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Isoptics of a closed strictly convex curve. - II

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Isoptics of a Closed Strictly Convex Curve. - II.

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1. - Introduction.

This article is concerned with some geometric properties of isoptics which complete and deepen the results obtained in our earlier paper [3]. We therefore begin by recalling the basic notions and necessary results concerning isoptics.

An α-isoptic C_α of a plane, closed, convex curve C consists of those points in the plane from which the curve is seen under the fixed angle $\pi - \alpha$.

We shall denote by $\mathcal{C}$ the set of all plane, closed, strictly convex curves. Choose an element $C \in \mathcal{C}$ and a coordinate system with the origin O in the interior of C. Let $p(t)$, $t \in [0, 2\pi]$, denote the support function of the curve C. It is well known [2] that the support function is differentiable and that C can be parametrized by

$$z(t) = p(t)\, e^{it} + \dot{p}(t)\, ie^{it} \qquad \text{for } t \in [0, 2\pi]. \tag{1.1}$$

We recall that the equation of C_α has the form

$$\begin{aligned} z_\alpha(t) &= p(t)\, e^{it} + \left(-p(t) \cot\alpha + \frac{1}{\sin\alpha}\, p(t+\alpha) \right) ie^{it} = \\ &= z(t) + \lambda(t, \alpha)\, ie^{it} = z(t+\alpha) + \mu(t, \alpha)\, ie^{i(t+\alpha)}, \end{aligned} \tag{1.2}$$

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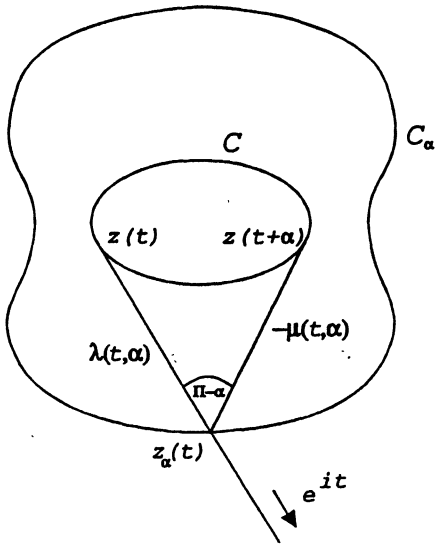

Fig. 1.

where

$$\lambda(\alpha, t) = \frac{1}{\sin\alpha}\,(p(t+\alpha) - p(t)\cos\alpha - \dot{p}(t)\sin\alpha)\,,$$

$$\mu(\alpha, t) = \frac{1}{\sin\alpha}\,(p(t+\alpha)\cos\alpha - \dot{p}(t+\alpha)\sin\alpha - p(t))\,,$$

and the tangent vector to C_α is given by the formula

$$\dot{z}_\alpha(t) = \left(-p(t)\cot\alpha + \frac{p(t+\alpha)}{\sin\alpha} - \dot{p}(t)\right)e^{it} + \left(p(t) - \dot{p}(t)\cot\alpha + \frac{\dot{p}(t+\alpha)}{\sin\alpha}\right)ie^{it} \tag{1.3}$$

for $t \in [0, 2\pi]$.

Moreover, the mapping F: $]0, \pi[\times]0, 2\pi[\to$ {the exterior of C}\{a certain support half-line} defined by $F(\alpha, t) = z_\alpha(t)$ is a diffeomorphism and the jacobian determinant $F'(\alpha, t)$ of F at (α, t) is equal to

$$F'(\alpha, t) = \frac{-\lambda(\alpha, t)\mu(\alpha, t)}{\sin\alpha}\,. \tag{1.4}$$

2. – Crofton-type formulae for annuli.

In this section we take C_β to be an arbitrary fixed isoptic, and we shall consider an annulus CC_β formed by C and C_β. Let $t_1(x, y)$ denote the distance between a point $(x, y) \in CC_\beta$ and a support point of C determined by the first, with respect to the orientation of C, support line of C passing by (x, y), (see fig. 2).

