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Global existence results for first-order integrodifferential identification problems

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Global Existence Results for First-Order Integrodifferential Identification Problems.

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ABSTRACT - We determine sufficient conditions in order that the solutions of abstract first-order linear integrodifferential equations be positive. Then, using a previouos result concerning the Fredholmness of linear identification problems related to integrodifferential equations, we prove, for such problems, an existence, uniqueness and stability theorem. Some applications are given to the case of materials with memory with unknown sources.

0. - Introduction.

In this paper we consider the identification problem consisting in determining a function $u: [0, T] \to X$ and an element $z \in Z$ (X and Z being two *Banach spaces*) satisfying the abstract integrodifferential equation

$$(0.1) \qquad u'(t) = Au(t) + \int_0^t H(t-s)Au(s)\,ds + \int_0^t H_0(t-s)u(s)\,ds + E(t)z + f(t), \qquad 0 \leqslant t \leqslant T,$$

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the initial condition

$$u(0) = \psi_0 \tag{0.2}$$

and the additional information

$$\Phi(u) = \psi_1 . \tag{0.3}$$

Here we limit ourselves to stating that A is a linear closed (unbounded) operator, while H, H_0, E and Φ are bounded linear operators. In particular, we shall consider the cases where Φ admits one of the following representations:

H0*a* $\Phi(u) = u(T)$ (*final determination*);

H0*b* $\Phi(u) = \int_0^T \omega(t)\, u(t)\, dt$, where $\omega\colon (0, T) \to \boldsymbol{R}$ is a prescribed non-negative, measurable, scalar function (*integral information*);

H0*c* $\Phi(u) = \int_0^T u(s)\, d\mu(s)$, where μ is a finite and positive Borel measure (*measure integral information*).

Actually, in the present paper we assume that X is a *Banach lattice* and Z a *Banach space* continuously embedded in X. First we determine sufficient conditions on the triplet of linear operators (A, H, H_0) assuring that any solution u to the integrodifferential equation

$$u'(t) = Au(t) + \int_0^t H(t-s)\, Au(s)\, ds + \int_0^t H_0(t-s)\, u(s)\, ds + g(t), \qquad 0 \leq t \leq T, \tag{0.1$'$}$$

satisfying (0.2) is positive whenever the pair (g, ψ_0) is positive.

Then we determine sufficient conditions on the quintuplet of linear operators (A, H, H_0, E, Φ) assuring the uniqueness of the solution to problem (0.1)-(0.3).

Finally, suitably combining such a result with the Fredholm character of problem (0.1)-(0.3) proved by the authors in [13], we can show the existence *in the large* of a unique solution to our identification problem.

We now observe that our problem is an *inverse problem*. As is well-known, such problems are (generally speaking) *ill-posed* (cf., e.g., Tikhonov-Arsenin [34], Ivanov-Vasin-Tanana [8], Lavrent'ev-Romanov-Shishatskij [10]).

Our basic task consists in determining suitable and sufficiently general conditions concerning the operator A, H, H_0, E, Φ and the data f, ψ_0, ψ_1 in order that our identification problem turn out to be *well-posed* in the sense of Hadamard (e.g. the solution exists, is unique and continuously depends on the data).

We note that, when $f = 0$, the «source» term of our equation (0.1) is a member of the class of source functions $\mathcal{S}$, which can be represented as products of two functions, the first depending on time and all the space variables and the second on a lesser number of variables. Such a class $\mathcal{S}$ was introduced by the second author [18] in 1967 to recover the density of an unapproachable body in terms of an outer potential of its. The function class $\mathcal{S}$ allows to reduce the problem of recovering a coefficient entering the differential operators to the problem of determining one of the two members in a source function in $\mathcal{S}$ (cf. also [19]). In [18] the uniqueness of the solution to the inverse problem in Potential Theory was proved under suitable assumptions involving the sign and the monotonicity of the unknown function.

Surveys concerning inverse problems in the case of both inner ($\Phi(u) = u(t_0)$ $t_0 \in (0, T)$) and final determinations are the papers by Prilepko-Orlovsky-Vasin [24], Prilepko-Kostin-Tikhonov [23] (for the case $H = H_0 = 0$) and Lorenzi [11] (for the case $(H, H_0) \neq (0, 0)$). Identification problems with final determination related to the heat equation, corresponding to problem (0.1)-(0.3) with $H = H_0 = 0$, were investigated by Iskenderov [7] (E being a scalar function), by Rundell [31] ($E(t) = I$, $\forall t \in [0, T]$). In the case of parabolic equations ($H = H_0 = 0$) and final determination Isakov [6] studied the question of uniqueness of the identification problem in the framework of Hölder spaces, when E is a function depending on time and all the space variables, while the unknown coefficient z is assumed to depend on a space variable only.

In the case of parabolic equations with Dirichlet boundary value conditions and a source function $E(t)z(x) = e(t, x)z(x)$ (e being a scalar function) subject to final determination, Prilepko-Solovyev [25] showed, in the framework of Hölder spaces, the Fredholm character of the corresponding identification problem under the assumption that $e(T, \cdot)$ is bounded away from zero. In the same paper they proved also the wellposedness of that problem under the additional assumptions $e(t, x) \geqslant 0$, $D_t e(t, x) \geqslant 0, \forall (t, x) \in [0, T] \times \overline{\Omega}$. Later on, in the case of final and integral determinations, conditions of the same type as the previous ones were used by Solovyev [32] (in the parabolic case), by Prilepko-Vasin [29], [30] (for the Navier-Stokes equation) and by Prilepko-Kostin [20], [21], [22] (in the framework of Sobolev spaces).

The abstract case with $H = H_0 = 0$ and final determination was considered by the second author [19] (where the general operator E was introduced), by Orlovsky [15], Prilepko-Tikhonov [26], [27], [28] (where the integral information was introduced).

The case of the identification problem (0.1)-(0.3) with $(H, H_0) \neq (0, 0)$ was investigated by Lorenzi-Prilepko [13], where the Fredholm character of such a problem was shown. We recall that the latter case is important for applications to the theory of heat conduction in materials with memory (cf., e.g. Lorenzi-Sinestrari [14], and Lorenzi-Paparoni [12], where a rather different additional information Φ was considered).

The plan of this paper is the following: Section 1 is devoted to showing a positivity result for the solution to problem (0.1′), (0.2); Section 2 proves a uniqueness result for the identification problem (0.1)-(0.3), while Section 3 proves the existence result for problem (0.1)-(0.3). Finally, Section 4 is devoted to some explicit parabolic identification problem.

1. – A positivity result.

First we state some of our basic assumptions, related to the positivity result, and observe that more precise conditions concerning our data will be introduced in the sequel:

H1 X is a Banach lattice with a positive cone X_+ [3];

H2 A: $\mathcal{D}(A) := Y \hookrightarrow X \to X$ (with Y dense in X) is a linear closed operator, whose resolvent set contains the angle $\Sigma_\phi = \{\lambda \in \boldsymbol{C}: |\arg \lambda| < \phi\}$ for some $\phi \in (\pi/2, \pi)$. Moreover, the resolvent operator $(\lambda - A)^{-1}$ satisfies the estimate

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \leqslant C_0 |\lambda|^{-1}, \qquad \forall \lambda \in \Sigma_\phi,$$

for some positive constant C_0;

H3 Y is endowed with the graph-norm of A;

H4 $H, H_0 \in W^{1,p}((0, T); \mathcal{L}(Y^{\theta,p}))$ (for any $\theta \in [0, \sigma]$ and some $\sigma \in (0, 1)\backslash\{1/p\}$, $p \in (1, +\infty])$;

H5 Z is a Banach space such that: i) $Z \hookrightarrow X$; ii) $\sup(z, 0) \in Y^{\theta,p}$ for any $z \in Z$ and some $\theta \in (0, \sigma]$;

H6 $\Phi \in \mathcal{L}(C([0, T]; X); X) \cap \mathcal{L}(C([0, T]; Y); Y)$;

H7 for any $\theta \in [0, \sigma]$, $E \in W^{1,p}((0, T); \mathcal{L}(Z; Y^{\sigma,p}))$ is an operator-valued function such that $E(0) \in \mathcal{L}(Y^{\theta,p}) \cap \mathcal{L}(Z; Y^{\sigma+1/p',p})$;

H8 $f \in W^{1,p}((0, T); Y^{\sigma,p})$;

H9 $\psi_0, \psi_1 \in Y^{2+\sigma-1/p,p}$.

We recall that in [13] we have denoted by $Y^{\sigma,p}$ ($\sigma \in (0, 1)$, $p \in (1, +\infty)$) the intermediate space between Y and X defined by

$$(1.1)\qquad Y^{\sigma,p} = \left\{ x \in X \colon\ |x|_{\sigma,p} := \left(\int_0^{+\infty} t^{(1-\sigma)p-1} \|AS_0(t)x\|^p \, dt \right)^{1/p} < +\infty \right\}$$

where $\{S_0(t)\}_{t \geq 0}$ denotes the semigroup of linear bounded operators generated by A. Moreover, we set

$$(1.2)\qquad Y^{n,p} = \mathcal{D}(A^n), \qquad n \in \mathbf{N},\ \ p \in (1, +\infty),$$

$$(1.3)\qquad Y^{\sigma,p} = \{x \in \mathcal{D}(A^n) \colon A^n x \in Y^{\sigma-n,p}\}, \qquad \sigma \in (n, n+1), \quad p \in (1, +\infty).$$

We observe that $Y^{0,p} = X$ and $Y^{1,p} = Y$ for any $p \in (1, +\infty)$ and we recall that $Y^{\sigma,p}$ is a Banach space when equipped with the norm

$$(1.4)\qquad \|x\|_{\sigma,p} = \sum_{j=1}^{n} \|A^j x\| + |A^n x|_{\sigma-n,p}, \qquad \sigma \in (n, n+1),\ \ p \in (1, +\infty).$$

In the case where $\sigma = n$ the seminorm $|A^n x|_{0,p}$ has to be dropped out.

