RENDICONTI del SEMINARIO MATEMATICO della UNIVERSITÀ DI PADOVA

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Rendiconti del Seminario Matematico della Università di Padova, tome 96 (1996), p. 99-104

<http://www.numdam.org/item?id=RSMUP_1996__96__99_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 96 (1996)

A Note on *IA*-Endomorphisms of Two-Generated Metabelian Groups.

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1. - Introduction and preliminaries.

An endomorphism of a group G is called an *IA-endomorphism* if it induces the identity mapping on the factor group G/G' of G over its derived subgroup G'. We write $ia(G)$ for the monoid of the IA-endomorphisms of G and $IA(G)$ for the group of invertible elements of $ia(G)$, the *IA-automorphisms* of G. It is easily verified that, $\mathrm{Inn}(G) \leq IA(G)$ where $\mathrm{Inn}(G)$ denotes the group of inner automorphisms of G.

The monoid $ia(G)$ has been studied in the case that G is a metabelian two-generated group, in [3], [4], [6]. In these papers, the description of the IA-endomorphisms is based on the construction of either a certain semigroup or a module.

Our approach in this note is of ring-theoretical nature, using techniques introduced by H. Laue [7]. We feel that certain key results of the paper [1], [2], [3], [4] gain in clarity and elegance by deriving them from those ideas as a starting point. This conviction was the main motivation for this unpretentious note. We first recall some definitions and results of [7].

Let G be a group and A be an abelian normal subgroup of G. A *cocycle* of G into A is a mapping f of G into A such that $(xy)^f = x^{fy} y^f$ for all $x, y \in G$. Let R denote the set of all cocycles of G into A. With respect to the usual addition of mappings into an abelian group and composition of mappings as multiplication, R is an associative ring.

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The authors acknowledge support from the Italian MURST (40%).

In [7], H. Laue establishes a close connection between $C_{\mathrm{Aut}G}(G/A)$ and R. More, precisely, for any $h \in C_{\mathrm{End}G}(G/A)$ let

$$f_h\colon G \to A\,, \qquad x \mapsto x^{-1}x^h\,.$$

Then $f_h \in R$ and

$$\psi\colon C_{\mathrm{End}G}(G/A) \to R\,, \qquad h \mapsto f_h\,,$$

is an isomorphism of the monoid $C_{\mathrm{End}G}(G/A)$ onto $(R, *)$, where $f * g = f + g + fg$ for all $f, g \in R$. Moreover, if the group of quasi regular elements of the ring R is denoted by $Q(R)$, then $(C_{\mathrm{Aut}G}(G/A))^{\psi} = Q(R)$. If $f \in Q(R)$, then we write f^- for the inverse of f with respect to $*$.

Let $\mathrm{End}_G A$ be the ring of G-endomorphisms of A. The restriction of any $f \in R$ to A is an element of $\mathrm{End}_G A$, and the mapping

$$\varrho\colon R \to \mathrm{End}_G A\,, \qquad f \mapsto f_{|A}\,,$$

is a ring homomorphism such that $\ker \varrho = \mathrm{Ann}_R(A) = \{f \mid f \in R, \forall x \in A \ x^f = 1\}$.

We remark that if $f \in R$, $\alpha \in \mathrm{End}_G A$, then $f\alpha \in R$ and $(f\alpha)^{\varrho} = f^{\varrho}\alpha$. Thus R is a $\mathrm{End}_G A$-module and ϱ is an $\mathrm{End}_G A$-module homomorphism. Moreover we have

$$Q(R)^{\varrho} = Q(R^{\varrho}) \tag{1.1}$$

PROOF. Obviously $Q(R^{\varrho}) \subseteq Q(R^{\varrho})$. Now, let $f \in R$ such that $f_{|A} \in Q(R^{\varrho})$. Then there exists $g \in R$ such that $g_{|A} * f_{|A} = 0 = f_{|A} * g_{|A}$. Therefore we have $f * g$, $g * f \in \mathrm{Ann}_R(A) \subseteq Q(R)$. Hence there exist $h, h' \in Q(R)$ such that

$$(f * g) * h = 0 = h' * (g * f)\,.$$

Therefore $f \in Q(R)$, hence $f_{|A} \in Q(R)^{\varrho}$. ■

We also observe

$$h \in C_{\mathrm{Aut}G}(G/A) \Leftrightarrow h_{|A} \in \mathrm{Aut}\, A\,. \tag{1.2}$$

2. – *IA*-endomorphisms of two-generated metabelian groups.

Let G be a two-generated metabelian group with generators a and b. Then $\mathrm{End}_G G'$ is a finitely generated commutative ring, generated by automorphisms of G' induced by conjugation by a, a^{-1}, b, b^{-1}, and G'

is a cyclic $\mathrm{End}_G G'$-module generated by c, where $c = [a, b]$ (see, for example [3, 2]). These properties will be used freely in the sequel without further reference.

Now an application of our introductory general remarks leads to a short proof of the result that

(2.1) $IA(G)$ *is a metabelian group* ([4, 2.3], [2, Cor. 1]).