THEOREM 2.1. *If L is the length of $C \in \mathcal{C}$, then*

$$\iint_{CC_\beta} \frac{dx\,dy}{t_1(x, y)} = L \tan\frac{\beta}{2}. \tag{2.1}$$

PROOF. Using the diffeomorphism F we get

$$\iint_{CC_\beta} \frac{dx\,dy}{t_1(x, y)} = \int_0^{2\pi}\int_0^{\beta} \frac{1}{\lambda(t, \alpha)} \cdot \frac{-\lambda(t, \alpha)\mu(t, \alpha)}{\sin\alpha}\, d\alpha\, dt =$$

$$= \int_0^{\beta} \frac{1}{\sin^2\alpha} \int_0^{2\pi} (p(t) - p(t+\alpha)\cos\alpha + \dot{p}(t+\alpha)\sin\alpha)\, dt\, d\alpha =$$

$$= \int_0^{\beta} \frac{1}{\sin\alpha}(L - L\cos\alpha)\, d\alpha = L\int_0^{\beta} \frac{1}{2\cos^2\frac{\alpha}{2}}\, d\alpha = L\tan\frac{\beta}{2}. \quad \blacksquare$$

An application of this formula will be given in the next paragraph.

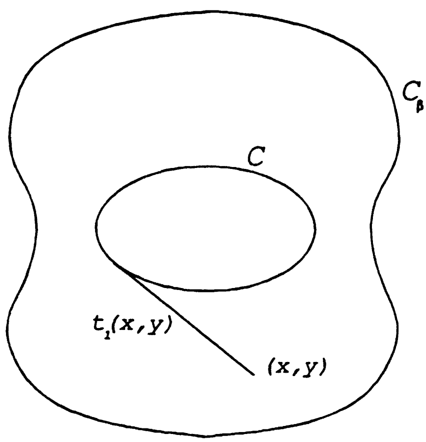

Fig. 2.

3. – Area of the annulus.

We shall now consider the expression $\{z_\alpha, \dot{z}_\alpha\}$, where $\{a+bi, c+ + di\} = ad - bc$. From (1.2), (1.3) we get

$$(3.1)\qquad \{z_\alpha(t), \dot{z}_\alpha(t)\} = \frac{1}{\sin^2\alpha}\,(p^2(t) + p^2(t+\alpha) - 2p(t)\,p(t+\alpha)\cos\alpha - \\ - \dot{p}(t)\,p(t+\alpha)\sin\alpha + p(t)\,\dot{p}(t+\alpha)\sin\alpha)\,.$$

Let $A(\alpha)$ denote the area of the region bounded by C_α. Using the Green formula

$$A(\alpha) = \frac{1}{2}\int_0^{2\pi}\{z_\alpha(t), \dot{z}_\alpha(t)\}\,dt$$

and next integrate by parts we get

$$(3.2)\quad A(\alpha)\sin^2\alpha = \int_0^{2\pi}(p^2(t) - p(t+\alpha)(\dot{p}(t)\sin\alpha + p(t)\cos\alpha))\,dt\,.$$

It follows that for an arbitrary strictly convex set C the function A is differentiable of class C^1.

THEOREM 3.1. *The function A satisfies the following differential equation*

$$(3.3)\qquad A'\sin\alpha + 2A\cos\alpha = G(\alpha)$$

and

$$(3.4)\qquad A'(0_+) = 0\,,$$

where

$$(3.5)\quad G(\tau) = \int_0^{2\pi}(p(t)\,p(t+\tau) - \dot{p}(t)\,\dot{p}(t+\tau))\,dt \qquad \text{for } \tau\in[0, 2\pi]\,.$$

PROOF. Differentiating (3.2) we obtain

$$(3.6)\qquad (\sin^2\alpha\,A(\alpha))' = G(\alpha)\sin\alpha\,.$$

Hence we get (3.3). The Crofton-type formula (2.1) implies

$$(3.7) \qquad L \tan\frac{\beta}{2} = \iint_{CC_\beta} \frac{dxdy}{t_1(x, y)} \geqslant$$

$$\geqslant \frac{1}{\max_{0 \leqslant t \leqslant 2\pi} \lambda(t, \beta)} \iint_{CC_\beta} dx\, dy = \frac{A(\beta) - A(0)}{\max_{0 \leqslant t \leqslant 2\pi} \lambda(t, \beta)} .$$

Thus we have

$$0 < \frac{A(\beta) - A(0)}{\beta} \leqslant L \frac{\tan \beta/2}{\beta} \max_{0 \leqslant t \leqslant 2\pi} \lambda(t, \beta).$$

This inequality leads us to (3.4). ■

REMARK 3.1. *If a convex curve C contains a segment and $A'(0)$ exists, then $A'(0) > 0$.*