Finally, we introduce the following Banach spaces related to the quadruplet $(s, p, \sigma, T) \in \mathbf{N} \times (1, +\infty) \times (0, 1) \times (0, +\infty)$:

$$(1.5)\qquad U^{s,p}_{\sigma,T} = W^{s,p}((0, T); Y^{\sigma,p}) \cap W^{s-1,p}((0, T); Y^{\sigma+1,p}).$$

Then we state some results concerning the resolvents of linear operator-valued convolution integral equations.

LEMMA 1.1. *Let* $H \in L^p((0, T); \mathcal{L}(Y^{\theta,p}))$ *for some* $p \in (1, +\infty)$ *and* $\theta \in [0, 1]$. *Then the integral equation*

$$(1.6) \qquad u(t) + \int_0^t H(t-s)\,u(s)\,ds = f(t) \quad \text{for a.e. } t \in (0, T),$$

admits, for any $f \in L^q((0, T); Y^{\theta,p})$ $(q \in (1, +\infty))$, *a unique solution* $u \in L^q((0, T); Y^{\theta,p})$ *represented by*

$$(1.7) \qquad u(t) = f(t) + \int_0^t J(t-s) f(s)\,ds \quad \text{for a.e. } t \in (0, T).$$

The resolvent kernel J *belongs to* $L^p((0, T); \mathcal{L}(Y^{\theta,p}))$ *and solves the operator-valued convolution integral equations*

$$(1.8) \qquad H(t) + J(t) + \int_0^t H(t-s) J(s)\,ds = 0 \quad \text{for a.e. } t \in (0, T),$$

$$(1.9) \qquad H(t) + J(t) + \int_0^t J(t-s) H(s)\,ds = 0 \quad \text{for a.e. } t \in (0, T).$$

LEMMA 1.2. *Let* $H \in W^{1,p}((0, T); \mathcal{L}(Y^{\theta,p}))$ *for some* $p \in (1, +\infty)$ *and* $\theta \in [0, 1]$. *Then the resolvent kernel* J *belongs to* $W^{1,p}((0, T); \mathcal{L}(Y^{\theta,p}))$ *and solves the operator-valued integral equations*

$$(1.10) \qquad H'(t) + J'(t) + \int_0^t H(t-s) J'(s)\,ds - H(t)H(0) = 0 \quad \text{for a.e. } t \in (0, T),$$

$$(1.11) \qquad H'(t) + J'(t) + \int_0^t J'(t-s) H(s)\,ds - H(0)H(t) = 0 \quad \text{for a.e. } t \in (0, T).$$

PROOF OF LEMMA 1.1. Assume that $H, K \in W^{1,p}((0, T); \mathcal{L}(Y^{\theta,p}))$. According to Lemma 2.5 in Lorenzi-Paparoni [12], which extends to our operator case, we get the estimates

$$(1.12)\qquad \|(H*)^j K\|_{L^p((0,T);\, \mathcal{L}(Y^{\theta,p}))} \leqslant$$

$$\leqslant \|(H*)^{j-1} H\|_{L^1((0,T);\, \mathcal{L}(Y^{\theta,p}))} \|K\|_{L^p((0,T);\, \mathcal{L}(Y^{\theta,p}))} \leqslant$$

$$\leqslant 2^{-1/p}(j!)^{-1/p'} t^{j/p'} \|H\|^j_{L^p((0,T);\, \mathcal{L}(Y^{\theta,p}))} \|K\|_{L^p((0,T);\, \mathcal{L}(Y^{\theta,p}))}, \qquad j \geqslant 2 .$$

Hence, the solution to equation (1.8) is given by

$$(1.13)\qquad J = \sum_{j=0}^{+\infty} (-1)^{j+1} (H*)^j H .$$

Owing to (1.12), we immediately deduce that J belongs to $L^p((0,T);\, \mathcal{L}(Y^{\theta,p}))$. Moreover, it is an easy task to check that J solves equation (1.9) too.

The remaining part of the proof is well-known. ■

PROOF OF LEMMA 1.2. The differentiability of J is an immediate consequence of Lemma 1.1 and well-known properties of convolutions. Moreover, equations (1.10)-(1.11) can be derived using well-known formulas regarding the differentiation of convolutions. ■

Suppose now that u is a solution to the Cauchy problem (0.1)-(0.2), which we rewrite in the following operator form, $*$ denoting convolution with respect to time:

$$(1.14)\qquad u' = (I + H*)Au + H_0 * u + g ,$$

$$(1.15)\qquad u(0) = \psi_0 ,$$

where

$$(1.16)\qquad g \in L^p((0,T);\, Y^{\sigma,p}) .$$

Apply the operator $I + J*$ to both members in (1.14) and recall the formula

$$(1.17)\qquad \int_0^t J(t-s)\, u'(s)\, ds = J(0)\, u(t) - J(t)\, u(0) + \int_0^t J'(t-s)\, u(s)\, ds , \qquad t \in (0,T) .$$

Since $J(0) = -H(0)$ (cf. (1.8)), we derive that u solves the following Cauchy problem, equivalent to the previous one:

$$u' = [A + H(0)]u + B * u + J\psi_0 + (I + J*)g, \tag{1.18}$$

$$u(0) = \psi_0, \tag{1.19}$$

where

$$B(t) = -J'(t) + (I + J*)H_0(t) \quad \text{for a.e. } t \in (0, T). \tag{1.20}$$

We now prove the following Lemma 1.3.

LEMMA 1.3. *Let H, H_0 satisfy assumption* H4. *Then for any $\theta \in [0, \sigma]$ operator B belongs to $L^p((0, T); \mathcal{L}(Y^{\theta,p}))$ and satisfies the integral equation*

$$B = B_1 - H * B, \tag{1.21}$$

where

$$B_1 = H' - HH(0) + H_0. \tag{1.22}$$

PROOF. From (1.8), (1.10), (1.22) we easily derive the assertion. For, we get the equations

$$B = -J' + (I + J*)H_0 = H' + H * J' - HH(0) + H_0 + \tag{1.23}$$

$$+(-H - H * J) * H_0 = B_1 + H * (J' - H_0 - J * H_0) = B_1 - H * B. \quad \blacksquare$$

In order to prove the positivity of the solution to problem (1.18)-(1.19) we make then the additional assumptions:

H10 $A + H(0)$ is the generator of a *positive* analytic semigroup $\{S(t)\}_{t \geq 0}$ of linear operators in $\mathcal{L}(X)$.

REMARK 1.1. We note that condition H10 is implied by the two following assumptions H10*a*, H10*b* or H10*a*, H10*b'*:

H10*a* A is the generator of a *positive* semigroup $\{S_0(t)\}_{t \geq 0}$ of linear operators in $\mathcal{L}(X)$;

H10*b* $H(0)$ is positive in $\mathcal{L}(X)$;

H10*b'* $H(0)$ commutes with A, preserves the space Y and $\exp[tH(0)]$ is positive in $\mathcal{L}(X)$ for any $t \in [0, +\infty)$.

In fact, according to H2 and the results in Pazy [16, section 2], we deduce that $A + H(0)$ satisfies H10 and generates the semigroup

$$S(t) = \sum_{j=0}^{+\infty} [S_0 H(0) *]^j S_0(t). \tag{1.24}$$

This implies that $\{S(t)\}_{t \geq 0}$ is positive in $\mathcal{L}(X)$, if $H(0)$ is positive in $\mathcal{L}(X)$. On the other hand, if $H(0)$ satisfies H10b', we easily deduce the equation

$$S(t) = \exp[tH(0)] S_0(t), \qquad \forall t \in [0, +\infty). \tag{1.25}$$

From the positivity of $\exp[tH(0)]$ in $\mathcal{L}(X)$ we immediately derive the positivity of $S(t)$ in $\mathcal{L}(X)$ for any $t \in [0, +\infty)$.

We now prove

LEMMA 1.4. *Let $H, K \in L^p((0, T); \mathcal{L}(X))$ $(p \in (1, +\infty))$ satisfy either of the properties:*

$$H * H(t) \geq 0 \quad \text{and} \quad K(t) - K * H(t) \geq 0 \quad \text{in } \mathcal{L}(X) \text{ for a.e. } t \in (0, T), \tag{1.26a}$$

$$H * H(t) \geq 0 \quad \text{and} \quad K(t) - H * K(t) \geq 0 \quad \text{in } \mathcal{L}(X) \text{ for a.e. } t \in (0, T). \tag{1.26b}$$

Then either $\sum_{j=0}^{+\infty} (-1)^j K(* H)^j \geq K - K * H \geq 0$ or $\sum_{j=0}^{+\infty} (-1)^j (H *)^j K \geq$ $\geq K - H * K \geq 0$ in $\mathcal{L}(X)$, according as either property (1.26a) *or property* (1.26b) *is satisfied.*

PROOF OF LEMMA 1.4. We limit ourselves to dealing with assumption (1.26a).

The hypotheses on H and K and estimate (1.12) imply the convergence of the series $\sum_{j=0}^{+\infty} \|(H *)^j K\|_{L^p((0, T); \mathcal{L}(X))}$. Hence, we can derive the equation

$$\sum_{j=0}^{+\infty} (-1)^j (H *)^j K = \sum_{j=0}^{+\infty} (H *)^{2j} (K - H * K). \tag{1.27}$$

From (1.12) (with $\theta = 0$), (1.26a), (1.27) we easily deduce the assertion. ■

LEMMA 1.5. *Let* $H \in L^p((0, T); \mathcal{L}(X))$ $(p \in (1, +\infty))$ *satisfy the property*:

$$H * H(t) \geqslant 0 \quad \text{and} \quad S(t) - S * H(t) \geqslant 0 \qquad \text{for a.e. } t \in (0, T). \tag{1.28}$$

Then $S(t) + S * J(t) \geqslant 0$ *in* $\mathcal{L}(X)$ *for a.e.* $t \in (0, T)$.

