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A Capacity Method for the Study of Dirichlet Problems for Elliptic Systems in Varying Domains.

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ABSTRACT - The asymptotic behaviour of solutions of second order linear elliptic systems with Dirichlet boundary conditions on varying domains is studied by means of a suitable notion of capacity.

Introduction.

Let Ω be a bounded open subset of $\mathbf{R}^n$ and let $\mathcal{A}: H_0^1(\Omega, \mathbf{R}^m) \to \to H^{-1}(\Omega, \mathbf{R}^m)$ be an elliptic operator of the form

$$\langle \mathcal{A}u, v \rangle = \int_\Omega (ADu, Dv)\, dx ,$$

where $A(x)$ is a fourth order tensor and $(\cdot, \cdot)$ denotes the scalar product between matrices. Given a sequence (Ω_j) of open subsets of Ω, we consider for every $f \in H^{-1}(\Omega, \mathbf{R}^m)$ the sequence (u_j) of the solutions of the Dirichlet problems

$$\begin{cases} u_j \in H_0^1(\Omega_j, \mathbf{R}^m), \\ \mathcal{A}u_j = f \quad \text{in } \Omega_j , \end{cases} \tag{0.1}$$

extended to Ω by setting $u_j = 0$ on $\Omega \backslash \Omega_j$. We want to describe the asymptotic behaviour of (u_j) as $j \to \infty$. As in the scalar case, a relaxation phenomenon may occur. Namely, if (u_j) converges weakly in $H_0^1(\Omega, \mathbf{R}^m)$ to some function u, then there exist an $m \times m$ matrix $B(x)$,

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with $|B(x)| = 1$, and a measure μ, not charging polar sets, such that u is the solution of the relaxed Dirichlet problem

$$\text{(0.2)} \qquad \begin{cases} u \in H_0^1(\Omega, \boldsymbol{R}^m) \cap L_\mu^2(\Omega, \boldsymbol{R}^m), \\ \int_\Omega (ADu, Dv)\, dx + \int_\Omega (Bu, v)\, d\mu = \langle f, v \rangle, \\ \forall v \in H_0^1(\Omega, \boldsymbol{R}^m) \cap L_\mu^2(\Omega, \boldsymbol{R}^m), \end{cases}$$

where, in the second integral, $(\cdot, \cdot)$ denotes the scalar product in $\boldsymbol{R}^m$, while $\langle \cdot, \cdot \rangle$ is the duality pairing between $H^{-1}(\Omega, \boldsymbol{R}^m)$ and $H_0^1(\Omega, \boldsymbol{R}^m)$. Compactness and localization results for the relaxed Dirichlet problems are established in [8] for symmetric A and B, and in [4] in the general case.

The problem we consider in this paper is the identification of the pair (B, μ) which appears in the limit problem (0.2). To this aim we introduce a suitable notion of capacity. If K is a compact subset of Ω and $\xi, \eta \in \boldsymbol{R}^m$, then the $\mathcal{A}$-capacity of K in Ω relative to ξ and η is defined as

$$C_{\mathcal{A}}(K, \xi, \eta) = \int_{\Omega \setminus K} (ADu^\xi, Du^\eta)\, dx\,,$$

where, for every $\zeta \in \boldsymbol{R}^m$, u^ζ is the weak solution in $\Omega \setminus K$ of the Dirichlet problem

$$\begin{cases} u^\zeta \in H^1(\Omega \setminus K, \boldsymbol{R}^m), \qquad u^\zeta = \zeta \quad \text{on } \partial K, \qquad u^\zeta = 0 \quad \text{on } \partial\Omega\,, \\ \int_{\Omega \setminus K} (ADu^\zeta, Dv)\, dx = 0\,, \qquad \forall v \in H_0^1(\Omega \setminus K, \boldsymbol{R}^m)\,. \end{cases}$$

For every $x \in \boldsymbol{R}^n$ let $D_\varrho(x)$ be the closed ball with centre x and radius ϱ. Assume that the limit

$$\lim_{j \to +\infty} C_{\mathcal{A}}(D_\varrho(x) \setminus \Omega_j, \xi, \eta) = \alpha(D_\varrho(x), \xi, \eta)$$

exists for every $x \in \Omega$ and for almost every $\varrho > 0$ such that $D_\varrho(x) \subset \Omega$. Our main result, Theorem 3.7, shows that, if α can be majorized by a Kato measure λ (Definition 1.1), then for λ-almost every $x \in \Omega$ there exists an $m \times m$ matrix $G(x)$ such that

$$\operatorname*{ess\,lim}_{\varrho \to 0} \frac{\alpha(D_\varrho(x), \xi, \eta)}{\lambda(D_\varrho(x))} = (G(x)\,\xi, \eta)\,, \qquad \forall \xi, \ \eta \in \boldsymbol{R}^m\,.$$

Moreover, for every $f \in H^{-1}(\Omega, \mathbf{R}^m)$, the sequence (u_j) of the solutions of (0.1) converges weakly in $H_0^1(\Omega, \mathbf{R}^m)$ to the solution u of (0.2) with $B(x) = (G(x))/|G(x)|$ and $\mu(E) = \int_E |G|\,d\lambda$. If $\mathcal{a}$ is symmetric, the same result (Theorem 4.3) holds whenever λ is a bounded measure.

1. – Notation and preliminaries.

Let $\mathbf{M}^{m \times n}$ be the space of all real $m \times n$ matrices $\xi = (\xi_j^\alpha)$ endowed with the scalar product

$$(\zeta, \xi) = \sum_{\alpha=1}^{m} \sum_{j=1}^{n} \zeta_j^\alpha \xi_j^\alpha$$

and with the corresponding norm $|\xi|^2 = (\xi, \xi)$. As usual, $\mathbf{R}^m$ is identified with $\mathbf{M}^{m \times 1}$. Let Ω be a bounded open subset of $\mathbf{R}^n$, $n \geqslant 3$. The case $n = 2$ can be treated in a similar way by using the logarithmic potentials. We assume that the boundary $\partial\Omega$ of Ω is of class C^1. The Sobolev space $H^1(\Omega, \mathbf{R}^m)$ is defined as the space of all functions u in $L^2(\Omega, \mathbf{R}^m)$ whose first order distribution derivatives $D_j u$ belong to $L^2(\Omega, \mathbf{R}^m)$, endowed with the norm

$$\|u\|^2_{H^1(\Omega, \mathbf{R}^m)} = \int_\Omega |Du|^2\,dx + \int_\Omega |u|^2\,dx\,,$$

where $Du = (D_j u^\alpha)$ is the Jacobian matrix of u. The space $H_0^1(\Omega, \mathbf{R}^m)$ is the closure of $C_0^1(\Omega, \mathbf{R}^m)$ in $H^1(\Omega, \mathbf{R}^m)$, and $H^{-1}(\Omega, \mathbf{R}^m)$ is the dual of $H_0^1(\Omega, \mathbf{R}^m)$. The symbol $\mathbf{R}^m$ will be omitted when $m = 1$.

For every subset E of Ω the (harmonic) capacity of E with respect to Ω is defined by $\operatorname{cap}(E) = \inf \int_\Omega |Du|^2\,dx$, where the infimum is taken over all functions $u \in H_0^1(\Omega)$ such that $u \geqslant 1$ almost everywhere in a neighbourhood of E, with the usual convention $\inf \emptyset = +\infty$.

A function $u\colon \Omega \to \mathbf{R}^m$ is said to be quasicontinuous if for every $\varepsilon > 0$ there exists a set $E \subset \Omega$, with $\operatorname{cap}(E) \leqslant \varepsilon$, such that the restriction of u to $\Omega \backslash E$ is continuous. We recall that for every $u \in H_0^1(\Omega, \mathbf{R}^m)$ there exists a quasicontinuous function $\tilde{u}$, unique up to sets of capacity zero, such that $u = \tilde{u}$ almost everywhere in Ω. We shall always identify u with $\tilde{u}$.