Indeed, assume that the length of this interval is m. It is evident that the area bounded by the isoptic C_α is greater then the area of C plus the area of the triangle (cf. fig. 3). Thus we have

$$A(\alpha) - A(0) > \frac{m^2}{4} \tan\frac{\alpha}{2},$$

that is

$$(3.8) \qquad A'(0_+) \geqslant \frac{m^2}{8} > 0 .$$

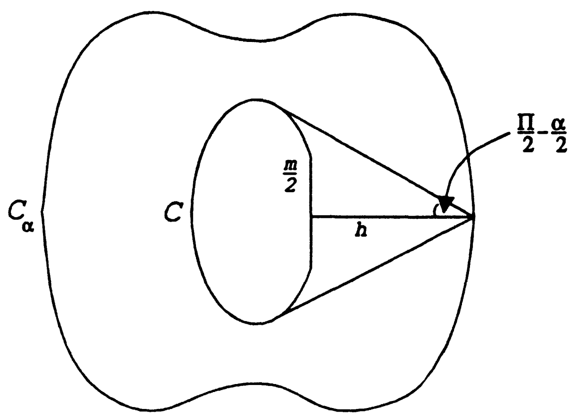

Fig. 3.

4. – Theorem on tangents to isoptic.

Let us fix an isoptic C_α of the curve C,

$$z_\alpha(t) = p(t)\, e^{it} + \left(- p(t) \cot \alpha + \frac{1}{\sin \alpha} p(t + \alpha) \right) ie^{it}.$$

We recall the following notations (cf. [3]):

$$\text{(4.1)} \qquad \begin{cases} b(t, \alpha) = p(t + \alpha) \sin \alpha + \dot{p}(t + \alpha) \cos \alpha - \dot{p}(t), \\ B(t, \alpha) = p(t) - p(t + \alpha) \cos \alpha + \dot{p}(t + \alpha) \sin \alpha, \\ q(t, \alpha) = z(t) - z(t + \alpha). \end{cases}$$

We have

$$\text{(4.2)} \qquad \begin{cases} q(t, \alpha) = B(t, \alpha)\, e^{it} - b(t, \alpha)\, ie^{it}, \\ \lambda(t, \alpha) = b(t, \alpha) - B(t, \alpha) \cot \alpha, \\ \mu(t, \alpha) = - \dfrac{B(t, \alpha)}{\sin \alpha} \end{cases}$$

and

$$\text{(4.3)} \qquad \dot{z}_\alpha(t) = - \lambda(t, \alpha)\, e^{it} + \varrho(t, \alpha)\, ie^{it},$$

where

$$\text{(4.4)} \qquad \varrho(t, \alpha) = B(t, \alpha) + b(t, \alpha) \cot \alpha.$$

Let us fix $\tau \in (0, 2\pi)$. We denote by $h^\tau(t, \alpha)$ the function $h(t + \tau, \alpha)$. Let $\angle(v, w)$ denote the angle between v and w.

THEOREM 4.1. *Let C_α be the α-isoptic of $C \in \mathcal{C}$. The following relation holds*

$$\text{(4.5)} \qquad \angle(\dot{z}_\alpha, \dot{z}_\alpha^\tau) + \angle(q, q^\tau) = 2\tau.$$

PROOF. We have

$$\dot{z}_\alpha^\tau = - (\lambda^\tau + \varrho^\tau \sin \tau)\, e^{it} + (\varrho^\tau \cos \tau - \lambda^\tau \sin \tau)\, ie^{it}$$

and

$$\langle \dot{z}_\alpha, \dot{z}_\alpha^\tau \rangle = \frac{bb^\tau + BB^\tau}{\sin^2 \alpha} \cos \tau + \frac{bB^\tau - b^\tau B}{\sin^2 \alpha} \sin \tau,$$

where $\langle\, ,\rangle$ is the canonical euclidean scalar product.