REMARK 1.2. Condition (1.28) is trivially satisfied if

$$H(t) \leqslant 0 \quad \text{for a.e. } t \in (0, T). \tag{1.29}$$

PROOF. Apply the convolution operator $S*$ to both members in equation (1.9). We get the new equation

$$S * J = -S * H - S * J * H. \tag{1.30}$$

Observe then that the operator $L = S + S * J$ satisfies

$$L = S - L * H. \tag{1.31}$$

Recall now that, according to assumption H2, the semigroup $\{S_0(t)\}_{t \geqslant 0}$ generated by A belongs to $C_b((0, +\infty); \mathcal{L}(X))$, the space of all operators-valued functions, which are continuous and bounded in $(0, +\infty)$. From formula (1.24) we deduce that $S \in C_b((0, T); \mathcal{L}(X))$ for any $T > 0$. Hence, according to Lemma 2.5 in Lorenzi-Paparoni [12], we conclude that the solution L to equation (1.31) is assigned by the formula

$$L = \sum_{j=0}^{+\infty} (-1)^j S(*H)^j \tag{1.32}$$

where the series converges in $L^p((0, T); \mathcal{L}(X))$. From (1.28) and Lemma 1.4 we deduce that L is positive. ■

THEOREM 1.1. *Assume that operator* $B \in L^p((0, T); \mathcal{L}(Y^{\theta, p}))$ $([0, 1] \setminus \{1/p\})$ *is positive in* $\mathcal{L}(X)$ *and* (1.28) *is satisfied. Then for any pair* $(g, \psi_0) \in L^p((0, T); Y^{\theta, p}) \times Y^{\theta, p}$ *such that* $g \geqslant 0$, $\psi_0 \geqslant 0$ *problem* (1.18)-(1.19) *admits a unique positive solution* u, *which is classical if* $\theta \in (0, 1/p)$ *and strict if* $\theta \in (1/p, 1)$.

PROOF. Owing to Theorem 29 in Di Blasio [2], the integrodifferential problem (1.18)-(1.19) is equivalent to the integral equation

$$u(t) = v(t) + S * B * u(t), \qquad t \in (0, T), \tag{1.33}$$

where

(1.34) $v(t) = (S + S * J)(t)\psi_0 + (S + S * J) * g(t), \qquad t \in (0, T).$

Observe that the function v and the linear operator $S * B$ are positive in X and $\mathcal{L}(X)$, according to our assumptions and Lemma 1.5. This implies that the solution u to (1.33) is positive.

A standard fixed-point theorem procedure assures the existence of a unique solution $u \in C([0, T]; \mathcal{L}(Y^{\theta, p}))$. ■

The next Lemma 1.6 gives some sufficient conditions on H in order that operator B be positive, commute with $A + H(0)$ and its convolution with J be commutative. For this purpose, we need the assumptions

H11 $[H' - HH(0) + H_0] * H = H * [H' - HH(0) + H_0]$;

H12 $A + H(0)$ commutes with $H(t)$ and $H'(t) - H(t)H(0) + H_0(t)$ for all $t \in [0, T]$ and for a.e. $t \in (0, T)$, respectively.

LEMMA 1.6. *Assume that for some* $p \in (1, +\infty)$ H, $H_0 \in W^{1,p}((0, T); \mathcal{L}(X))$ *enjoy properties* H11, H12 *and the following*:

(1.35) $H * H(t) \geq 0$ *and* $B_1(t) - H * B_1(t) \geq 0$

in $\mathcal{L}(X)$ *for a.e.* $t \in (0, T)$

where B_1 *is defined by* (1.22). *Then operator* B *is positive in* $\mathcal{L}(X)$ *and satisfies*

(1.36) $$B * J = J * B,$$

(1.37) $B(t)[A + H(0)] = [A + H(0)]B(t) \qquad$ *for a.e.* $t \in (0, T)$.

REMARK 1.2. Condition (1.35) is trivially satisfied if

(1.38) $H(t) \leq 0$ and $B_1(t) \geq 0 \qquad$ for a.e. $t \in (0, T)$.

PROOF. The solution B to equation (1.21) is given by the formula

(1.39) $$B = \sum_{j=0}^{+\infty} (-1)^j (H *)^j B_1$$

where the series converges in $L^p((0, T); \mathcal{L}(X))$. Hence, from Lemma 1.4 we conclude that B is positive in $\mathcal{L}(X)$. In order to show that B satisfies (1.36)-(1.37), observe that H11 implies the equations

$$(1.40)\qquad B_1*(H*)^j H = [H' - HH(0) - H_0]*(H*)^j H =$$

$$= (H*)^j H*[H' - HH(0) - H_0] = (H*)^j H*B_1, \qquad j\in \mathbb{N}\setminus\{0\}.$$

Whence and from (1.13) we easily derive (1.36).

Finally, H12 and (1.39) immediately imply (1.37). ■

We can now prove now Theorem 1.2 concerning the resolvent of the Cauchy problem (1.18)-(1.19).

THEOREM 1.2. *Let assumptions* H1-H4, H7-H12, (1.28), (1.35) *be satisfied. Then the operator-valued function*

$$(1.41)\qquad R = \sum_{j=0}^{+\infty}(S*B*)^j(S+S*J)$$

where the operator B is defined by formula (1.20), *belongs to* $\mathcal{L}(X;\ C([0,T];\ X))$ *and $R(t)$ is positive in $\mathcal{L}(X)$ for any $t\in[0,T]$. Moreover, for any $\psi_0\in Y^{\sigma+1/p',p}$ and $g\in L^p([0,T];\ Y^{\sigma,p})$ $(\sigma\in(0,1)\setminus\{1/p\}$, $p\in(1,+\infty))$ satisfying*

$$(1.42)\qquad A\psi_0 + g(0)\in Y^{\sigma-1/p,p} \qquad \textit{if } \sigma\in(1/p,1)$$

problem (1.14)-(1.15) *admits a unique solution $u\in U^{1,p}_{\sigma,T}$ given by*

$$(1.43)\qquad u(t) = R(t)\psi_0 + R*g(t), \qquad \forall t\in[0,T].$$

PROOF. First we observe that $R(t)$ is positive in $\mathcal{L}(X)$ for any $t\in[0,T]$ according to Theorem 1.1 and Lemmas 1.5, 1.6. Then we note that formulas (1.41)-(1.42) are easily implied by equations (1.34)-(1.35) and the equivalence between problems (1.18)-(1.19) and (1.33). The membership of the function u in the Banach space pointed out is a consequence of the following relationships concerning operator R (cf. Lemma 2.5 in Lorenzi-Paparoni [12], which applies also to the present operator case):

$$(1.44)\qquad R(t)\in\mathcal{L}(X;\ Y),$$

$$(1.45)\qquad \|R(t)\|_{\mathcal{L}(X)}\leq M_0, \qquad \forall t\in[0,T],$$

$$(1.46)\qquad \|R'(t)\|_{\mathcal{L}(X)}\leq M_1 t^{-1} \qquad \text{for a.e. } t\in(0,T),$$

for some positive constants M_0 and M_1. ■

We conclude this section by proving Lemmas 1.7 and 1.8 which give some sufficient conditions on the pair (H, E) in order that operators $E' - B*E$ and $(I + J*)E$ be positive in $\mathcal{L}(X)$.

Lemmas 1.7 and 1.8 will be used in Section 2, only, but we prove them here, since their proofs are very similar to the ones of Lemmas 1.5 and 1.6.

LEMMA 1.7. *Assume that for some* $p \in (1, +\infty)$ $E, H \in W^{1,p}((0, T); \mathcal{L}(X))$ *satisfy the property*:

$$H*H(t) \geq 0 \quad \textit{and} \quad E_1 - H*E_1 \geq 0 \quad \textit{in } \mathcal{L}(X) \textit{ for a.e. } t \in (0, T) \tag{1.47}$$

where

$$E_1 := E' + H*E' - [H' - HH(0) + H_0]*E. \tag{1.48}$$

Then operator $E' - B*E$ *is positive in* $\mathcal{L}(X)$.

REMARK 1.3. Condition (1.47) is trivially satisfied if

$$H(t) \leq 0 \quad \text{and} \quad E_1(t) \geq 0 \qquad \text{for a.e. } t \in (0, T). \tag{1.49}$$

PROOF. Apply the convolution operator $*E$ to both members in equation (1.21). We get the new equation

$$B*E = B_1*E - H*B*E. \tag{1.50}$$

Hence operator $L := E' - B*E$ satisfies (cf. (1.21))

$$\begin{aligned} L = E' - (B_1 - H*B)*E = E' - B_1*E + H*(E' - L) = \\ = E' - B_1*E + H*E' - H*L = E_1 - H*L. \end{aligned} \tag{1.51}$$

From assumptions (1.47) and Lemma 1.4 we deduce that L is positive in $\mathcal{L}(X)$. ∎

LEMMA 1.8. *Assume that for some* $p \in (1, +\infty)$ $E, H \in W^{1,p}((0, T); \mathcal{L}(X))$ *satisfy the property*:

$$H*H(t) \geq 0 \quad \textit{and} \quad E(t) - H*E(t) \geq 0 \quad \textit{in } \mathcal{L}(X) \textit{ for a.e. } t \in (0, T). \tag{1.52}$$

Then

(1.53) $$(I + J*)E(t) \geq E(t) - H*E(t) \geq 0 \quad \text{in } \mathcal{L}(X) \text{ for a.e. } t \in (0, T).$$

REMARK 1.4. Condition (1.52) is trivially satisfied if

(1.54) $$H(t) \leq 0 \quad \text{and} \quad E(t) \geq 0 \qquad \text{for a.e. } t \in (0, T).$$

PROOF. The assertion is an immediate consequence of Lemma 1.4, formula (1.13) and the following equation

(1.55) $$(I + J*)E = \sum_{j=0}^{+\infty} (-1)^j (H*)^j E. \qquad \blacksquare$$

2. - A uniqueness result.