By a Borel measure on Ω we mean a positive, countably additive set function with values in $[0, +\infty]$ defined on the σ-field of all Borel subsets of Ω; by a Radon measure on Ω we mean a Borel measure which is finite on every compact subset of Ω. By $\mathcal{M}_0(\Omega)$ we denote the set of all positive Borel measures μ on Ω such that $\mu(E) = 0$ for every Borel set $E \subset \Omega$ with $\operatorname{cap}(E) = 0$. If E is μ-measurable in Ω, we define the Borel measure $\mu \llcorner E$ by $(\mu \llcorner E)(B) = \mu(E \cap B)$ for every Borel set $B \subset \Omega$, while $\mu_{|_E}$ is the measure on E given by $\mu_{|_E}(B) = \mu(B)$ for every Borel subset B of E.

For every $x \in \mathbf{R}^n$ and $\varrho > 0$ we set $U_\varrho(x) = \{y \in \mathbf{R}^n : |x - y| < \varrho\}$ and $D_\varrho(x) = \bar{U}_\varrho(x)$. A special class of measures we shall frequently use is the Kato space.

DEFINITION 1.1. The Kato space $K^+(\Omega)$ is the cone of all positive Radon measures μ on Ω such that

$$\lim_{\varrho \to 0^+} \sup_{x \in \Omega} \int_{\Omega \cap U_\varrho(x)} |y - x|^{2-n} d\mu(y) = 0 .$$

We recall that every measure in $K^+(\Omega)$ is bounded and belongs to $H^{-1}(\Omega)$. For more details about Kato measures we refer to [10] and [6].

Let $A(x) = (a_{\alpha\beta}^{ij}(x))$, with $1 \leqslant i, j \leqslant n$ and $1 \leqslant \alpha, \beta \leqslant m$, be a family of functions in $C(\bar{\Omega})$ satisfying the following conditions: there exist two constants $c_1 > 0$ and $c_2 > 0$ such that

$$(1.1) \qquad \begin{cases} c_1 |\xi|^2 \leqslant \sum_{i,j} \sum_{\alpha,\beta} a_{\alpha\beta}^{ij}(x) \xi_j^\beta \xi_i^\alpha , & \forall x \in \Omega, \ \forall \xi \in \mathbf{M}^{m \times n} , \\ \sum_{i,j} \sum_{\alpha,\beta} |a_{\alpha\beta}^{ij}(x)| \leqslant c_2 , & \forall x \in \Omega , \end{cases}$$

and let $\mathcal{A}: H_0^1(\Omega, \mathbf{R}^m) \to H^{-1}(\Omega, \mathbf{R}^m)$ be the elliptic operator defined by

$$\langle \mathcal{A}u, v \rangle = \int_\Omega (ADu, Dv)\, dx ,$$

where ADu is the $m \times n$ matrix defined by

$$(ADu)_i^\alpha = \sum_j \sum_\beta a_{\alpha\beta}^{ij} D_j u^\beta .$$

For fixed $x \in \Omega$ the Green's function $G(x, y) = G^x(y)$ is the solution of the problem

$$\begin{cases} \mathcal{A}^* G^x = \delta_x I \quad \text{in } \Omega, \\ G^x \in H_0^{1,p}(\Omega, \boldsymbol{M}^{m \times m}), \qquad 1 < p < \dfrac{n}{n-1}, \end{cases}$$

where $\mathcal{A}^*$ is the adjoint operator of $\mathcal{A}$, δ_x is the Dirac distribution at x, and I is the $m \times m$ identity matrix. Since the coefficients are continuous the existence of the Green's function can be obtained by a classical duality argument. It is well-known that, as the boundary of Ω is of class C^1, there exists a constant $c_3 > 0$ such that

$$|G(x, y)| \le c_3 |x - y|^{2-n}, \qquad \forall x, y \in \Omega. \tag{1.2}$$

This estimate can be proved by using classical regularity results, as in [1]. For any $\boldsymbol{R}^m$-valued bounded Radon measure μ, the solution u of the problem

$$\begin{cases} \mathcal{A}u = \mu \quad \text{in } \Omega, \\ u \in H_0^{1,p}(\Omega, \boldsymbol{R}^m), \qquad 1 < p < \dfrac{n}{n-1}, \end{cases}$$

can be represented for almost every $x \in \Omega$ as

$$u(x) = \int_\Omega G(x, y)\, d\mu(y). \tag{1.3}$$

If, in addition, $\mu \in H^{-1}(\Omega, \boldsymbol{R}^m)$, then this formula provides the quasi-continuous representative of the solution u.

2. Definition and properties of the μ-capacity.

We introduce now two notions of capacity associated with the operator $\mathcal{A}$.

DEFINITION 2.1. Let $\xi, \eta \in \boldsymbol{R}^m$ and let K be a compact subset of Ω. The capacity of K in Ω relative to the operator $\mathcal{A}$ and to the vectors ξ and η is defined by

$$C_{\mathcal{A}}(K, \xi, \eta) = \int_{\Omega \setminus K} (ADu^\xi, Du^\eta)\, dx, \tag{2.1}$$

where, for every $\zeta \in \boldsymbol{R}^m$, u^ζ is the weak solution in $\Omega \setminus K$ of the Dirichlet problem

$$\text{(2.2)}\quad \begin{cases} u^{\zeta} \in H^1(\Omega\backslash K, \boldsymbol{R}^m), & u^{\zeta} = \zeta \quad \text{on } \partial K, \qquad u^{\zeta} = 0 \quad \text{on } \partial\Omega\,, \\ \displaystyle\int_{\Omega\backslash K} (ADu^{\zeta}, Dv)\,dx = 0\,, & \forall v \in H_0^1(\Omega\backslash K, \boldsymbol{R}^m)\,. \end{cases}$$

We extend u^{ζ} to Ω by setting $u^{\zeta} = \zeta$ in K. In (2.2) the boundary conditions are understood in the following sense: for every $\varphi \in C_0^{\infty}(\Omega, \boldsymbol{R}^m)$ with $\varphi = \zeta$ on K we have $u^{\zeta} - \varphi \in H_0^1(\Omega\backslash K, \boldsymbol{R}^m)$.

REMARK 2.2. For every $\psi \in C_0^{\infty}(\Omega, \boldsymbol{R}^m)$ with $\psi = \eta$ on K we have

$$C_{\mathcal{a}}(K, \xi, \eta) = \int_{\Omega} (ADu^{\xi}, D\psi)\,dx\,.$$

This can be easily seen by taking $u^{\eta} - \psi$, which belongs to $H_0^1(\Omega\backslash K, \boldsymbol{R}^m)$, as test function in the equation (2.2) satisfied by u^{ξ}.

REMARK 2.3. The function $C_{\mathcal{a}}(K, \xi, \eta)$ is bilinear with respect to ξ and η. Moreover there exist two constants $c_4 > 0$ and $c_5 > 0$, depending on n, m, and on the constants c_1 and c_2 which appear in (1.1), such that

$$C_{\mathcal{a}}(K, \xi, \xi) \geqslant c_4 \operatorname{cap}(K)|\xi|^2 \qquad \text{and} \qquad |C_{\mathcal{a}}(K, \xi, \eta)| \leqslant c_5 \operatorname{cap}(K)|\xi|\,|\eta|\,,$$

for every compact set $K \subset \Omega$ and for every $\xi, \eta \in \boldsymbol{R}^m$. For the proof see Proposition 2.7.

