(T^*_T Y) \subset \bigcup_{i=1}^r T^*_{T_i} X \cup T^*_X X.$$

This also implies that  $f$  is non-characteristic for  $\mathcal{N}$ . We have that  $C_{\bar{Y}}[n]$  is a perverse sheaf (see [K-S1, Chapter 10]) and  $\eta$  is a finite map, so that  $L$  is a perverse sheaf (see [K-S1, Prop. 10.3.7]). Applying Lemma 2.3 we have that  $\mathfrak{E}\mu\text{hom}(L, \mathcal{O}_Y)$  is concentrated in one degree. As above, we have that  $C_{\bar{X}_i}[n + 1]$  is a perverse sheaf for every  $i$ , and being  $\eta_i$  a finite map also  $K_i$  is a perverse sheaf (see [K-S1, Prop. 10.3.7]); we can then apply Lemma 2.3 (see also [A, Cor. 5.6.1]) to get that  $\mathfrak{E}\mu\text{hom}(K_i, \mathcal{O}_X)$  is concentrated in one degree. We have that, out of the zero section  $T^*_X X$ :

$$(4.2) \quad \mathfrak{E}\mu\text{hom}(K, \mathcal{O}_X) \xrightarrow{\sim} \bigoplus_{i=1}^r \mathfrak{E}\mu\text{hom}(K_i, \mathcal{O}_X)$$

so that, out of the zero section,  $\mathfrak{E}\mu\text{hom}(K, \mathcal{O}_X)$  is concentrated in one degree.

**THEOREM 4.2.** *In the above situation, the natural morphism*

$$(4.3) \quad f^{-1} R \mathfrak{H}om_{\mathcal{O}_X} \left( \mathcal{N}, \sum_i \mathcal{O}^{\text{ram}}_{[T_i|X]} \right) |_T \simeq R \mathfrak{H}om_{\mathcal{O}_Y} (\mathcal{N}_Y, \mathcal{O}^{\text{ram}}_{[T|Y]}) |_T$$

*is an isomorphism.*

As a result we have the following formulation:

**THEOREM 4.3.** *Let  $\forall i = 1, \dots, r$   $f_i$  be an holomorphic function on  $X \setminus T_i$ , with meromorphic ramification at  $T_i$ . Let  $P$  be a partial differential operator of degree  $m$ , with simple characteristics transversal to  $Y \times_X T^* X$ , let  $\forall h = 0, \dots, m - 1$ ,  $w_h$  be an holomorphic function on  $Y \setminus T$  with meromorphic ramification at  $T$ . Then the Cauchy problem:*

$$(4.4) \quad \begin{cases} Pu = \sum_{i=1}^r f_i, \\ D_1^h u|_Y = w_h, \quad h = 0, \dots, m - 1, \end{cases}$$

*is locally well posed.*

We skip the details (see § 2) and just note that this theorem proves existence and uniqueness for the solution of the Cauchy problem, when holomorphic tempered data and traces ramified along  $T$  and the  $T_i$ 's are considered; the solution itself being a holomorphic tempered function ramified along the  $T_i$ 's. Theorem 4.2 generalizes to general (overdetermined) systems the result of Theorem 4.3.

## A. - Appendix.

As usual, we suppose that  $X$  is a complex manifold. Let  $X \times \bar{X}$  be a complexification of  $X$  and  $\delta: X \rightarrow X \times \bar{X}$  the diagonal immersion. The following result was communicated to be by P. Schapira.

LEMMA A.1. *Suppose  $U \subset X$  is open with real analytic boundary. Then  $\mu\text{hom}(C_U, \mathcal{O}_X)$  is a well defined object in  $D^b(\delta_X)$ .*

PROOF. By [S2] we have that

$$(A.1) \quad H^j(\mu\text{hom}(\delta_* C_U, \mathcal{O}_{X \times \bar{X}})) = 0 \quad \forall j \neq 2\dim X.$$

Hence by [K-S1, Chapter 11]  $\mu\text{hom}(\delta_* C_U, \mathcal{O}_{X \times \bar{X}})$  is a well defined object of  $D^b(\delta_{X \times \bar{X}})|_{T_X \times \bar{X}}$ . Since

$$(A.2) \quad \mu\text{hom}(C_U, \mathcal{O}_X) \simeq R\mathcal{H}om_{\mathcal{O}_{\bar{X}}}(\mathcal{O}_{\bar{X}}, \mu\text{hom}(\delta_* C_U, \mathcal{O}_{X \times \bar{X}})),$$

we get  $\mu\text{hom}(C_U, \mathcal{O}_X) \in D^b(\delta_X)$ . ■

Remark that Lemma A.1 holds true replacing  $\mu\text{hom}$  by  $\mathfrak{L}\mu\text{hom}$ . We make use of this lemma in § 4.

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