In order to prove the uniqueness of the solution to the identification problem (0.1)-(0.3) we need to introduce

DEFINITION 2.1. Two elements z, w belonging to a Banach lattice X are called *disjoint* if, and only if, $\inf(|z|, |w|) = 0$. An operator $L \in \mathcal{L}(Z; X)$ is said to preserve disjointness if, and only if, $\inf(|z|, |w|) = 0$ implies $\inf(|Lz|, |Lw|) = 0$.

We now write down some additional properties concerning operators A, B, H, H_0, E:

H13 there exists an (at most) denumerable subset F^* in the cone X_+^* of positive functionals on the Banach lattice X such that

$$x \in X_+, \quad \langle x, x^* \rangle = 0, \qquad \forall x^* \in F^* \Rightarrow x = 0;$$

H14 $\Phi \in \mathcal{L}(C([0, T]; X); X)$ is a positive operator;

H15 $\Phi([A + H(0)]u) = [A + H(0)]\Phi(u)$, $\forall u \in C([0, T]; Y)$;

H16 $H*H$, $(I - H*)[H' - HH(0) + H_0]$, $(I - H*H*)E' - (I - H*)\cdot[H' - HH(0) - H_0]*E$ and $E - H*E$ are positive operators in $\mathcal{L}(X)$;

H17 $\Phi(u) = 0 \Rightarrow u(\tau) = 0$, $\forall u \in C([0, T]; X_+)$ and some $\tau \in (0, T]$;

H18 $Z_+ \cap \ker[E(\tau) - H*E(\tau)] = \{0\}$, where $Z_+ = \{z \in Z : z \geq 0\}$;

H19 $\Phi\left[\sum_{j=0}^{+\infty} (-1)^j (H*)^j E\right] := \Lambda$ preserves disjointness;

H20 $A + H(0)$ admits an inverse $(A + H(0))^{-1}$, which is negative in $\mathcal{L}(X)$.

REMARK 2.1. If we assume that $\{S(t)\}_{t \geq 0}$ is an *equibounded*, positive, analytic semigroup of linear operators in $\mathcal{L}(X)$, we can drop out in H20 the requirement concerning the negativity of $A + H(0)$. For, from Corollary 1 in [35, chapt. 9] we deduce that the resolvent of $A + H(0)$ contains (at least) the half plane $\Re\lambda > 0$ and the following relationships hold

$$(2.1) \qquad (\lambda - A - H(0))^{-1} = \int_0^{+\infty} e^{-\lambda t} S(t)\, dt \geq 0\,, \qquad \lambda > 0\,.$$

On the other hand, since $A + H(0)$ is invertible, we can find $r > 0$ such that $\lambda - A - H(0)$ is invertible for any $\lambda \in (0, r)$.

Finally, from the relationship

$$(2.2) \quad (\lambda - A - H(0))^{-1} \to -(A + H(0))^{-1} \quad \text{in } \mathcal{L}(X) \text{ as } \lambda \to 0+$$

we easily deduce that $(A + H(0))^{-1}$ is negative in $\mathcal{L}(X)$.

THEOREM 2.1. *Let properties* H1-H20 *be satisfied. Then the solution* (u, z) *to problem* (1.18)-(1.19), (0.3) *is unique in* $U_{\sigma,T}^{2,p} \times Z$.

PROOF. Let $(u, z) \in U_{\sigma,p}^{2,p} \times Z$ be a solution to the linear problem (1.18)-(1.19), (0.3) with $f = 0$ and $\psi_0 = \psi_1 = 0$. Represent z as

$$(2.3) \qquad z = z^+ - z^-$$

where $z^+ = \sup(z, 0)$ and $z^- = \sup(z, 0)$. According to assumption H5 we deduce that $z^+, z^- \in Y^{\theta,p}$ for some $\theta \in (0, \sigma]$. Consider then the solutions $u_1 \in U_{\theta,T}^{1,p}$ and $u_2 \in U_{\theta,T}^{1,p}$ to the following Cauchy problems, where $z_1 = z^+$, $z_2 = z^-$ and $j = 1, 2$:

$$(2.4) \quad u_j'(t) = [A + H(0)]\, u_j(t) + B * u_j(t) + (I + J*) E(t) z_j\,, \quad t \in (0, T),$$

$$(2.5) \qquad u_j(0) = 0\,.$$

According to Theorem 1.1, Lemmas 1.6, 1.8 and assumptions H5, H16 we deduce that u_j is positive ($j = 1, 2$).

Then from (2.3) and the uniqueness of the solution to problem (1.18)-(1.19) (with $g = Ez$ and $\psi_0 = 0$) we deduce the equation $u(t) = u_1(t) - u_2(t)$ for any $t \in [0, T]$. Since $\Phi(u) = 0$, then $\Phi(u_1) = \Phi(u_2) = \phi$, where $\phi \in Y$.

Using a standard procedure we can prove that u_j' is once more differentiable and solves the Cauchy problem

$$(2.6)\qquad u_j'' = [A + H(0)]\,u_j' + B * u_j' + $$

$$+J(t)\,E(0)\,z_j + (I + J*)E'(t)z_j\,,\qquad t \in (0, T)\,,$$

$$(2.7)\qquad u_j'(0) = E(0)\,z_j\,,$$

at least in a classical sense, since $(I + J*)E'z_j \in L^p((0, T); Y^{\theta, p})$ and $E(0)z_j \in Y^{\theta, p}$ $(j = 1, 2)$ according to H7 and Lemma 1.1. More precisely, taking advantage of Theorem 29 in Di Blasio [2], we can prove that $u_j' \in C([0, T]; Y^{\theta, p}) \cap W^{1, p}((0, T]; Y^{\theta, p})$.

Introduce then the functions $v_j = u_j' - B * u_j$ $(j = 1, 2)$. From (2.4), (2.5), (2.7), (2.8) and Lemma 1.6 we easily deduce that v_j solves the Cauchy problem

$$(2.8)\qquad v_j'(t) = [A + H(0)]\,v_j(t) + B * v_j(t) + J(t)\,E(0)\,z_j +$$

$$+(I + J*)[E'(t) - B * E(t)]\,z_j\,,\qquad t \in (0, T)\,,$$

$$(2.9)\qquad v_j(0) = E(0)\,z_j\,.$$

Observe then that, owing to Lemma 1.7, assumption H16 implies that $[E'(t) - B * E(t)]\,z_j$ is positive for a.e. $t \in (0, T)$. Moreover, taking the limit as $t \to 0+$ in (1.53), we easily deduce $E(0)\,z_j \geqslant 0$. Applying Theorem 1.1 we deduce that v_j is positive.

Applying operator Φ to both members in (2.4) and using H6 and H15, we get the equations

$$(2.10)\qquad \Phi(v_j) = \Phi(u_j' - B * u_j) = [A + H(0)]\,\phi + \Phi[(I + J*)Ez_j]\,,\qquad j = 1,2.$$

Consequently, from equations (1.13), (2.6) we derive the inequalities

$$(2.11)\qquad 0 \leqslant [A + H(0)]\,\phi + \Phi\left[\sum_{j=0}^{+\infty}(-1)^j (H*)^j Ez_j\right],\qquad j = 1, 2\,.$$

Recalling that $z_1 = z^+$ and $z_2 = z^-$, from (2.11) we immediately obtain

$$(2.12)\qquad -\,[A + H(0)]\,\phi \leqslant$$

$$\leqslant \inf\left\{\Phi\left[\sum_{j=0}^{+\infty}(-1)^j (H*)^j Ez^+\right],\ \Phi\left[\sum_{j=0}^{+\infty}(-1)^j (H*)^j Ez^-\right]\right\} = 0$$

by virtue of assumptions H14, H16, H19 and Lemma 1.8. Since operator $-[A + H(0)]^{-1}$ is positive owing to H20, we get $\phi \leqslant 0$.

Observe then that from Theorem 1.2 we derive the representation $u_1 = R*(I + J*)Ez^+$. Moreover, from the definition of ϕ, Theorem 1.2 and assumption H14 we deduce the relationships

$$0 \geqslant \phi = \Phi(u_1) = \Phi(R*(I + J*)Ez^+) \geqslant 0 . \tag{2.13}$$

Hence, we get $\Phi(R*(I + J*)Ez^+) = 0$. From assumption H17 we derive the equation

$$\int_0^\tau R(\tau - s)(I + J*)E(s)z^+ \, ds = 0 \tag{2.14}$$

for some $\tau \in (0, T]$. We introduce now the following basic lemma, whose proof is postponed:

LEMMA 2.1. *Let X_+ and μ denote the positive cone of a Banach lattice X and a positive measure, respectively. Let E^* be a subset in the cone X_+^* of the positive functionals enjoying property* H13. *Then*

$$f \in L^1((a, b); X_+; \mu) \quad \text{and} \quad \int_a^b f(t)\, d\mu = 0 \Rightarrow f(t) = 0 , \quad \text{for } \mu\text{-a.e. } t \in (a, b). \tag{2.15}$$

From Lemmas 1.8, 2.3 and the continuity of the integrand in (2.14) we derive

$$R(\tau - s)(I + J*)E(s)z^+ = 0 , \qquad \forall s \in [0, \tau]. \tag{2.16}$$

Taking the limit as $s \to \tau -$ and recalling that Y is dense in X, we get

$$(I + J*)E(\tau)z^+ = 0 . \tag{2.17}$$

Then, from (1.53) we deduce

$$0 = (I + J*)E(\tau)z^+ \geqslant [E(\tau) - H*E(\tau)]z^+ \geqslant 0 . \tag{2.18}$$

This implies

$$[E(\tau) - H*E(\tau)]z^+ = 0 . \tag{2.19}$$

Then, from assumption H18 we derive $z^+ = 0$. Likewise, we can show that $z^- = 0$. Hence we get $z = 0$.