Let $\mu \in \mathcal{M}_0(\Omega)$ and let $B = (b_{\alpha\beta})$ be an $m \times m$ matrix of Borel functions satisfying the following conditions: there exist two constants $c_6 > 0$ and $c_7 > 0$ such that

$$\text{(2.3)}\qquad c_6|\xi|^2 \leqslant \sum_{\alpha,\beta} b_{\alpha\beta}(x)\xi^{\alpha}\xi^{\beta}\,, \qquad \sum_{\alpha,\beta} |b_{\alpha\beta}(x)| \leqslant c_7\,,$$

for μ-almost every $x \in \Omega$ and every $\xi \in \boldsymbol{R}^m$.

DEFINITION 2.4. Let $\xi, \eta \in \boldsymbol{R}^m$. For every Borel set $E \subset\subset \Omega$ the (B, μ)-capacity of E in Ω relative to $\mathcal{a}$, ξ, and η is defined by

$$C_{\mathcal{a}}^{B,\mu}(E, \xi, \eta) = \int_{\Omega} (ADu^{\xi}, Du^{\eta})\,dx + \int_{E} (B(u^{\xi} - \xi), (u^{\eta} - \eta))\,d\mu\,,$$

where, for every $\zeta \in \boldsymbol{R}^m$, u^ζ is the solution of

$$
(2.4)\qquad \begin{cases} u^\zeta \in H_0^1(\Omega, \boldsymbol{R}^m), \qquad u^\zeta - \zeta \in L^2_\mu(E, \boldsymbol{R}^m), \\ \displaystyle\int_\Omega (ADu^\zeta, Dv)\,dx + \int_E (B(u^\zeta - \zeta), v)\,d\mu = 0, \\ \forall v \in H_0^1(\Omega, \boldsymbol{R}^m) \cap L^2_\mu(E, \boldsymbol{R}^m). \end{cases}
$$

The existence and the uniqueness of the solution u^ζ of problem (2.4) follow from the Lax-Milgram Lemma.

REMARK 2.5. For any $\psi \in H_0^1(\Omega, \boldsymbol{R}^m)$ with $\psi - \eta \in L^2_\mu(E, \boldsymbol{R}^m)$, we have

$$
(2.5)\qquad C_a^{B,\mu}(E, \xi, \eta) = \int_\Omega (ADu^\xi, D\psi)\,dx + \int_E (B(u^\xi - \xi), (\psi - \eta))\,d\mu .
$$

To prove this fact it is enough to take $u^\eta - \psi$, which belongs to $H_0^1(\Omega, \boldsymbol{R}^m) \cap L^2_\mu(E, \boldsymbol{R}^m)$, as test function in the equation (2.4) satisfied by u^ξ. In particular (2.5) gives

$$
C_a^{B,\mu}(E, \xi, \eta) = \int_\Omega (ADu^\xi, D\psi)\,dx ,
$$

if $\psi = \eta$ μ-almost everywhere on E.

REMARK 2.6. If μ is bounded, then $u^\eta \in L^2_\mu(E, \boldsymbol{R}^m)$, thus we may take u^η as test function in the equation satisfied by u^ξ and we obtain

$$
C_a^{B,\mu}(E, \xi, \eta) = -\int_E (B(u^\xi - \xi), \eta)\,d\mu .
$$

We shall compare now the capacity $C_a^{B,\mu}$ with the μ-capacity C^μ relative to the Laplacian, introduced in [7], Definition 5.1.

PROPOSITION 2.7. *There exist two constants $c_8 > 0$ and $c_9 > 0$, depending on n, m, and on c_1, c_2, c_6, c_7, such that for every Borel set $E \subset\subset \Omega$*

$$
(2.6)\qquad c_8 C^\mu(E)|\xi|^2 \leq C_a^{B,\mu}(E, \xi, \xi), \qquad \forall \xi \in \boldsymbol{R}^m ,
$$

$$
(2.7)\qquad |C_a^{B,\mu}(E, \xi, \eta)| \leq c_9 C^\mu(E)|\xi|\,|\eta| , \qquad \forall \xi, \eta \in \boldsymbol{R}^m .
$$

PROOF. To prove (2.6), let $v^\alpha = (u^\xi)^\alpha/\xi^\alpha$, if $\xi^\alpha \neq 0$, and $v^\alpha = 0$ otherwise. Then, using the ellipticity of A and B, for every Borel subset

$E \subset\subset \Omega$ and for every $\xi \in \boldsymbol{R}^m$ we obtain

$$\int_\Omega (ADu^\xi, Du^\xi)\, dx + \int_E (B(u^\xi - \xi), u^\xi - \xi)\, d\mu \geqslant$$

$$\geqslant k\left(\int_\Omega |Du^\xi|^2 dx + \int_E |u^\xi - \xi|^2 d\mu \right) \geqslant$$

$$\geqslant k|\xi|^2 \sum_{\alpha=1}^m \left(\int_\Omega |Dv^\alpha|^2 dx + \int_E |v^\alpha - 1|^2 d\mu \right),$$

where $k = \min\{c_1, c_6\}$. This implies that

$$C_\Omega^{B,\mu}(E, \xi, \xi) \geqslant mkC^\mu(E)|\xi|^2 .$$

Using Hölder Inequality it can be easily proved that

$$|C_\Omega^{B,\mu}(E, \xi, \eta)| \leqslant (C_\Omega^{B,\mu}(E, \xi, \xi))^{1/2}(C_\Omega^{B,\mu}(E, \eta, \eta))^{1/2} .$$

Hence it suffices to prove (2.7) for $\xi = \eta$. Let v_E be the C^μ-capacitary potential of E in Ω (see [6], Definition 3.1). Define $\psi^\alpha = (1 - v_E)\xi^\alpha$. By (2.5), using the boundedness of A and B, Young Inequality, and then Poincaré Inequality we get

$$C_\Omega^{B,\mu}(E, \xi, \xi) \leqslant M\left(\int_\Omega |Du^\xi|\,|D\psi|\, dx + \int_E |u^\xi - \xi|\,|\psi - \xi|\, d\mu \right) \leqslant$$

$$\leqslant \frac{M}{2}\left(\varepsilon \int_\Omega |Du^\xi|^2 dx + \frac{1}{\varepsilon} \int_\Omega |D\psi|^2 dx + \right.$$

$$\left. + \varepsilon \int_E |u^\xi - \xi|^2 d\mu + \frac{1}{\varepsilon} \int_E |\psi - \xi|^2 d\mu \right).$$

For a suitable choice of ε the sum of the terms containing u^ξ can be majorized by $(1/M)C_\Omega^{B,\mu}(E, \xi, \xi)$, hence there exists a constant K such that

$$C_\Omega^{B,\mu}(E, \xi, \xi) \leqslant K\left(\int_\Omega |D\psi|^2 dx + \int_E |\psi - \xi|^2 d\mu \right) \leqslant$$

$$\leqslant K|\xi|^2 \left(\int_\Omega |Dv_E|^2 dx + \int_E |v_E|^2 d\mu \right) = K|\xi|^2 C^\mu(E). \qquad \blacksquare$$

PROPOSITION 2.8. *For every Kato measure μ, the solution u^{ξ} of (2.4) corresponding to a Borel subset E of Ω of sufficiently small diameter belongs to $L^{\infty}(\Omega, \boldsymbol{R}^m)$ and tends to 0 in $L^{\infty}(\Omega, \boldsymbol{R}^m)$ as the diameter of E tends to zero.*

PROOF. Let E be a Borel subset of Ω and let u^{ξ} be the solution of (2.4). If $u^{\xi} \in L^{\infty}_{\mu}(\Omega, \boldsymbol{R}^m)$, then the representation formula (1.3) for the solution of a linear system of second order partial differential equations gives

$$u^{\xi}(x) = -\int_E G(x, y) B(y)(u^{\xi}(y) - \zeta)\, d\mu(y) \qquad \text{for a.e. } x \in \Omega, \tag{2.8}$$

where $G(x, y)$ is the Green's function associated with the operator $\mathcal{A}$ and with the domain Ω. In this case the measure $B(u^{\xi} - \zeta)\mu \llcorner E$ belongs to $H^{-1}(\Omega, \boldsymbol{R}^m)$ and (2.8) provides the quasicontinuous representative of u^{ξ}.