On the other hand

$$q^\tau = B^\tau(\cos\tau + i\sin\tau)e^{it} - b^\tau(i\cos\tau - \sin\tau)ie^{it}$$

and

$$\langle q, q^\tau\rangle = (BB^\tau + bb^\tau)\cos\tau - (bB^\tau - Bb^\tau)\sin\tau . \tag{4.6}$$

By the above consideration we get

$$\{q, q^\tau\} = (bB^\tau - Bb^\tau)\cos\tau + (bb^\tau + BB^\tau)\sin\tau . \tag{4.7}$$

By the above formulae we have

$$\sin^2\alpha\langle \dot z_\alpha, \dot z_\alpha^\tau\rangle = \{q, q^\tau\}\sin 2\tau + \langle q, q^\tau\rangle\cos\tau .$$

Taking into account that $|\dot z_\alpha|\sin\alpha = |q|$ we get

$$\cos\angle(\dot z_\alpha, \dot z_\alpha^\tau) = \cos(2\tau - \angle(q, q^\tau)) .$$

This shows that either

$$\angle(\dot z_\alpha, \dot z_\alpha^\tau) + \angle(q, q^\tau) = 2\tau \tag{4.8}$$

or

$$\angle(\dot z_\alpha, \dot z_\alpha^\tau) + 2\tau = \angle(q, q^\tau) \tag{4.9}$$

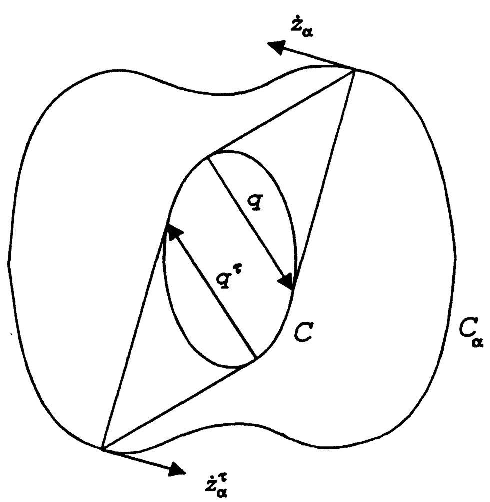

Fig. 4.

or

$$\angle(\dot{z}_\alpha, \dot{z}_\alpha^\tau) = 2\pi - 2\tau + \angle(q, q^\tau). \tag{4.10}$$

If $\tau \to 0$, then $\angle(\dot{z}_\alpha, \dot{z}_\alpha^\tau) \to 0$ and $\angle(q, q^\tau) \to 0$, on the other hand if $\tau \to \to 2\pi$, then $\angle(\dot{z}_\alpha, \dot{z}_\alpha^\tau) \to 2\pi$ and $\angle(q, q^\tau) \to 2\pi$. This implies relation (4. 8). ■

If $\tau = \pi$, then we get

COROLLARY 4.1.

$$\angle(\dot{z}_\alpha(t), \dot{z}_\alpha(t+\pi)) + \angle(q(t, \alpha), q(t+\pi, \alpha)) = 2\pi. \tag{4.11}$$

COROLLARY 4.2. *Vector $\dot{z}_\alpha$ is parallel to $\dot{z}_\alpha^\tau$ if and only if q is parallel to q^τ.*

5. - Isoptics of curves of constant width.

Let C: $z(t) = p(t)e^{it} + \dot{p}(t)ie^{it}$ be a curve of constant width d. Then its width is given by $d = p(t) + p(t+\pi)$. If $t \mapsto z_\alpha(t)$ is the parametrization of its α-isoptic then

$$z_\alpha(t) - z_\alpha(t+\pi) = de^{it} + \frac{d}{\sin\alpha}(1-\cos\alpha)ie^{it}. \tag{5.1}$$

It follows that

$$|z_\alpha(t) - z_\alpha(t+\pi)| = \frac{d}{\cos(\alpha/2)}. \tag{5.2}$$

Thus we get

THEOREM 5.1. *If $C \in \mathcal{C}$ is of constant width d then the distance between the points z_α and $z_\alpha(t+\pi)$ of its α-isoptic C_α is constant and equal to $d/\cos(\alpha/2)$.*

Now we prove the following

THEOREM 5.2. *Let $C \in \mathcal{C}$ and let α be linearly independent of π over $\boldsymbol{Q}$. If the distance between the points $z_\alpha(t)$ and $z_\alpha(t+\pi)$ on the α-isoptic C_α is constant then C is a curve of constant width.*