Finally, from the Cauchy problem (1.18)-(1.19) (with $z=0$, $f=0$, $\psi_0=\psi_1=0$) we deduce $u=0$. ■

PROOF OF LEMMA 2.1. From $\int_a^b f(t)\,dt = 0$ it follows

$$(2.20)\qquad 0=\left\langle \int_a^b f(t)\,dt,\, x^*\right\rangle = \int_a^b \langle f(t), x^*\rangle\,dt\,, \qquad \forall x^*\in F^*\,.$$

Since

$$(2.21)\qquad \langle f(t), x^*\rangle \geq 0 \quad \text{for a.e. t} \in (a,b),\ \forall x^* \in F^*\,,$$

we deduce

$$(2.22)\qquad \langle f(t), x^*\rangle = 0 \quad \text{for a.e. } t \in (a,b),\ \forall x^* \in F^*\,.$$

Observe, then, that, owing to H13, the set $\mathcal{E}=\bigcup_{x^*\in F^*}\{t\in(0,T)\colon \langle f(t), x^*\rangle \neq 0\}$ is a denumerable union of null sets. Hence $m(\mathcal{E})=0$. This implies

$$(2.23)\qquad \langle f(t), x^*\rangle = 0\,, \qquad \forall t\in(a,b)\backslash\mathcal{E},\ \forall x^*\in F^*$$

By virtue of H13 we deduce $f(t)=0$ $\forall t\in(a,b)\backslash\mathcal{E}$. ■

We conclude this section by proving the following lemma basic for our applications.

LEMMA 2.2. *Let Φ be defined as in* H0*a*)-*c*) *and assume that ω and μ satisfy the properties*:

$$(2.24)\qquad \omega(t)\geq m>0 \ \textit{a.e. in a left neighbourhood of } t=\tau\in(0,T]\,,$$

$$(2.25)\qquad \mu(\tau-\varepsilon,\tau)>0\,, \qquad \forall\varepsilon\in(0,\varepsilon_0)\subset(0,\tau),\ \tau\in(0,T]\,.$$

Then Φ satisfies H17.

PROOF. H0*a*) $\Phi(u)$ trivially coincides with $u(T)=0$: hence we Choose $\tau=T$.

H0*b*) $\Phi(u)=0$ and $u\in C([0,T];X_+)$ imply (by virtue of Lemma 2.1 and the positivity of ω)

$$(2.26)\qquad \omega(t)\,u(t)=0 \quad \text{for a.e. } t\in(0,T)\,.$$

From (2.24) and (2.26) we deduce $u(\tau)=0$.

H0*c*) $\Phi(u)=0$ and $u\in C([0,T];X_+)$ imply (by virtue of Lemma

2.1 and the positivity of μ)

$$u(t) = 0 \quad \text{for } \mu\text{-a.e. } t \in (0, T). \tag{2.27}$$

From assumption (2.25) we deduce that there exists a sequence $\{t_n\} \subset \subset (\tau - \varepsilon, \tau)$ converging to τ such that

$$u(t_n) = 0, \qquad \forall n \in \boldsymbol{N}.$$

We have thus proved that $u(\tau) = 0$. ■

3. – Solving the identification problem (0.1)-(0.3).

In order to apply Theorem 2.1 in [13], which assures that *uniqueness implies existence* we have to choose $Z = Y^{\sigma + 1/p', p}$ and to identify the operator-valued B in [13] with $HA + H_0$. Moreover, we have to add to our previous assumptions H1-H20 the following ones, which imply hypotheses H1, H3-H10 in [13, Section 1] with $B = HA + H_0$:

H21 A admits an inverse $A^{-1} \in \mathcal{L}(X; Y) \cap \mathcal{L}(Y^{\sigma, p}; Y^{\sigma+1, p})$;

H22 $(\lambda_0 - A)^{-1} \in \mathcal{C}(Y^{\sigma, p}; Y^{1+\sigma, p})$ for some $\lambda_0 \in \Sigma_\phi$;

H23 $H + H_0 A^{-1}$ commutes with A^{-1} in $\mathcal{L}(X)$;

H24 $E \in W^{1,p}((0, T); \mathcal{L}(Y^{\sigma + 1/p', p}))$;

H25 $\Phi \in \mathcal{L}(C_b((0, T]; Y^{\theta, p}); Y^{\theta, p})$ for $\theta \in \{\sigma, \sigma + 1/p', 1, 1 + \sigma\}$;

H26 $[A + H(0)]\Phi(u) = \Phi([A + H(0)]u)$, $\forall u \in U^{2,p}_{\sigma, T}$;

H27 $-H(0)\Phi + \Phi H(0) \in \mathcal{C}(U^{2,p}_{\sigma, T}; Y^{\sigma + 1/p'})$;

H28 $\Phi\left[\sum_{j=0}^{+\infty} (-1)^j (H*)^j E\right]$ admits an inverse $\Psi \in \mathcal{L}(Y^{\sigma + 1/p', p})$;

H29 $\lim_{n \to +\infty} \sup\{\|\Phi[(1 - \chi_n) S_0 z\|_{\sigma + 1/p', p} : \|z\|_{\sigma + 1/p', p} = 1\} = 0$,
where S_0 is the analytic semigroup generated by A and χ_n is the scalar continuous function defined by

$$\chi_n(t) = \begin{cases} 0 & t \in [0, T/(2n)], \\ (2nt - T)/T & t \in (T/(2n), T/n), \\ 1 & t \in [T/n, +\infty). \end{cases} \tag{3.1}$$

We recall that $\mathcal{C}(X_1; X_2)$ and $C_b((0, T]; Y^{\theta, p})$ denote, respectively, the Banach spaces of all compact linear operator from X_1 to X_2 and of all $Y^{\theta, p}$-valued functions which are continuous and bounded on $(0, T]$.

REMARK 3.1. We observe that our assumption H2 is weaker than the corresponding H2 in [13]. However, this change does not affect at all the Fredholm result proved in [13].

REMARK 3.2. We note that our assumption H26 is stronger in comparison with H8 in [13], but this gives place to no additional trouble, since it is satisfied also in [13, Section 6].

REMARK 3.3. Assumptions H27 and H28 imply that the linear operator $K(u) = A\Phi(u) - \Phi(Au)$ belongs to $\mathcal{C}(U^{2,p}_{\sigma,T}; Y^{\sigma+1/p'})$ (cf. H9 in [13, Section 1]).

REMARK 3.4. We observe that, instead of assumption H10 in [13], which in our case reads as *the linear operator* $\Phi\left[\sum_{j=0}^{+\infty}(-1)^j[(H+H_0A^{-1})*]^j E\right]$ *admits an inverse* $\Psi \in \mathcal{L}(Y^{\sigma+1/p',p})$ we assume the simpler hypothesis H28.

We can now state our existence result.

THEOREM 3.1. *Let assumptions* H1-H6, H8-H29 *and* (1.28) *hold and assume*

$$f(0) \in Y^{\sigma+1/p',p} \qquad \text{if } \sigma \in (1/p, 1). \tag{3.2}$$

Then problem (0.1)-(0.3) *admits a unique solution* $(u, z) \in U^{2,p}_{\sigma,T} \times Z$.

We premise the proof of Theorem 3.1 with the following Lemma 3.1, whose proof will be given at the end of this section.

LEMMA 3.1. *Let assumptions* H10*a and* H29 *hold. Then* $\Phi S_0 \in \mathcal{C}(Y^{\sigma+1/p',p})$.

PROOF OF THEOREM 3.1. Using (almost) the same notations as in [13, Sections 4, 5, 6], we can rewrite problem (0.1)-(0.3) in the equivalent form

$$\Phi Q z = W_3(f, \psi_1, \psi_0) + W_4 z \tag{3.3}$$

where

$$Q = \sum_{j=0}^{+\infty}(-1)^j[(H+H_0A^{-1})*]^j E\,. \tag{3.4}$$

We recall that operators W_3 and W_4 are defined by the formulas

$$W_3(f, \psi_1, \psi_0) = \Phi[V_1 v_0 - f + \mathcal{R} * (V_1 v_0 - f)] - A\psi_1, \tag{3.5}$$

$$W_4 z = \Phi[(V_2 + V_3) z + \mathcal{R} * (V_2 + V_3) z], \tag{3.6}$$

where

$$V_1 v_0 = v_0 + L(H + H_0 A^{-1}) * (A v_0 + \mathcal{K} * A v_0), \tag{3.7}$$

$$V_2 z = [L(E') + L(H + H_0 A^{-1}) * (W_0 + \mathcal{K} * W_0)] z, \tag{3.8}$$

$$V_3 z = S_0(t) E(0) z, \tag{3.9}$$

$$v_0(t) = A S_0(t) \psi_0 + S_0(t) f(0) + \int_0^t S_0(t-s)\{[H(s)A + H_0(s)]\psi_0 + f'(s)\}\, ds, \qquad t \in [0, T], \tag{3.10}$$

$$L(N)(t) x = \int_0^t S_0(t-s) N(s) x\, ds, \qquad t \in [0, T], \qquad N \in L^1((0, T); \mathcal{L}(Y^{\sigma, p})), \quad x \in Y^{\sigma, p}, \tag{3.11}$$

$$W_0 = A S_0(t) E(0) + A L(E'). \tag{3.12}$$

Moreover, operators $\mathcal{R}$ and $\mathcal{K}$ are solutions to the following operator equations, respectively:

$$\mathcal{R}(t) + H(t) + H_0(t) A^{-1} + \int_0^t [H(t-s) + H_0(t-s) A^{-1}] \mathcal{R}(s)\, ds = 0, \qquad t \in [0, T], \tag{3.13}$$

$$\mathcal{K}(t) - \int_0^t A L(H + H_0 A^{-1})(t-s) \mathcal{K}(s)\, ds = A L(H + H_0 A^{-1})(t), \qquad t \in [0, T]. \tag{3.14}$$