Let us consider the operator T: $L^{\infty}_{\mu}(\Omega, \boldsymbol{R}^m) \to L^{\infty}_{\mu}(\Omega, \boldsymbol{R}^m)$ defined by

$$Tf(x) = -\int_E G(x, y) B(y)(f(y) - \zeta)\, d\mu(y).$$

Since the functions $b_{\alpha\beta}$ are bounded, we may apply estimate (1.2) for the Green's function and we obtain

$$\|Tf_1 - Tf_2\|_{L^{\infty}_{\mu}(\Omega, \boldsymbol{R}^m)} \leq c_3 c_7 \|f_1 - f_2\|_{L^{\infty}_{\mu}(\Omega, \boldsymbol{R}^m)} \sup_{x \in \Omega} \int_E |x - y|^{2-n}\, d\mu(y).$$

As $\mu \in K^+(\Omega)$, the integral in the above formula tends to zero as diam(E) tends to zero, so that for sets E of sufficiently small diameter the operator T is a contraction, hence it has a unique fixed point w in $L^{\infty}_{\mu}(\Omega, \boldsymbol{R}^m)$. By (1.3), for $f \in L^{\infty}_{\mu}(\Omega, \boldsymbol{R}^m)$ the function $w_f = Tf$ is the solution of the Dirichlet problem

$$\begin{cases} w_f \in H^1_0(\Omega, \boldsymbol{R}^m), \\ Aw_f = -B(f - \zeta)\mu \llcorner E \qquad \text{in } \Omega, \end{cases}$$

so that the fixed point w belongs to $H^1_0(\Omega, \boldsymbol{R}^m)$ and is a solution in the sense of distributions of $Aw = -B(w - \zeta)\mu \llcorner E$, and hence a solution of

(2.4). Therefore $u^\zeta = w$ and we conclude that for sets E of sufficiently small diameter $u^\zeta \in L^\infty_\mu(\Omega, \boldsymbol{R}^m)$. Then, from (2.8), for the quasicontinuous representative of u^ζ we have

$$|u^\zeta(x)| = \left| \int_E G(x, y) B(y)(u^\zeta(y) - \zeta)\, d\mu(y) \right| \leqslant$$

$$\leqslant \int_E |G(x, y)|\, |B(y)|\, |u^\zeta(y) - \zeta|\, d\mu(y) \leqslant$$

$$\leqslant c_3 c_7 \|u^\zeta - \zeta\|_{L^\infty_\mu(\Omega, \boldsymbol{R}^m)} \sup_{x \in \Omega} \int_E |x - y|^{2-n}\, d\mu(y),$$

which implies that $\|u^\zeta\|_{L^\infty(\Omega, \boldsymbol{R}^m)} \leqslant c_E \|u^\zeta\|_{L^\infty_\mu(\Omega, \boldsymbol{R}^m)} + c_E |\zeta|$, where the coefficient c_E is given by $c_3 c_7 \sup_{x \in \Omega} \int_E |x - y|^{2-n}\, d\mu$ and tends to zero as the diameter of E tends to zero. As $u^\zeta \in H^1_0(\Omega, \boldsymbol{R}^m)$ and μ vanishes on sets of capacity zero, $\|u^\zeta\|_{L^\infty_\mu(\Omega, \boldsymbol{R}^m)} \leqslant \|u^\zeta\|_{L^\infty(\Omega, \boldsymbol{R}^m)}$ and from the previous inequality we obtain that $\|u^\zeta\|_{L^\infty(\Omega, \boldsymbol{R}^m)}$ tends to zero as the diameter of E tends to zero. ∎

THEOREM 2.9. *If μ is a Kato measure then*

$$\lim_{\varrho \to 0+} \frac{C^{B,\mu}_a(D_\varrho(x), \xi, \eta)}{\mu(D_\varrho(x))} = (B(x)\xi, \eta)$$

for μ-almost every $x \in \Omega$ and for every $\xi, \eta \in \boldsymbol{R}^m$.

PROOF. Let $x \in \Omega$. Since every $\mu \in K^+(\Omega)$ is bounded, by Remark 2.6 we have

$$C^{B,\mu}_a(D_\varrho(x), \xi, \eta) = -\int_{D_\varrho(x)} (B(y)(u^\xi(y) - \xi), \eta)\, d\mu(y). \tag{2.9}$$

By the Besicovitch Differentiation Theorem (see, e.g., [9], 1.6.2),

$$\lim_{\varrho \to 0+} \frac{1}{\mu(D_\varrho(x))} \int_{D_\varrho(x)} (B(y)\xi, \eta)\, d\mu(y) = (B(x)\xi, \eta) \tag{2.10}$$

for μ-almost every $x \in \Omega$ and for every $\xi, \eta \in \boldsymbol{R}^m$. The conclusion follows now from (2.9), (2.10), and Proposition 2.8. ∎

3. $\gamma^{\mathcal{A}}$-convergence.

In order to study the asymptotic behaviour of sequences of solutions of Dirichlet problems in varying domains we introduce the notion of $\gamma^{\mathcal{A}}$-convergence and show that under certain hypotheses the $\gamma^{\mathcal{A}}$-limit can be identified.

DEFINITION 3.1. Let (Ω_j) be a sequence of open subsets of Ω, let $\mu \in \mathcal{M}_0(\Omega)$, and let B be an $m \times m$ matrix of Borel functions satisfying (2.3). We say that (Ω_j) $\gamma_\Omega^{\mathcal{A}}$-converges to (B, μ), and we use the notation $\Omega_j \xrightarrow{\gamma_\Omega^{\mathcal{A}}} (B, \mu)$, if for every $f \in H^{-1}(\Omega, \boldsymbol{R}^m)$ the sequence (u_j) of the solutions of the problems

$$\begin{cases} u_j \in H_0^1(\Omega_j, \boldsymbol{R}^m), \\ \displaystyle\int_{\Omega_j} (ADu_j, Dv)\,dx = \langle f, v\rangle, \qquad \forall v \in H_0^1(\Omega_j, \boldsymbol{R}^m), \end{cases}$$

extended by zero on $\Omega \backslash \Omega_j$, converges weakly in $H_0^1(\Omega, \boldsymbol{R}^m)$ to the solution of the relaxed Dirichlet problem

$$\text{(3.1)} \qquad \begin{cases} u \in H_0^1(\Omega, \boldsymbol{R}^m) \cap L_\mu^2(\Omega, \boldsymbol{R}^m), \\ \displaystyle\int_\Omega (ADu, Dv)\,dx + \int_\Omega (Bu, v)\,d\mu = \langle f, v\rangle, \\ \forall v \in H_0^1(\Omega, \boldsymbol{R}^m) \cap L_\mu^2(\Omega, \boldsymbol{R}^m). \end{cases}$$

REMARK 3.2. Let $\mu \in \mathcal{M}_0(\Omega)$, let B be an $m \times m$ matrix of Borel functions satisfying (2.3), and let ν and C be defined by

$$\nu(E) = \int_E |B|\,d\mu, \qquad C(x) = \frac{B(x)}{|B(x)|}.$$

Then the measure ν belongs to $\mathcal{M}_0(\Omega)$ and the matrix C satisfies (2.3). Moreover $\Omega_j \xrightarrow{\gamma_\Omega^{\mathcal{A}}} (B, \mu)$ if and only if $\Omega_j \xrightarrow{\gamma_\Omega^{\mathcal{A}}} (C, \nu)$. This shows that, in Definition 3.1, it is not restrictive to assume $|B(x)| = 1$ for every $x \in \Omega$. However, it is sometimes useful to consider also matrices B which do not satisfy this condition.