PROOF. The zero ideal $\mathrm{Ann}_R(G')$ is contained in $Q(R)$ and therefore $\mathrm{Ann}_R(G')$ is an abelian normal subgroup of $Q(R)$. Since $Q(R)/\mathrm{Ann}_R(G') \cong Q(R)^{\varrho} \subseteq \mathrm{End}_G G'$, we have $Q(R)' \subseteq \mathrm{Ann}_R(G')$. It follows that $Q(R)$ is a metabelian group and, hence $IA(G)$ is metabelian group. ■

If $h \in ia(G)$ then $a^h = au$, $b^h = bv$ for suitable elements $u, v \in G'$. Viceversa, we have the following

(2.2) *For all* $(u, v) \in G' \times G'$ *there exists an endomorphism* h *of* G *such that* $a^h = au$, $b^h = bv$ ([3, 3.1(i)]).

PROOF. Let $(u, v) \in G' \times G'$ and $\alpha, \beta \in \mathrm{End}_G G'$ such that $u = c^{\alpha}$, $v = c^{\beta}$. Then the application $h\colon G \to G$, $x \mapsto x[x, a]^{-\beta}[x, b]^{\alpha}$ is an element of $ia(G)$ and $a^h = au$, $b^h = bv$. ■

If z is any group element, we write $\bar{z}$ for the inner automorphism induced by z. For later reference we remark as a consequence of the foregoing proof:

(1) *For all* $g \in R$ *there are* $\alpha, \gamma \in \mathrm{End}_G G'$ *such that* $g = f_{\bar{a}}\gamma + f_{\bar{b}}\alpha$.

Moreover, an application of (1.2) yields a criterion for h to be an automorphism ([3, 3.1(ii)]).

For all $f, g \in R$ we set $f \circ g =: fg - gf$. It is well known that $(R, +, \circ)$ is a Lie ring. As $\mathrm{End}_G G'$ is commutative, we have $g * f * (f \circ g) = f * g$ for all $f, g \in R$. In particular, for all $f, g \in Q(R)$

(2) $$f \circ g = [f, g]$$

where $[f, g] = f^- * g^- * f * g$.

It is readily verified that for arbitrary metabelian groups the Witt identity for group commutators reduces to the simple Jacobi-like equation $[x, y, z][y, z, x][z, x, y] = 1$. In terms of cocycles, this rule may be expressed as follows:

(3) $$f_{\bar{y}} \circ f_{\bar{z}} = f_{\overline{[y,z]}}$$

for all $y, z \in G$.

From these remarks we may deduce the following description of the descending central series of $IA(G)$:

(2.3) $$\gamma_k(IA(G)) = \gamma_k(\mathrm{In}(G)) \quad ([3,\ 4.1])$$

for all $k \geqslant 2$.

PROOF. We set $D := (\mathrm{In}(G))^{\psi}$ and have to show that $\gamma_k(Q(R)) \subseteq \gamma_k(D)$ for $k \geqslant 2$. We proceed by induction on k. If $g_1, g_2 \in Q(R)$ and $\alpha, \beta, \gamma, \delta \in \mathrm{End}_G G'$ such that $g_1 = f_{\bar{a}}\alpha + f_{\bar{b}}\beta$, $g_2 = f_{\bar{a}}\gamma + f_{\bar{b}}\delta$, then (2) and (3) show that

$$[g_1, g_2] = f_{\overline{([b,a])^{\beta\gamma}}} + f_{\overline{([a,b])^{\alpha\delta}}} = f_{\overline{c^{\alpha\delta-\beta\gamma}}} \in ((\mathrm{In}(G))')^{\psi} = \gamma_2(D)$$

which settles the case of $k = 2$.

Now let $k > 2$, $g_1 \in \gamma_{k-1}(Q(R))$, $g_2 \in Q(R)$. Inductively we assume that $g_1 = f_{\bar{z}}$ for an element $z \in \gamma_{k-1}(G)$, and we write $g_2 = f_{\bar{a}}\gamma + f_{\bar{b}}\delta$ for some $\gamma, \delta \in \mathrm{End}_G G'$. Then

$$[g_1, g_2] = f_{\overline{([z,a])^{\alpha}}} + f_{\overline{([z,b])^{\beta}}} = f_{\overline{[z^{\alpha},a][z^{\beta},b]}} \in (\gamma_k(\mathrm{In}(G)))^{\psi} = \gamma_k(D). \quad \blacksquare$$

The nilpotent case allows a neat characterization which has been useful in various places (see, for example, [8])

(2.4) *$IA(G)$ nilpotent $\Leftrightarrow$ G nilpotent $\Leftrightarrow$ $ia(G) = IA(G)$* ([3, 3.1]).