We now observe that operator Q, defined by (3.4), is a solution to the integral equation

$$Q = -H * Q - H_0 A^{-1} * Q + E. \tag{3.15}$$

Hence, we deduce the representation

$$Q = \sum_{j=0}^{+\infty} (-1)^j (H*)^j E - \sum_{j=0}^{+\infty} (-1)^j (H*)^j H_0 * QA^{-1}. \tag{3.16}$$

From (3.3) and (3.16) we deduce the equation

$$\Phi\left(\sum_{j=0}^{+\infty} (-1)^j (H*)^j Ez\right) - \Phi\left(\sum_{j=0}^{+\infty} (-1)^j (H*)^j H_0 * QA^{-1} z\right) = \tag{3.17}$$

$$= W_3(f, \psi_1, \psi_0) + W_4 z.$$

Owing to (3.16) and assumption H29, equation (3.17) can be rewritten in the equivalent fixed-point form

$$z = \Psi W_3(f, \psi_1, \psi_0) + \Psi W_4 z + \Psi W_5 z \tag{3.18}$$

where

$$W_5 = \Phi\left(\sum_{j=0}^{+\infty} (-1)^j (H*)^j H_0 * QA^{-1}\right). \tag{3.19}$$

Since, owing to Lemma 6.1 in [13], $A^{-1} \in \mathcal{C}(Y^{\sigma + 1/p', p})$, it is immediate to check that $W_5 \in \mathcal{C}(Y^{\sigma + 1/p', p})$. Then, arguing as in [13, Section 6] and taking advantage of Lemma 3.1, we deduce that $W_4 \in \mathcal{C}(Y^{\sigma + 1/p', p})$. Consequently, $W_4 + W_5 \in \mathcal{C}(Y^{\sigma + 1/p', p})$. Hence, according to Theorem 2.1 in [13], equation (3.18) admits a unique solution for any choice of the triplet (f, ψ_0, ψ_1) satisfying H8-9 and (3.2).

This concludes the proof of our theorem. ■

Proof of Lemma 3.1. Consider the identity

$$\Phi(S_0 z) = \Phi(\chi_n S_0 z) + \Phi[(1 - \chi_n) S_0 z], \qquad \forall n \in \boldsymbol{N}. \tag{3.20}$$

We observe also that, owing to Lemma 6.1 in [13] and to Remark 3.1, $S_0(t) \in \mathcal{C}(Y^{\sigma + 1/p', p})$ for any $t \in \boldsymbol{R}_+$ and $S_0 \in C_b((0, T]; Y^{\sigma + 1/p', p})$. Then by virtue of Lemma 6.2 in [13] from any bounded sequence $\{z_m\} \subset \subset Y^{\sigma + 1/p', p}$, we can extract a subsequence $\{z_{m_k}\} \subset \{z_m\}$ and a function $u_0 \in C_b((0, T]; Y^{\sigma + 1/p', p})$ such that

$$S_0 z_{m_k} \to u_0 \text{ in } C([T/(2n), T]; Y^{\sigma + 1/p', p}) \text{ as } k \to +\infty, \ \forall n \in \boldsymbol{N}. \tag{3.21}$$

This implies

$$\chi_n S_0 z_{m_k} \to \chi_n u_0 \tag{3.22}$$

$$\text{in } C([T/(2n), T]; Y^{\sigma + 1/p', p}) \text{ as } k \to +\infty, \quad \forall n \in \boldsymbol{N}.$$

Hence

$$\Phi[\chi_n S_0 z_{m_k}] \to \Phi[\chi_n u_0] \quad \text{in } Y^{\sigma+1/p',p} \text{ as } k \to +\infty, \ \forall n \in \mathbf{N}. \tag{3.23}$$

We have thus proved that the sequence $\{\Phi[\chi_n S_0]\}$ belongs to $\mathcal{C}(Y^{\sigma+1/p',p})$, $\forall n \in \mathbf{N}$.

Finally, according to H29, we deduce that $\{\Phi[(1-\chi_n)S_0]\} \to 0$ in $\mathcal{L}(Y^{\sigma+1/p',p})$ as $n \to +\infty$. This, along with (3.20) and a classical result in Functional Analysis, implies that $\Phi S_0 \in \mathcal{C}(Y^{\sigma+1/p',p})$. ∎

We conclude this section proving the following Lemma 3.2.

LEMMA 3.2. *Let operator Φ be defined by* H0c, *where μ is a positive and finite Borel measure satifying*

$$\mu\{0\} = 0. \tag{3.24}$$

Then assumption H29 *is satisfied.*

PROOF. Under assumption H2 we deduce that S_0 satisfies

$$\|S_0(t)\|_{\mathcal{L}(Y^{\sigma+1/p',p})} \leq M, \qquad \forall t \in (0, +\infty), \tag{3.25}$$

for some positive constant M. Then we note that for any $z \in Y^{\sigma+1/p',p}$ with $\|z\|_{\sigma+1/p',p} = 1$, according to Lebesgue's dominated convergence theorem, the following relationships hold true:

$$\begin{aligned} &\|\Phi[(1-\chi_n)S_0 z]\|_{\sigma+1/p',p} \leq \\ &\leq \|\Phi\|_{\mathcal{L}(Y^{\sigma+1/p',p})} \int_0^T [1-\chi_n(t)] \|S_0(t) z\|_{\sigma+1/p',p} \, d\mu(t) \leq \\ &\leq M \|\Phi\|_{\mathcal{L}(Y^{\sigma+1/p',p})} \int_0^T [1-\chi_n(t)] \, d\mu(t) \to \\ &\to M \|\Phi\|_{\mathcal{L}(Y^{\sigma+1/p',p})} \int_0^T \varrho(t) \, d\mu(t) = M\|\Phi\| \mu\{0\} = 0, \qquad \text{as } n \to +\infty \end{aligned} \tag{3.26}$$

where we have set

$$\varrho(t) = \begin{cases} 1 & t = 0, \\ 0 & t \in (0, T]. \end{cases} \tag{3.27}$$

We have thus proved that assumption H29 is satisfied. ∎

REMARK 3.5. If Φ is defined by H0a or H0b, obviously Φ satisfies H29 by virtue of Lemmas 3.1 and 3.2.

4. – Some applications.

We consider in this section the explicit operator

$$A = \sum_{i,j=1}^{n} a_{i,j}(x) D_i D_j + \sum_{j=1}^{n} a_j(x) D_j + a_0(x), \qquad (D_j = \partial/\partial x_j), \tag{4.1}$$

related to an open, bounded, connected set $\Omega \subset \boldsymbol{R}^n$ of class $C^{2+\delta}$ ($\delta \in (0, 1)$).

We assume

$$a_{i,j} \in C^1(\overline{\Omega}) \quad (i, j = 1, \dots, n), \qquad a_j, a_0 \in C(\overline{\Omega}) \quad (j = 1, \dots, n), \tag{4.2}$$

where the $a_{i,j}$'s satisfy the uniform ellipticity condition

$$\sum_{i,j=1}^{n} a_{i,j}(x) \xi_i \xi_j \geqslant \mu |\xi|^2, \qquad \forall x \in \overline{\Omega}, \ \forall \xi \in \boldsymbol{R}^n, \tag{4.3}$$

for some positive constant μ. Moreover, we choose $X = L^p(\Omega)$ ($1 < p < + \infty$) and

$$Y_j = \mathcal{D}(A) = \{u \in W^{2,p}(\Omega) \colon D_\nu^j u = 0 \text{ on } \partial\Omega\}, \qquad j = 0, 1, \tag{4.4}$$

where ν denotes the outward normal unit vector on $\partial\Omega$.

We recall that X is a Banach lattice when equipped with the usual order relation ($f \geqslant 0$ in $X \Rightarrow f(x) \geqslant 0$ for a.e. $x \in \Omega$).

Using Lemma 3.8.1 in [33], we easily deduce that A satisfies assumptions H2.

Moreover, according to the results in Grisvard [5], we derive the following equations for the intermediate spaces:

$$Y_0^{\theta,p}(\Omega) = \begin{cases} W^{2\theta,p}(\Omega) & \text{if } \theta \in (0, 1/(2p)), \\ \{z \in W^{2\theta,p}(\Omega) \colon z = 0 \text{ on } \partial\Omega\} & \text{if } \theta \in (1/(2p), 1), \end{cases} \tag{4.5}$$

$$Y_1^{\theta,p} = \begin{cases} W^{2\theta,p}(\Omega) & \text{if } \theta \in (0, (p+1)/(2p)), \\ \{z \in W^{2\theta,p}(\Omega) \colon D_\nu z = 0 \text{ on } \partial\Omega\} & \text{if } \theta \in ((p+1)/(2p), 1). \end{cases} \tag{4.6}$$

Then, recalling that the functions $y \to \max(y, 0)$ and $y \to \max(-y, 0)$ are Lipschitz continuous, we deduce that assumption H5 is satisfied if we choose $\theta = \sigma + 1/p'$, when $0 < \sigma + 1/p' < 1/2$, and $\theta \in (0, 1/2)$, when $1/2 < \sigma + 1/p' < 1$.