If $m = 1$ and $\mathcal{A} = -\Delta$, we shall always assume that $B(x) = 1$ for every $x \in \Omega$. In this case we use the notation $\Omega_j \xrightarrow{\gamma_\Omega} \mu$.

The following compactness result is proved in [4].

THEOREM 3.3. *For every sequence* (Ω_j) *of open subsets of* Ω *there exist a subsequence* (Ω_{j_k}), *a measure* $\mu \in \mathcal{M}_0(\Omega)$, *and an* $m \times m$ *matrix of Borel functions satisfying* (2.3), *such that* $\Omega_{j_k} \xrightarrow{\gamma_\Omega} \mu$ *and* $\Omega_{j_k} \xrightarrow{\gamma^a_\Omega} (B, \mu)$.

The localization property of the γ^a-convergence is also proved in [4].

THEOREM 3.4. *If* $\Omega_j \xrightarrow{\gamma^a_\Omega} (B, \mu)$ *then* $\Omega_j \cap U \xrightarrow{\gamma^a_U} (B_{|U}, \mu_{|U})$ *for every open subset* U *of* Ω.

PROPOSITION 3.5. *Suppose that* $\Omega_j \xrightarrow{\gamma^a_\Omega} (B, \mu)$ *and* $\Omega_j \xrightarrow{\gamma^a_\Omega} (\tilde{B}, \tilde{\mu})$. *If* $\mu = \tilde{\mu}$ *and* $\mu(\Omega) < +\infty$, *then* $B(x) = \tilde{B}(x)$ *for* μ*-almost every* $x \in \Omega$.

PROOF. Let $f \in H^{-1}(\Omega, \mathbf{R}^m)$ and let u be the solution of the relaxed Dirichlet problem (3.1). Then we have

$$\int_\Omega ((B - \tilde{B})u, v)\, d\mu = 0\,, \qquad \forall v \in H_0^1(\Omega, \mathbf{R}^m) \cap L^2_\mu(\Omega, \mathbf{R}^m)\,.$$

In particular, since $\mu(\Omega) < +\infty$, this equality holds true for every $v \in C_0^\infty(\Omega, \mathbf{R}^m)$. So, varying v, we obtain that $(B - \tilde{B})u = 0$ μ-almost everywhere in Ω. Since $\mu(\Omega) < +\infty$, the set of all solutions u of (3.1) corresponding to different data $f \in H^{-1}(\Omega, \mathbf{R}^m)$ is dense in $H_0^1(\Omega, \mathbf{R}^m)$. This implies that $B = \tilde{B}$ μ-almost everywhere in Ω. ■

For every $x \in \Omega$ let $d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$.

THEOREM 3.6. *If* $\Omega_j \xrightarrow{\gamma_\Omega} \mu$, *with* $\mu(\Omega) < +\infty$, *and* $\Omega_j \xrightarrow{\gamma^a_\Omega} (B, \mu)$, *then for every* $x \in \Omega$ *there exists a countable set* $N(x) \subset \mathbf{R}$ *such that*

$$C_a(D_\varrho(x) \backslash \Omega_j, \xi, \eta) \to C_a^{B,\mu}(D_\varrho(x), \xi, \eta)$$

for every $\varrho \in (0, d_\Omega(x)) \backslash N(x)$.

PROOF. Let us fix $x \in \Omega$. It is proved in [7] that there exists a countable set $N_1(x) \subset \mathbf{R}$ such that for all $\varrho \in (0, d_\Omega(x)) \backslash N_1(x)$

$$\Omega \backslash (D_\varrho(x) \backslash \Omega_j) \xrightarrow{\gamma_\Omega} \mu \llcorner D_\varrho(x)\,.$$

Then, applying Theorem 3.3 to the sequence $\tilde{\Omega}_j = \Omega \backslash (D_\varrho(x) \backslash \Omega_j)$, we obtain that there exist a subsequence, still denoted by the same index j, and an $m \times m$ matrix $\tilde{B}$ of Borel functions satisfying (2.3) such that

$$\tilde{\Omega}_j \xrightarrow{\gamma_\Omega^a} (\tilde{B}, \mu \llcorner D_\varrho(x)).$$

Now we apply the localization result (Theorem 3.4) to the sequence (Ω_j) and we obtain

$$\Omega_j \cap U_\varrho(x) \xrightarrow{\gamma U_\varrho(x)} \mu_{|U_\varrho(x)} \quad \text{and} \quad \Omega_j \cap U_\varrho(x) \xrightarrow{\gamma^a_{U_\varrho(x)}} (B_{|U_\varrho(x)}, \mu_{|U_\varrho(x)}).$$

The same localization result applied to the sequence $\tilde{\Omega}_j$ gives

$$\Omega_j \cap U_\varrho(x) = \tilde{\Omega}_j \cap U_\varrho(x) \xrightarrow{\gamma^a_{U_\varrho(x)}} (\tilde{B}_{|U_\varrho(x)}, \mu_{|U_\varrho(x)}),$$

hence $B = \tilde{B}$ μ-almost everywhere in $U_\varrho(x)$ by Proposition 3.5. On the other hand, since $\mu(\Omega) < +\infty$, for every $x \in \Omega$ there exists a countable set $N_2(x) \subset \boldsymbol{R}$ such that $\mu(\partial D_\varrho(x)) = 0$ for all $\varrho \in (0, d_\Omega(x)) \setminus N_2(x)$. Together with the previous results this implies that

$$\tilde{\Omega}_j \xrightarrow{\gamma_\Omega^a} (B, \mu \llcorner D_\varrho(x)), \qquad \forall \varrho \in (0, d_\Omega(x)) \setminus (N_1(x) \cup N_2(x)).$$

Let $K_j = D_\varrho(x) \setminus \Omega_j = \Omega \setminus \tilde{\Omega}_j$ and let u_j be the weak solution in $\tilde{\Omega}_j$ of the problem

$$\begin{cases} u_j \in H^1(\tilde{\Omega}_j), \qquad u_j = \xi \quad \text{on } \partial K_j, \qquad u_j = 0 \text{ on } \partial\Omega, \\ \displaystyle\int_{\tilde{\Omega}_j} (ADu_j, Dv)\, dx = 0, \qquad \forall v \in H_0^1(\tilde{\Omega}_j, \boldsymbol{R}^m). \end{cases}$$

As usual we extend u_j to Ω by setting $u_j = \xi$ on K_j. Let $\varphi \in C_0^\infty(\Omega, \boldsymbol{R}^m)$ with $\varphi = \xi$ on $D_\varrho(x)$, and let $z_j = u_j - \varphi$. Then z_j is the solution of the problem

$$\begin{cases} z_j \in H_0^1(\tilde{\Omega}_j, \boldsymbol{R}^m), \\ \displaystyle\int_{\tilde{\Omega}_j} (ADz_j, Dv)\, dx = \langle f, v \rangle, \qquad \forall v \in H_0^1(\tilde{\Omega}_j, \boldsymbol{R}^m), \end{cases}$$

where f is the element of $H^{-1}(\Omega, \boldsymbol{R}^m)$ defined by $\langle f, v \rangle = -\int_\Omega (AD\varphi, Dv)\, dx$. By Definition 3.1 the sequence (z_j) converges weakly in $H_0^1(\Omega, \boldsymbol{R}^m)$ to the solution z of the problem

$$\begin{cases} z \in H_0^1(\Omega, \boldsymbol{R}^m) \cap L_\mu^2(D_\varrho(x), \boldsymbol{R}^m), \\ \int_\Omega (ADz, Dv)\,dx + \int_{D_\varrho(x)} (Bz, v)\,d\mu = \langle f, v\rangle, \\ \forall v \in H_0^1(\Omega, \boldsymbol{R}^m) \cap L_\mu^2(D_\varrho(x), \boldsymbol{R}^m). \end{cases}$$