PROOF. First, we note that

$$z_\alpha(t) - z_\alpha(t+\pi) = d(t)e^{it} + \left\{ -d(t)\cot\alpha + \frac{d(t+\alpha)}{\sin\alpha} \right\} ie^{it},$$

where $d(t) = p(t) + p(t+\pi)$. Let

$$D = |z_\alpha(t) - z_\alpha(t+\pi)|.$$

Then there exists a function $t \mapsto \xi(t)$, $0 < \xi(t) < \pi$ such that

$$d(t) = D\sin\xi(t),$$

$$-d(t)\cot\alpha + \frac{d(t+\alpha)}{\sin\alpha} = D\cos\xi(t).$$

From these formulae it follows that

$$d(t+\alpha) = D\sin(\alpha + \xi(t)).$$

On the other hand we have

$$d(t+\alpha) = D\sin\xi(t+\alpha).$$

Thus we can write

$$\xi(t+\alpha) = \xi(t) + \alpha + 2\pi j$$

or

$$\xi(t+\alpha) = \pi - (\xi(t) + \alpha) + 2\pi k$$

for some $k, j \in \mathbf{Z}$. Since $0 < \xi(t) < \pi$, then

$$\xi(t+\alpha) = \xi(t) + \alpha \tag{5.3}$$

or

$$\xi(t+\alpha) + \xi(t) + \alpha = \pi. \tag{5.4}$$

The function $d(t) = p(t) + p(t+\pi)$ is periodic of period 2π. Thus

$$\xi(t+2\pi) = \xi(t) + 2\pi m, \tag{5.5}$$

but since $0 < \xi(t) < \pi$ then

$$\xi(t+2\pi) = \xi(t). \tag{5.6}$$

The conditions (5. 3) and (5. 6) are contradictory because

$$\xi(t) = \xi(t + 2\pi) = \xi\left(t + 4 \cdot \frac{\pi}{2}\right) = \xi(t) + 4 \cdot \frac{\pi}{2}.$$

This means that (5.4) and (5.6) must hold. By (5.4) we have

$$\xi(t + 2\alpha) + \xi(t + \alpha) + \alpha = \pi. \tag{5.7}$$

Thus subtracting (5.4) from (5.7) we get

$$\xi(t + 2\alpha) = \xi(t).$$

This means that the function ξ has two periods 2π and 2α. Since α is linearly independent of π over $\boldsymbol{Q}$, then ξ has to be constant. ■

In the above theorem α has to be necessarily linearly independent of π over $\boldsymbol{Q}$. This condition can not be removed as shows the example of an ellipse and its $(\pi/2)$-isoptic which is a circle.

6. - Differential equations related to isoptics.

In this paragraph we shall consider a curve $C \in \mathcal{C}$ satisfying the following condition:

$$\begin{cases} p \in C^2, \\ R(t) = p(t) + \ddot{p}(t) > 0, \end{cases} \tag{6.1}$$

where R is the radius of curvature. The curve C will be then called an oval.

Let us fix an oval C and consider a family of its isoptics $\{C_\alpha : 0 < < \alpha < \pi\}$, where C_α is an isoptic given by $z_\alpha(t) = z(t, \alpha) = z(t) + + \lambda(t, \alpha)\, ie^{it}$. We shall now find a differential equation which is satisfied by the function λ. Let us note that

$$\begin{cases} \dfrac{\partial b}{\partial \alpha} = R(t + \alpha)\cos\alpha, \\ \dfrac{\partial B}{\partial \alpha} = R(t + \alpha)\sin\alpha, \end{cases} \tag{6.2}$$

and

$$(6.3)\qquad \begin{cases} \dfrac{\partial b}{\partial t} = B(t, \alpha) + R(t + \alpha)\cos\alpha - R(t), \\ \dfrac{\partial B}{\partial t} = -\, b(t, \alpha) + R(t + \alpha)\sin\alpha . \end{cases}$$

THEOREM 6.1. *Let C be an oval and let p denote its support function. Let*

$$t \mapsto z_\alpha = p(t)\, e^{it} + (\dot{p}(t) + \lambda(t, \alpha))\, ie^{it}$$

be an α-isoptic of the oval C. Then the function $\lambda(t, \alpha) > 0$ satisfies the partial differential equation

$$(6.4)\qquad \frac{\partial \lambda}{\partial \alpha} - \frac{\partial \lambda}{\partial t} + \lambda(t, \alpha)\cot\alpha = R(t).$$

Moreover

$$(6.5)\qquad \lambda(t, 0) = 0 \text{ and } \lambda(t, -) \text{ is an increasing function.}$$

PROOF. We have $\lambda = b - B\cot\alpha$. Using (6.2) and (6.3) we get

$$(6.6)\qquad \frac{\partial \lambda}{\partial \alpha} = \frac{B(t, \alpha)}{\sin^2\alpha}, \qquad \frac{\partial \lambda}{\partial t} = B(t, \alpha) + b(t, \alpha)\cot\alpha - R(t).$$