We also assume that the linear operators H, H_0, E admit the representations

$$(4.7)\qquad H(t) = h(t)I, \quad H_0(t) = h_0(t)I, \quad E(t) = e(t, \cdot)I, \quad \forall t \in [0, T],$$

where I is the identity operator and h, h_0, e denote *scalar* functions satisfying

$$(4.8)\qquad h, h_0 \in W^{1,p}(0, T), \quad e \in W^{1,p}((0, T); C^{\alpha}(\overline{\Omega})), \; (\alpha > \sigma + 1/p').$$

As far as the functional Φ is concerned, we assume that it admits one of the representations H0*a*-H0*c*.

Whence and from (4.7)-(4.8), it is an easy task to check that assumptions H6, H10-H12, H14-15, H23-H28 are satisfied. Further, we observe that H13 is trivially satisfied when F^* reduces to the positive functional generated by any positive constant function.

Assumption H16 is implied by the following:

$$(4.9)\qquad h * h(t) \geq 0, \qquad \forall t \in [0, T],$$

$$(4.10)\qquad [h' - hh(0) + h_0](t) - h*[h' - hh(0) + h_0](t) \geq 0, \qquad \forall t \in [0, T],$$

$$(4.11)\qquad e(t, x) - h * e(t, x) \geq 0, \qquad \forall (t, x) \in [0, T] \times \Omega,$$

$$(4.12)\qquad \frac{\partial e}{\partial t}(t, x) - h * h * \frac{\partial e}{\partial t}(t, x) - [h' - hh(0) + h_0] * e(t, x) + \\ + h*[h' - hh(0) + h_0] * e(t, x) \geq 0, \qquad \forall (t, x) \in [0, T] \times \Omega.$$

Then we note that assumption H17 is nothing but Lemma 2.2: hence we have to require that kernel ω and measure μ satisfy properties (2.24) and (2.25), respectively. In order to derive H18 with the same τ related to Lemma 2.2 we require that the pair (h, e) satisfy the inequality

$$(4.13)\qquad e(\tau, x) - h * e(\tau, x) > 0 \quad \text{for a.e. } x \in \Omega.$$

Then, we observe that the linear operator Λ defined in H19, in our case, is the multiplication operator by the function $\phi(x) = |\Phi[k * e(\cdot, x)]|$, where

$$(4.14)\qquad k = \sum_{j=0}^{+\infty} (-1)^{j+1} (h*)^j h.$$

Consequently, Λ preserves disjointness.

We now note that properties H10, H20, H21 are implied by (4.2) and the following condition, when $\mathcal{D}(A) = Y_0$ (cf. Gilbarg-Trudinger [4, Section 8]):

(4.15a) $$a_0(x) \leqslant -\max(h(0), 0), \qquad \forall x \in \overline{\Omega}.$$

When $\mathcal{D}(A) = Y_1$ we must assume (cf. Chicco [1, Corollary 2])

(4.15b) $$a_0(x) < -\max(h(0), 0), \qquad \forall x \in \overline{\Omega}.$$

Owing to equations (4.5)-(4.6) and Lemma 3.1 in [13], property H25 is satisfied owing to (4.8).

Then we observe that property H29 unconditionally holds, when Φ is defined by H0a, H0b, while we need the additional assumptions $\mu\{0\} = 0$ when Φ is defined by H0c. Finally, we observe that in our case assumption H28 is implied by the following condition (cf. (4.14))

(4.16) $$|\Phi[k * e(\cdot, x)]| \geqslant m > 0, \qquad \forall x \in \overline{\Omega}.$$

The next Lemma 4.1 exhibits some sufficient conditions on the pair (h, e) assuring that condition (4.16) is satisfied.

LEMMA 4.1. *Let the pair* $(h, e) \in C([0, T]) \times C([0, T] \times \overline{\Omega})$ *satisfy either of the conditions*:

(4.17a) $$\begin{cases} h(t) \leqslant 0, \quad \forall t \in [0, T], \quad e(t, x) \geqslant 0, \quad \forall (t, x) \in [0, T] \times \overline{\Omega}, \\ \Phi[|h| * e(\cdot, x)] \geqslant m > 0, \quad \forall x \in \overline{\Omega}; \end{cases}$$

(4.17b) $$\begin{cases} h * h(t) \geqslant 0, \quad \forall t \in [0, T], \quad (h - h*h)*e(t, x) \geqslant 0, \quad \forall (t, x) \in [0, T] \times \overline{\Omega}, \\ \Phi[(h - h * h) * e(\cdot, x)] \geqslant m > 0, \quad \forall x \in \overline{\Omega}. \end{cases}$$

Then (h, e) *satisfies* (4.16) *with* k *defined by* (4.14).

REMARK 4.1. The first two conditions in (4.17b) are trivially satisfied if we assume

(4.18) $$\begin{cases} h(t) \geqslant 0, \quad h(t) - h * h(t) \geqslant 0, \quad \forall t \in [0, T], \\ e(t, x) \geqslant 0, \quad \forall (t, x) \in [0, T] \times \overline{\Omega}. \end{cases}$$

PROOF OF LEMMA 4.1. From (4.14), assumption (4.17a) and Lemma 1.4 we easily derive the chain of inequalities

$$(4.19)\qquad k*e = \sum_{j=0}^{+\infty}(-1)^{j+1}(h*)^{j+1}*e =$$

$$= \sum_{j=0}^{+\infty}(|h|*)^{j+1}*e \geq |h|*e \geq 0 \quad \text{in } [0, T]\times\overline{\Omega}.$$

Since Φ is a linear positive operator, we deduce

$$(4.20)\qquad |\Phi[k*e(\cdot, x)]| = \Phi[k*e(\cdot, x)] \geq \Phi[|h|*e(\cdot, x)] \geq m, \quad x\in\overline{\Omega}.$$

Reasoning as above and taking (4.11) into account, from assumption (4.18b) we derive the inequalities

$$(4.21)\qquad -k*e = \sum_{j=0}^{+\infty}(-1)^{j}(h*)^{j}*(h*e) \geq h*e - h*h*e \geq 0$$

$$\text{in } [0, T]\times\overline{\Omega},$$

which imply

$$(4.22)\qquad |\Phi[k*e(\cdot, x)]| =$$

$$= \Phi[-k*e(\cdot, x)] \geq \Phi[(h-h*h)*e(\cdot, x)] \geq m, \qquad x\in\overline{\Omega}. \qquad ■$$

It remains to investigate under which assumptions on functions $a_{i,j}$, a_j, a_0, h the basic condition (1.28) is satisfied. For the moment we take only the trivial case $h \leq 0$ in $[0, T]$ into account. The more general case $h*h \geq 0$ will be dealt with in the second part of this section in the one-dimensional case, only, and under the assumption $\mathcal{D}(A) = Y_2$.

We now consider the following parabolic integrodifferential identification problem: *determine a pair of functions* $u: [0, T]\times\overline{\Omega}\to\boldsymbol{R}$ *and* $z: \overline{\Omega}\to\boldsymbol{R}$ *such that*

$$(4.23)\qquad \frac{\partial u}{\partial t}(t, x) = Au(t, x) + \int_0^t h(t-s)Au(s, x)\,ds +$$

$$+ \int_0^t h_0(t-s)u(s, x)\,ds + e(t, x)z(x) + f(t, x), \qquad \forall (t, x)\in[0, T]\times\overline{\Omega},$$

$$(4.24)\qquad u(0, x) = \psi_0(x), \qquad \forall x\in\overline{\Omega},$$

$$(4.25)\qquad B_j u(t, x) = 0, \qquad \forall (t, x)\in[0, T]\times\overline{\Omega},$$

(4.26) $$\Phi[u(\cdot, x)] = \psi_1(x), \qquad \forall x \in \overline{\Omega},$$

where $j \in \{0, 1\}$ is fixed and

(4.27) $$B_0 z = z \quad \text{on } \partial\Omega, \qquad B_1 z = D_\nu z \quad \text{on } \partial\Omega.$$

Taking advantage of the abstract Theorem 3.1, from the results proved in this section, we deduce the following existence theorem.

THEOREM 4.1. *Assume that operator Φ is assigned by either* H0a *or* H0b *or* H0c *and let the functions $a_{i,j}$, a_j, a_0, h, h_0, e, ω and the positive Borel measure μ satisfy properties* (4.2), (4.3), (4.8), (4.10), (4.12), (4.15), (4.17*a*), (2.24), (2.25), (3.24) *and the following*

(4.28) $$h(t) \leqslant 0, \qquad \forall t \in [0, T].$$

Assume further that the data f, ψ_0, ψ_1 satisfy properties H8-H9. *Then each of the identification problems* (4.23)-(4.26) ($j = 0, 1$) *admits a unique solution $(u, z) \in U^{2,p}_{\sigma, T} \times Y^{\sigma + 1/p', p}_j$, which continuously depends on the data with respect to the norms related to assumptions* H8-H9.

We now consider now condition (1.28) in the more general case $h * h \geqslant 0$ under the following restrictions: $n = 1$, $\Omega = (0, l)$ $(l > 0)$, $j = 1$, we are given the Neumann boundary conditions and the coefficients of the operator A, rewritten in the simpler form

(4.29) $$A = \sum_{j=0}^{2} a_j(x) D^j$$

are assumed to enjoy the following properties:

(4.30) $$a_j \in C^j([0, l]), \quad (j = 0, 1, 2), \qquad a_2(x) \geqslant \mu > 0, \qquad \forall x \in [0, 1],$$

(4.31) $$a_2'(kl) = 2a_1(kl), \qquad k = 0, 1.$$

Consequently, proving (1.28) is equivalent to proving that the solution u to the Cauchy-Neumann problem

(4.32) $$D_t u(t, x) - Au(t, x) - h(0)u(t, x) = = -h(t)g(x), \qquad (t, x) \in (0, T) \times (0, l),$$

(4.33) $$u(0, x) = g(x), \qquad x \in [0, l],$$

(4.34) $$D_x u(t, kl) = 0, \qquad t \in [0, T], \quad k = 0, 1,$$

is positive for any positive g, when the positive part of the function h is assumed not to vanish everywhere in $[0, T]$ and to satisfy some additional requirement. More exactly, we can prove the following Theorem 4.2.