This implies that (u_j) converges weakly in $H_0^1(\Omega, \boldsymbol{R}^m)$ to the solution u^ξ of (2.4) corresponding to $\zeta = \xi$ and $E = D_\varrho(x)$. Consequently (ADu_j) converges to ADu^ξ weakly in $L^2(\Omega, \boldsymbol{M}^{m\times n})$. Let us fix now $\psi \in C_0^\infty(\Omega, \boldsymbol{R}^m)$ with $\psi = \eta$ on $D_\varrho(x)$. Then, by Remarks 2.2 and 2.5,

$$C_{\mathfrak{a}}(D_\varrho(x)\backslash\Omega_j, \xi, \eta) = \int_\Omega (ADu_j, D\psi)\,dx,$$

$$C_{\mathfrak{a}}^{B,\mu}(D_\varrho(x), \xi, \eta) = \int_\Omega (ADu^\xi, D\psi)\,dx,$$

and the conclusion follows from the weak convergence of (ADu_j). ■

Given a family $(f_\varrho)_{\varrho>0}$ of real numbers, we say that $\operatorname{ess}\lim_{\varrho\to 0} f_\varrho = a$ if for every neighbourhood V of a there exists a neighbourhood U of 0 such that $f_\varrho \in V$ for almost every $\varrho \in U$. Let (Ω_j) be a sequence of open subsets of Ω. For every closed ball $D_\varrho(x) \subset \Omega$ and for every $\xi, \eta \in \boldsymbol{R}^m$ we define

$$\begin{cases} \alpha'(D_\varrho(x), \xi, \eta) = \liminf_{j\to\infty} C_{\mathfrak{a}}(D_\varrho(x)\backslash\Omega_j, \xi, \eta), \\ \alpha''(D_\varrho(x), \xi, \eta) = \limsup_{j\to\infty} C_{\mathfrak{a}}(D_\varrho(x)\backslash\Omega_j, \xi, \eta). \end{cases} \tag{3.2}$$

We are now in a position to prove the main result of the paper.

THEOREM 3.7. *Assume that there exists a measure $\lambda \in K^+(\Omega)$ such that*

$$\alpha''(D_\varrho(x), \xi, \xi) \leq \lambda(D_\varrho(x))\,|\xi|^2 \tag{3.3}$$

for every closed ball $D_\varrho(x) \subset \Omega$ and for every $\xi \in \boldsymbol{R}^m$. Assume, in addition, that for every $x \in \Omega$

$$\alpha'(D_\varrho(x), \xi, \eta) = \alpha''(D_\varrho(x), \xi, \eta) \quad \textit{for a.e. } \varrho \in (0, d_\Omega(x)). \tag{3.4}$$

Then there exists an $m \times m$ matrix $G(x)$ of bounded Borel functions such that

$$\operatorname*{ess\,lim}_{\varrho \to 0} \frac{\alpha'(D_\varrho(x), \xi, \eta)}{\lambda(D_\varrho(x))} = \operatorname*{ess\,lim}_{\varrho \to 0} \frac{\alpha''(D_\varrho(x), \xi, \eta)}{\lambda(D_\varrho(x))} = (G(x)\xi, \eta) \tag{3.5}$$

for λ-almost every $x \in \Omega$ and for every $\xi, \eta \in \mathbf{R}^m$. Let B and μ be defined by

$$B(x) = \frac{G(x)}{|G(x)|} \quad \text{for } \lambda\text{-a.e. } x \in \Omega\,,$$

$$\mu(E) = \int_E |G|\, d\lambda \quad \text{for every Borel set } E \subset \Omega\,,$$

with the convention that $0/0$ is the $m \times m$ identity matrix I. Then B satisfies (2.3) *and $\Omega_j \xrightarrow{\gamma_\Omega^{\mathfrak{a}}} (B, \mu)$.*

REMARK 3.8. Theorems 3.3 and 3.6 imply that every sequence (Ω_j) has a subsequence which satisfies (3.4). Therefore condition (3.3) is the only non-trivial hypothesis of Theorem 3.7.

REMARK 3.9. For every closed ball $D_\varrho(x) \subset \Omega$ let

$$\beta''(D_\varrho(x)) = \limsup_{j \to +\infty} \operatorname{cap}(D_\varrho(x) \backslash \Omega_j)\,.$$

If there exists a measure $\lambda \in K^+(\Omega)$ such that $\beta''(D_\varrho(x)) \leq \lambda(D_\varrho(x))$ then the estimates in Remark 2.3 imply that (3.3) is satisfied with λ replaced by $c_5\lambda$. This condition is satisfied, for instance, in the periodic case with a critical size of the holes (see [5]) and for the sequences of domains considered in [11] and [12].

PROOF OF THEOREM 3.7. Let us fix $x \in \Omega$. From the compactness result (Theorem 3.3) we obtain that there exist a subsequence, still denoted by (Ω_j), and a pair $(\tilde{B}, \tilde{\mu})$, with $\tilde{B}$ satisfying (2.3) and $\tilde{\mu} \in \mathcal{M}_0(\Omega)$, such that $\Omega_j \xrightarrow{\gamma_\Omega} \tilde{\mu}$ and $\Omega_j \xrightarrow{\gamma_\Omega^{\mathfrak{a}}} (\tilde{B}, \tilde{\mu})$. Let us fix $x \in \Omega$. By Theorem 5.15 in [7] for almost every $\varrho \in (0, d_\Omega(x))$ we have $\operatorname{cap}(D_\varrho(x) \backslash \Omega_j) \to$ $\to C^{\tilde{\mu}}(D_\varrho(x))$. The first estimate in Remark 2.3 gives

$$c_4 |\xi|^2 \operatorname{cap}(D_\varrho(x) \backslash \Omega_j) \leq C_{\mathfrak{a}}(D_\varrho(x) \backslash \Omega_j, \xi, \xi)\,,$$

and passing to the limit we get

$$c_4 C^{\tilde\mu}(D_\varrho(x))|\xi|^2 \leq$$

$$\leq \limsup_{\varrho\to 0} C_{\mathfrak{a}}(D_\varrho(x)\setminus\Omega_j, \xi, \xi) = \alpha''(D_\varrho(x), \xi, \xi) \leq \lambda(D_\varrho(x))|\xi|^2 .$$

Applying now Theorem 2.3 in [3] we get that $\tilde\mu$ is absolutely continuous with respect to λ and that the density $(d\tilde\mu/d\lambda)(x)$ is bounded, hence $\tilde\mu \in K^+(\Omega)$. Let

$$G(x) = \tilde B(x)\frac{d\tilde\mu}{d\lambda}(x), \qquad B(x) = \frac{G(x)}{|G(x)|},$$

$$\mu(E) = \int_E |G|\,d\lambda = \int_E |\tilde B|\,d\tilde\mu,$$

with the convention that 0/0 is the $m \times m$ identity matrix I. Then

$$B(x) = \frac{\tilde B(x)}{|\tilde B(x)|}, \quad \text{if } \frac{d\tilde\mu}{d\lambda}(x) > 0, \quad \text{and} \quad B(x) = I, \quad \text{if } \frac{d\tilde\mu}{d\lambda}(x) = 0 .$$

As $\tilde B$ satisfies (2.3) $\tilde\mu$-almost everywhere, B satisfies (2.3) μ-almost everywhere. Since $\Omega_j \xrightarrow{\gamma^{\mathfrak{a}}_\Omega} (\tilde B, \tilde\mu)$ and $B(x) = \tilde B(x)/|\tilde B(x)|$ $\tilde\mu$-almost everywhere in Ω, by Remark 3.2 we have also $\Omega_j \xrightarrow{\gamma^{\mathfrak{a}}_\Omega} (B, \mu)$.