PROOF. The first equivalence follows from (2.3). Furthermore, we know that $ia(G) = IA(G)$ if and only if $R = Q(R)$. As $\ker \varrho \subseteq Q(R)$, this is equivalent to saying that $R^{\varrho} = Q(R^{\varrho})$ by (1.1), i.e. that R^{ϱ} is a radical ring.
As R^{ϱ} is an ideal of finitely generated commutative ring $\mathrm{End}_G G'$, an application of [9, 4.1(i)] shows that R^{ϱ} is a radical ring if and only if R^{ϱ} is nilpotent. An easy induction shows that $\gamma_k(G) = c^{(R^{\varrho})^{k-2}}$ for $k \geqslant 2$, where $(R^{\varrho})^0 = \mathrm{End}_G G'$. Hence the nilpotency of R^{ϱ} is equivalent to the nilpotency of G. $\blacksquare$

Moreover, the nilpotent case allows the following description of the ascending central serie of $IA(G)$ (cf. [3, 4])

(2.5) *If G is nilpotent then, for all $k \in N_0$*

$$\zeta_k(IA(G)) = C_{\mathrm{Aut}\,G}(G/(G' \cap \zeta_k(G))).$$

PROOF. We remark that, for all $n \in \boldsymbol{N}_0$, we have

$$(\zeta_k(IA(G)))^\psi = \zeta_k(Q(R))$$

and, by (2), $\zeta_k(Q(R)) = \zeta_k(R) = Z_k(R)$, where $Z_k(R)$ is the k-th term of the upper central series of the Lie ring $R(+, \circ)$. Moreover, if we put

$$R_k = (C_{\mathrm{Aut}\,G}(G/(G' \cap \zeta_k(G)))^\psi$$

then it suffices to show that $Z_k(R) = R_k$ for all $k \in \boldsymbol{N}_0$. As G is metabelian, one readily verifies that

$$x^{f \circ f_{\bar{y}}} = [y^f, x] \tag{4}$$

for any $f \in R$ and $x, y \in G$. The inclusion $Z_k(R) \subseteq R_k$ follows from (4) by an easy induction on k.
Now, we prove by induction that $R_k \subseteq Z_k(R)$ for all $k \in \boldsymbol{N}_0$, the case $k = 0$ being trivial. Let $n \in \boldsymbol{N}_0$ and assume that $R_k \subseteq Z_k(R)$. Let $f \in R_{n+1}$ and $g \in R$. By (1), there are $\alpha, \beta \in \mathrm{End}_G G'$ such that $g = f_{\bar{a}}\alpha + f_{\bar{b}}\beta$. By (4) and by our inductive hypothesis it follows that $f \circ f_{\bar{a}}$, $f \circ f_{\bar{b}} \in Z_k(R)$. Hence

$$f \circ g = (f \circ f_{\bar{a}})\alpha + (f \circ f_{\bar{b}})\beta \in Z_k(R).$$

Therefore $f \in Z_{n+1}(R)$. ■

The study of $IA(G)$ has been of particular interest in the case that G is *free* metabelian of rank two. Then G' is a free abelian group with the coniugates of as a set of generators, and there is a canonical isomorphism of $\boldsymbol{Z}[G/G']$ onto $\mathrm{End}_G G'$. We conclude this note by pointing out that the crucial step of the proof of the following well-known result is simplified considerably by a suitable application of (1):

(2.6) *If G is a free metabelian group of rank two, then* $IA(G) = \mathrm{Inn}(G)$ ([1, Theor.2], [4, 2.4], [2, Cor.3]).

PROOF. It suffices to show that every $h \in IA(G)$ is an inner automorphism of G. The main step of the proof is to show this in the case that $h_{|G'} = \mathrm{id}_{G'}$. Then, by (1), $f_{\bar{b}}^{\varrho}\alpha = f_{\bar{a}}^{\varrho}\beta$ for suitable elements $\alpha, \beta \in \mathrm{End}_G G'$. We know that we may identify $\mathrm{End}_G G'$ with the integral group ring of the free abelian group G/G'. By a well-known line of reasoning we obtain therefore an element $\gamma \in$

$\in \mathrm{End}_G G'$ such that $\alpha = f^{\varrho}_{\bar{a}} \gamma$, $\beta = f^{\varrho}_{\bar{b}} \gamma$. By (1) and (3) we now have

$$f_h = (f_{\bar{b}} \circ f_{\bar{a}}) \gamma = f_{\overline{c^{-1}}} \gamma = f_{\overline{c^{-\gamma}}}$$

hence $h = \overline{c^{-\gamma}} \in \mathrm{In}(G)$.

The reduction of the case an arbitrary $h \in IA(G)$ to the case just settled is standard. The group of units of $\mathbb{Z}[G/G']$ is $\pm G/G'$ (see [5]) whence $h_{|G'}$ is induced by an inner automorphism $\bar{g}$ of G, as h induces the identity automorphism on the non-trivial factor group $G'/\gamma_3(G)$. Therefore $h \bar{g}^{-1}_{|G'} = \mathrm{id}_{G'}$, hence $h \bar{g}^{-1} \in \mathrm{In}(G)$ by the part treated above. The claim follows. ■

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Manoscritto pervenuto in redazione il 29 novembre 1994
e, in forma revisionata, il 15 maggio 1995.