THEOREM 4.2. *Assume that coefficients* a_j $(j = 0, 1, 2)$ *satisfy properties* (4.30), (4.31) *and the positive part* h^+ *of the function* $h \in C([0, T])$ *satisfies the bound*

$$t^{1/2}\int_0^t (t-s)^{-1/2} e^{-s\lambda} h^+(s)\, ds \leq 1, \qquad \forall t \in [0, T], \tag{4.35}$$

where

$$\lambda = \min_{x \in [0, l]} \{a_2(x)[c''(x) + (c'(x))^2] + a_1(x)c'(x) + a_0(x) + h(0)\}, \tag{4.36}$$

$$c(x) = \frac{1}{4}\log\left(\frac{a_2(x)}{a_2(0)}\right) - \frac{1}{2}\int_0^x \frac{a_1(r)}{a_2(r)}\, dr, \qquad \forall x \in [0, l]. \tag{4.37}$$

Then for any $g \in L^p(\Omega; \overline{\boldsymbol{R}}_+)$ *the solution to problem* (4.32)-(4.34) *is nonnegative.*

PROOF. First we introduce the functions $b \in C^3([0, l])$ and $v: (0, T) \times (0, 1/2) \to \boldsymbol{R}$ defined, respectively, by

$$b(x) = \int_0^x a_2(r)^{-1/2}\, dr, \qquad \forall x \in [0, l], \tag{4.38}$$

$$u(t, x) = \exp[c(x) + t\lambda]\, v\left(t, \frac{b(x)}{2b(l)}\right). \tag{4.39}$$

Taking advantage of properties (4.30), it is easy to check that v solves the following Cauchy-Neumann problem

$$D_t v(t, \xi) - \frac{1}{4b(l)^2} D_\xi^2 v(t, \xi) - \alpha(\xi) v(t, \xi) = -e^{-t\lambda} h(t)\tilde{g}(\xi), \tag{4.40}$$

$$(t, \xi) \in (0, T) \times (0, 1/2),$$

$$v(0, \xi) = \tilde{g}(\xi), \qquad \xi \in [0, 1/2], \tag{4.41}$$

$$D_\xi v(t, k/2) = 0, \qquad t \in [0, T],\ k = 0, 1, \tag{4.42}$$

where

$$\alpha(\xi) = \{a_2[c'' + (c')^2] + a_1 c' + a_0 + h(0) - \lambda\} \circ b^{-1}(2b(l)\xi), \quad \forall \xi \in [0, 1/2], \tag{4.43}$$

$$\tilde{g}(\xi) = g(b^{-1}(2b(l)\xi)) \exp\{-[c \circ b^{-1}(2b(l)\xi) + t\lambda]\}, \quad \forall \xi \in [0, 1/2]. \tag{4.44}$$

We note that, owing to definitions (4.36), (4.37) and (4.45), functions α and $\tilde{g}$ are nonnegative on $[0, l]$.

Let G the Green function related to problem (4.40)-(4.42). Then (cf. Polozhiy [17]) the solution v is represented by

$$v(t, \xi) = \int_0^{1/2} G(t, \xi, \eta)\tilde{g}(\eta)\,d\eta - \int_0^t h(s)\,ds \int_0^{1/2} G(t-s, \xi, \eta)\,e^{-s\lambda}\tilde{g}(\eta)\,d\eta. \tag{4.45}$$

From formulas (4.39) and (4.45) we deduce that a sufficient condition for u to be positive for any positive g is that G should satisfy the following bound

$$G(t, \xi, \eta) - \int_0^t G(t-s, \xi, \eta)\,e^{-s\lambda}h(s)\,ds \geqslant 0, \quad \forall t \in (0, T],\ \forall \xi, \eta \in [0, 1/2]. \tag{4.46}$$

In order to derive such a result we recall that the Green function G_0 for the operator $D_t - \beta^2 D_\xi^2$, with $\beta = 1/(2b(l))$, related to the half-strip $\boldsymbol{R}_+ \times (0, 1/2)$ and the homogeneous Neumann conditions, is given (cf. Polozhiy [17]) by

$$G_0(t, \xi, \eta) = \theta(t, \xi - \eta) + \theta(t, \xi + \eta) \tag{4.47}$$

where

$$\theta(t, x) = \frac{1}{\beta^{5/2}t^{1/2}} \sum_{n=-\infty}^{+\infty} \exp\left(-\frac{(x-n)^2}{4\beta^2 t}\right). \tag{4.48}$$

Obviously, we get

$$G_0(t, \xi, \eta) > 0, \qquad \forall t \in \boldsymbol{R}_+,\ \forall \xi, \eta \in [0, 1/2], \tag{4.49}$$

$$(4.50)\qquad D_\xi G_0(t, k/2, \eta) = 0, \qquad \forall t \in \boldsymbol{R}_+, \ \forall \eta \in [0, 1/2], \ k = 0, 1.$$

We now look for the Green function G, related to problem (4.40)-(4.42), in the form

$$(4.51)\qquad G(t, \xi, \eta) = G_0(t, \xi, \eta) + \int_0^t ds \int_0^{1/2} G_0(t-s, \xi, \zeta)\varphi(s, \zeta, \eta)\, d\zeta.$$

Since G satisfies the boundary conditions (4.50) and solves the differential equation

$$(4.52)\qquad D_t G(t, \xi, \eta) - \frac{1}{\beta^2} D_\xi^2 G(t, \xi, \eta) - \alpha(\xi) G(t, \xi, \eta) = \delta(t)\delta(\xi - \eta)$$

where δ denotes the Dirac delta concentrated at the origin, we deduce that φ is a solution to the Volterra linear integral equation

$$(4.53)\qquad \varphi(t, \xi, \eta) = \alpha(\xi) \int_0^t ds \int_0^{1/2} G_0(t-s, \xi, \zeta)\varphi(s, \zeta, \eta)\, d\eta + \alpha(\xi) G_0(t, \xi, \eta), \qquad \forall t \in (0, T], \ \forall \xi, \eta \in [0, 1/2].$$

Since G_0 and α are positive, from well-known results (cf. Ladyzhenskaya-Solonnikov-Ural'ceva [9]) we obtain that equation (4.53) is uniquely solvable and

$$(4.54)\qquad \varphi(t, \xi, \eta) \geq 0, \qquad \forall t \in \boldsymbol{R}_+, \ \forall \xi, \eta \in [0, 1/2].$$

Consequently, owing to representation (4.51), it suffices to prove, instead of bound (4.46), the following:

$$(4.55)\qquad 0 \leq G_0(t, \xi, \zeta) - \int_0^t G_0(t-s, \xi, \zeta) e^{-s\lambda} h(s)\, ds = \theta(t, \xi - \zeta) + \theta(t, \xi + \zeta) - \int_0^t [\theta(t-s, \xi - \zeta) + \theta(t-s, \xi + \zeta)] \cdot e^{-s\lambda} h(s)\, ds, \qquad \forall t \in (0, T], \ \forall \xi, \zeta, \eta \in [0, 1/2].$$

Consequently, it suffices to show that the following bound holds true:

$$(4.56)\qquad 0 \leqslant \theta(t,\zeta) - \int_0^t \theta(t-s,\zeta)\,\mathrm{e}^{-s\lambda} h(s)\,ds = \frac{1}{\beta^{5/2}} \sum_{n=-\infty}^{+\infty} t^{1/2}\cdot$$

$$\cdot\exp\left(-\frac{(\zeta-n)^2}{4\beta^2 t}\right)\left\{1 - t^{1/2}\int_0^t (t-s)^{-1/2}\exp\left(-\frac{s(\zeta-n)^2}{4\beta^2 t(t-s)} - s\lambda\right)h(s)\,ds\right\},$$

$$\forall t \in (0, T],\ \ \forall \zeta \in [0, 1/2].$$

From (4.35) we easily derive the following inequalities, which show the positivity of the terms between curly brackets in (4.56):

$$(4.57)\qquad t^{1/2}\int_0^t (t-s)^{-1/2}\exp\left(-\frac{s(\zeta-n)^2}{4\beta^2 t(t-s)} - s\lambda\right)h(s)\,ds \leqslant$$

$$\leqslant t^{1/2}\int_0^t (t-s)^{-1/2}\mathrm{e}^{-s\lambda}h^+(s)\,ds \leqslant 1, \qquad \forall t \in (0, T],\ \ \forall \zeta \in [0, 1/2].$$

This concludes the proof of the theorem. ■

We conclude this section by stating the existence and uniqueness theorem for our identification problem in the case when h may be positive.

THEOREM 4.3. *In addition to the hypotheses listed in Theorem* 4.1 *assume that* $n = 1$, A *is defined by* (4.29) *with* $\mathcal{D}(A) = Y_2$ *and coefficients* a_j ($j = 0, 1, 2$) *enjoy properties* (4.30). *Moreover, assume that* h *satisfies properties* (4.35) *and* (4.9) (*instead of* (4.28)). *Then the identification problem* (4.23)-(4.26) *with* $j = 1$ *admits a unique solution* $(u, z) \in U^{2,p}_{\sigma,T} \times Y^{\sigma + 1/p', p}$, *which continuously depends on the data with respect to the norms related to assumptions* H8-H9.

PROOF. It is a straightforward consequence of the analysis developed in the first part of this section and of Theorem 4.2 assuring that the validity of the basic assumption (1.28). ■

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