Let us prove now (3.5). Applying Theorem 3.6 we obtain that $C_{\mathfrak{a}}(D_\varrho(x)\setminus\Omega_j, \xi, \eta) \to C^{\tilde B,\tilde\mu}_{\mathfrak{a}}(D_\varrho(x), \xi, \eta)$ for almost every $\varrho \in (0, d_\Omega(x))$. Thus

$$\alpha'(D_\varrho(x), \xi, \eta) = \alpha''(D_\varrho(x), \xi, \eta) = C^{\tilde B,\tilde\mu}_{\mathfrak{a}}(D_\varrho(x), \xi, \eta)$$

for almost every $\varrho \in (0, d_\Omega(x))$ and for every $\xi, \eta \in \boldsymbol{R}^m$. We may now apply Theorem 2.9 and the Besicovitch Differentiation Theorem to obtain

$$\operatorname*{ess\,lim}_{\varrho\to 0} \frac{\alpha'(D_\varrho(x), \xi, \eta)}{\lambda(D_\varrho(x))} = \operatorname*{ess\,lim}_{\varrho\to 0} \frac{C^{\tilde B,\tilde\mu}_{\mathfrak{a}}(D_\varrho(x), \xi, \eta)}{\tilde\mu(D_\varrho(x))} \operatorname*{ess\,lim}_{\varrho\to 0} \frac{\tilde\mu(D_\varrho(x))}{\lambda(D_\varrho(x))} =$$

$$= (\tilde B(x)\xi, \eta)\frac{d\tilde\mu}{d\lambda}(x) = (G(x)\xi, \eta)$$

for every ξ, $\eta \in \boldsymbol{R}^m$ and for λ-almost every $x \in \Omega$ such that $(d\tilde\mu/d\lambda)(x) > 0$. Since $C^{\tilde B,\tilde\mu}_{\mathfrak{a}}(D_\varrho(x), \xi, \eta) \leq c_9 C^{\tilde\mu}(D_\varrho(x))|\xi|\,|\eta| \leq$

$\leq c_9 \tilde{\mu}(D_\varrho(x))|\xi||\eta|$ by (2.7), we obtain that

$$\operatorname*{ess\,lim}_{\varrho \to 0} \frac{\alpha'(D_\varrho(x), \xi, \eta)}{\lambda(D_\varrho(x))} = 0 = (G(x)\xi, \eta)$$

for λ-almost every $x \in \Omega$ such that $(d\tilde{\mu}/d\lambda)(x) = 0$. This concludes the proof of (3.5). ∎

4. The symmetric case.

If the operator $\mathcal{A}$ is symmetric, then the $\mathcal{A}$-capacity can be obtained by solving a minimum problem. If $\Omega_j \xrightarrow{\gamma_\Omega^{\mathcal{A}}} (B, \mu)$, with $\mu(\Omega) < +\infty$, then the matrix B is symmetric (see [8], Corollary 5.4). In this case we have

$$C_{\mathcal{A}}^{B,\mu}(E, \xi, \xi) =$$

$$= \min_{u \in H_0^1(\Omega, \boldsymbol{R}^m)} \left\{ \int_\Omega (ADu^\xi, Du^\xi)\, dx + \int_E (B(u^\xi - \xi), (u^\xi - \xi))\, d\mu \right\}$$

for every measure $\mu \in \mathcal{M}_0(\Omega)$, for every $\xi \in \boldsymbol{R}^m$, and for every Borel set $E \subset\subset \Omega$.

REMARK 4.1. Assume that $\mathcal{A}$ and B are symmetric. If $\mu_1 \leq \mu_2$, then $C_{\mathcal{A}}^{B,\mu_1}(E, \xi, \xi) \leq C_{\mathcal{A}}^{B,\mu_2}(E, \xi, \xi)$ for every Borel set $E \subset\subset \Omega$ and every $\xi \in \boldsymbol{R}^m$.

This monotonicity property of the capacity with respect to the measure allows us to extend the derivation theorem to any bounded measure in $\mathcal{M}_0(\Omega)$.

THEOREM 4.2. *Assume that $\mathcal{A}$ is symmetric. Let $\mu, \nu \in \mathcal{M}_0(\Omega)$, with $\nu(\Omega) < +\infty$, and let B be an $m \times m$ symmetric matrix of Borel functions satisfying* (2.3). *For every $x \in \Omega$ and for every $\xi \in \boldsymbol{R}^m$ let*

(4.1) $f(x, \xi) =$

$$= \liminf_{\varrho \to 0} \frac{C_{\mathcal{A}}^{B,\mu}(D_\varrho(x), \xi, \xi)}{\nu(D_\varrho(x))} \qquad (\textit{with the convention that } 0/0 = 1).$$

Assume that there exists $\xi \in \boldsymbol{R}^m \setminus \{0\}$ *such that*

$$(4.2) \qquad f(x, \xi) < +\infty \quad \forall x \in \Omega \qquad \text{and} \qquad \int_\Omega f(x, \xi)\, d\nu < +\infty .$$

Then $\mu(\Omega) < +\infty$, μ *is absolutely continuous with respect to* ν, *and*

$$f(x, \xi) = (B(x)\xi, \xi) \frac{d\mu}{d\nu}(x) \qquad \text{for } \nu - \text{a.e. } x \in \Omega \text{ and } \forall \xi \in \boldsymbol{R}^m .$$

Moreover, the lim inf *in the definition of* f *is a limit for* ν*-almost every* $x \in \Omega$ *and for every* $\xi \in \boldsymbol{R}^m$.

PROOF. For every $x \in \Omega$ let

$$f_1(x) = \liminf_{\varrho \to 0} \frac{C^\mu(D_\varrho(x))}{\nu(D_\varrho(x))} .$$

The estimates in Proposition 2.7 give

$$(4.3) \qquad c_8 |\xi|^2 f_1(x) \leq f(x, \xi) \leq c_9 |\xi|^2 f_1(x), \qquad \forall x \in \Omega,\ \forall \xi \in \boldsymbol{R}^m ,$$

thus $f_1 \in L^1_\nu(\Omega)$ and $f_1(x) < +\infty$ for every $x \in \Omega$. Then from Proposition 2.3 in [3] we deduce that $\mu(\Omega) < +\infty$ and that $\mu = f_1 \nu$, i.e., $\mu(E) = \int_E f_1\, d\nu$ for every Borel set $E \subseteq \Omega$. By Proposition 2.5 of [2] there exist a measure $\lambda \in K^+(\Omega)$ and a Borel function g: $\Omega \to [0, +\infty]$ such that $\mu = g\lambda$. For every $k \in \boldsymbol{N}$ let $g_k(x) = \min\{g(x), k\}$. Since $g_k \lambda$ belongs to $K^+(\Omega)$, Theorem 2.9 implies the existence of a subset E_1 of Ω such that $\int_{E_1} g_k\, d\lambda = 0$ and

$$\lim_{\varrho \to 0} \frac{C^{B, g_k\lambda}_{\Omega}(D_\varrho(x), \xi, \xi)}{(g_k\lambda)(D_\varrho(x))} = (B(x)\xi, \xi), \quad \forall x \in \Omega \setminus E_1,\ \forall \xi \in \boldsymbol{R}^m,\ \forall k \in \boldsymbol{N}.$$

Since $\lambda + \nu$ is a bounded measure on Ω, by the Besicovitch Differentiation Theorem there exists a set $E_2 \subset \Omega$ such that $(\lambda + \nu)(E_2) = 0$ and

$$\lim_{\varrho \to 0} \frac{(g_k\lambda)(D_\varrho(x))}{(\lambda + \nu)(D_\varrho(x))} = g_k(x) \frac{d\lambda}{d(\lambda + \nu)}(x) < +\infty, \quad \forall x \in \Omega \setminus E_2,\ \forall k \in \boldsymbol{N},$$

$$\lim_{\varrho \to 0} \frac{\nu(D_\varrho(x))}{(\lambda + \nu)(D_\varrho(x))} = \frac{d\nu}{d(\lambda + \nu)}(x) \leq 1, \qquad \forall x \in \Omega \setminus E_2 .$$