Then formula (6.4) is easy to check. The condition (6.5) is obvious. ■

We shall find a partial differential equation for the function $v = |q| = \sqrt{b^2 + B^2}$. It follows from (6.2) and (6.3) that

$$(6.7)\qquad \begin{cases} \dfrac{\partial b}{\partial t} = B(t, \alpha) + \dfrac{\partial b}{\partial \alpha} - R(t), \\ \dfrac{\partial B}{\partial t} = -\, b(t, \alpha) + \dfrac{\partial B}{\partial \alpha} . \end{cases}$$

Differentiating the first equation with respect to α and then using the second one we get

$$\frac{\partial^2 b}{\partial \alpha^2} - \frac{\partial^2 b}{\partial \alpha \partial t} = R(t + \alpha)\sin\alpha ,$$

$$\frac{\partial^2 B}{\partial \alpha^2} - \frac{\partial^2 B}{\partial \alpha \partial t} = -\, R(t + \alpha)\cos\alpha .$$

Moreover, we have

$$\frac{\partial^2 B}{\partial t^2} - \frac{\partial^2 B}{\partial \alpha \partial t} + B(t, \alpha) = R(t) - R(t + \alpha)\cos\alpha .$$

We first find a differential equation for the function $u = (1/2)(b^2 + B^2)$. In view of the above calculation we get

$$\begin{cases} \dfrac{\partial^2 u}{\partial \alpha^2} = R^2(t+\alpha) + b(t, \alpha)\dfrac{\partial^2 b}{\partial \alpha^2} + B(t, \alpha)\dfrac{\partial^2 B}{\partial \alpha^2}, \\ \dfrac{\partial^2 u}{\partial \alpha \partial t} = \dfrac{\partial b}{\partial t}\dfrac{\partial b}{\partial \alpha} + b(t, \alpha)\dfrac{\partial^2 b}{\partial \alpha \partial t} + \dfrac{\partial B}{\partial t}\dfrac{\partial B}{\partial \alpha} + B(t, \alpha)\dfrac{\partial^2 B}{\partial \alpha^2}. \end{cases}$$

These equations imply

PROPOSITION 6.1. *The function* $u = (1/2)(b^2 + B^2)$ *satisfies the following differential equation*

$$\frac{\partial^2 u}{\partial \alpha^2} - \frac{\partial^2 u}{\partial t \partial \alpha} = R(t)R(t+\alpha)\cos\alpha . \tag{6.8}$$

In a similar way we find an equation for the function $v = \sqrt{2u}$. The function v satisfies the following partial differential equation

$$v\left(\frac{\partial^2 v}{\partial \alpha^2} - \frac{\partial^2 v}{\partial \alpha \partial t}\right) + \left(\frac{\partial v}{\partial \alpha}\right)^2 - \frac{\partial v}{\partial t}\frac{\partial v}{\partial \alpha} = R(t)R(t+\alpha)\cos\alpha . \tag{6.9}$$

Now we consider the function $F(\alpha) = \int_0^{2\pi} R(t)p(t+\alpha)\,dt$. Then

$$F''(\alpha) = \int_0^{2\pi} R(t)\ddot{p}(t+\alpha)\,dt ,$$

$$F(\alpha) + F''(\alpha) = \int_0^{2\pi} R(t)R(t+\alpha)\,dt .$$

By (6.8) we have

$$(F(\alpha) + F''(\alpha))\cos\alpha = \int_0^{2\pi}\left(\frac{\partial^2 u}{\partial \alpha^2} - \frac{\partial^2 u}{\partial t \partial \alpha}\right)(t, \alpha)\,dt = \tag{6.10}$$

$$= \frac{d^2}{d\alpha^2}\int_0^{2\pi} u(t, \alpha)\,dt = \frac{1}{2}\frac{d^2}{d\alpha^2}\int_0^{2\pi} |q(t, \alpha)|^2\,dt .$$

If we put

$$Q(\alpha) = \frac{1}{2} \int_0^{2\pi} |q(t, \alpha)|^2 \, dt, \tag{6.11}$$

then F satisfies the following differential equation

$$(F + F'') \cos \alpha = Q''. \tag{6.12}$$

This formula implies that if $C \in \mathcal{C}$ and $F \in C^2$, then

$$Q''\left(\frac{\pi}{2}\right) = 0. \tag{6.13}$$

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