By (4.2) and (4.3) we have $f_1(x) < +\infty$ and $f(x, \xi) < +\infty$ for every $x \in \Omega$ and for every $\xi \in \mathbf{R}^m$. Let $E = E_1 \cup E_2$. For $x \in \Omega \setminus E$ and $\xi \in \mathbf{R}^m$ we have

$$g_k(x)(B(x)\xi, \xi)\frac{d\lambda}{d(\lambda+\nu)}(x) =$$

$$= \lim_{\varrho \to 0} \frac{(g_k\lambda)(D_\varrho(x))}{(\lambda+\nu)(D_\varrho(x))} \lim_{\varrho \to 0} \frac{C_{\mathfrak{a}}^{B, g_k\lambda}(D_\varrho(x), \xi, \xi)}{(g_k\lambda)(D_\varrho(x))} =$$

$$= \lim_{\varrho \to 0} \frac{C_{\mathfrak{a}}^{B, g_k\lambda}(D_\varrho(x), \xi, \xi)}{(\lambda+\nu)(D_\varrho(x))} \leqslant$$

$$\leqslant \liminf_{\varrho \to 0} \frac{C_{\mathfrak{a}}^{B, g\lambda}(D_\varrho(x), \xi, \xi)}{\nu(D_\varrho(x))} \lim_{\varrho \to 0} \frac{\nu(D_\varrho(x))}{(\lambda+\nu)(D_\varrho(x))} = f(x, \xi)\frac{d\nu}{d(\lambda+\nu)}(x).$$

So, for every Borel set $F \subset \Omega \setminus E$ and for every $\xi \in \mathbf{R}^m$ we have

$$\int_F \left[g_k(x)(B(x)\xi, \xi)\frac{d\lambda}{d(\lambda+\nu)}(x)\right] d(\lambda+\nu) \leqslant$$

$$\leqslant \int_F \left[f(x, \xi)\frac{d\nu}{d(\lambda+\nu)}(x)\right] d(\lambda+\nu),$$

hence

$$\int_F g_k(x)(B(x)\xi, \xi)\, d\lambda \leqslant \int_F f(x, \xi)\, d\nu$$

for every Borel set $F \subset \Omega$. Passing now to the limit as $k \to +\infty$, by the monotone convergence theorem we have

$$\int_F (B(x)\xi, \xi)\, d\mu = \int_F g(x)(B(x)\xi, \xi)\, d\lambda \leqslant \int_F f(x, \xi)\, d\nu$$

for every Borel set $F \subset \Omega$ and every $\xi \in \mathbf{R}^m$. Thus, $f_1(x)(B(x)\xi, \xi) \leqslant$ $\leqslant f(x, \xi)$ for ν-almost every $x \in \Omega$ and for every $\xi \in \mathbf{R}^m$. Since

$$C_{\mathfrak{a}}^{B, \mu}(D_\varrho(x), \xi, \xi) \leqslant \int_{D_\varrho(x)} (B(y)\xi, \xi) f_1(y)\, d\nu(y),$$

by the Besicovitch Differentiation Theorem we obtain $f(x, \xi) \leqslant$ $\leqslant f_1(x)(B(x)\xi, \xi)$ for ν-almost every $x \in \Omega$ and for every every $\xi \in \mathbf{R}^m$. So we proved that $f(x, \xi) = f_1(x)(B(x)\xi, \xi)$ for every $\xi \in \mathbf{R}^m$ and ν-almost every $x \in \Omega$. Moreover, by the Besicovitch Differentiation Theorem for ν-almost every $x \in \Omega$ and for every $\xi \in \boldsymbol{R}^m$ we have

$$f(x, \xi) = \liminf_{\varrho \to 0} \frac{C_{\mathcal{a}}^{B,\mu}(D_\varrho(x), \xi, \xi)}{\nu(D_\varrho(x))} \leqslant \limsup_{\varrho \to 0} \frac{C_{\mathcal{a}}^{B,\mu}(D_\varrho(x), \xi, \xi)}{\nu(D_\varrho(x))} \leqslant$$

$$\leqslant \limsup_{\varrho \to 0} \frac{1}{\nu(D_\varrho(x))} \int_{D_\varrho(x)} (B(y)\xi, \xi) f_1(y)\, d\nu(y) = f_1(x)(B(x)\xi, \xi),$$

and this completes the proof. ■

The hypotheses in Theorem 3.7 can be weakened by using the monotonicity of the $\mathcal{a}$-capacity and the previous result.

THEOREM 4.3. *Assume that* $\mathcal{a}$ *is symmetric and that there exists a bounded Radon measure* λ *on* Ω *such that*

$$\alpha''(D_\varrho(x), \xi, \xi) \leqslant \lambda(D_\varrho(x))|\xi|^2$$

for every closed ball $D_\varrho(x) \subset \Omega$ *and for every* $\xi \in \boldsymbol{R}^m$. *Assume, in addition, that for every* $x \in \Omega$ *there exists a dense set* $D \subset (0, d_\Omega(x))$ *such that*

$$\alpha'(D_\varrho(x), \xi, \xi) = \alpha''(D_\varrho(x), \xi, \xi), \qquad \forall \varrho \in D,\ \forall \xi \in \boldsymbol{R}^m. \tag{4.4}$$

Then there exists an $m \times m$ *symmetric matrix* $G(x)$ *of bounded Borel functions such that*

$$\operatorname*{ess\,lim}_{\varrho \to 0} \frac{\alpha'(D_\varrho(x), \xi, \xi)}{\lambda(D_\varrho(x))} = \operatorname*{ess\,lim}_{\varrho \to 0} \frac{\alpha''(D_\varrho(x), \xi, \xi)}{\lambda(D_\varrho(x))} = (G(x)\xi, \xi)$$

for λ*-almost every* $x \in \Omega$ *and for every* $\xi \in \boldsymbol{R}^m$. *Let* B *and* μ *be defined by*

$$B(x) = \frac{G(x)}{|G(x)|} \qquad \textit{for } \lambda - \textit{a.e. } x \in \Omega,$$

$$\mu(E) = \int_E |G|\, d\lambda \qquad \textit{for every Borel set } E \subset \Omega,$$

with the convention that $0/0$ *is the* $m \times m$ *identity matrix* I. *Then* $\mu \in \mathcal{M}_0(\Omega)$, B *satisfies* (2.3), *and* $\Omega_j \xrightarrow{\gamma_\Omega^{\mathcal{a}}} (B, \mu)$.

PROOF. Since $C_{\mathcal{a}}(\cdot, \xi, \xi)$ is an increasing set function, $\alpha'(D_\varrho(x), \xi, \xi)$ and $\alpha''(D_\varrho(x), \xi, \xi)$ are increasing functions of ϱ, hence

(4.4) implies that $\alpha'(D_\varrho(x), \xi, \xi) = \alpha''(D_\varrho(x), \xi, \xi)$ for almost every $\varrho \in \in (0, d_\Omega(x))$. As in the proof of Theorem 3.7, we obtain that $\Omega_j \xrightarrow{\gamma^{\mathcal{A}}_{\Omega}} (\tilde{B}, \tilde{\mu})$, with $\tilde{\mu}$ absolutely continuous with respect to λ. Since $(d\tilde{\mu}/d\lambda)(x)$ is bounded, we have $\tilde{\mu}(\Omega) < +\infty$. Let $G(x) = \tilde{B}(x)(d\tilde{\mu}/d\lambda)(x)$. Since $\mu(E) = \int_E |G|\, d\lambda = \int_E |\tilde{B}|\, d\tilde{\mu}$, and $\tilde{\mu} \in \mathcal{M}_0(\Omega)$, we have $\mu \in \mathcal{M}_0(\Omega)$. The conclusion follows now by repeating the same arguments as in Theorem 3.7, the only difference being that now we apply Theorem 4.2 instead of Theorem 2.9. ■

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