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On the extension of a valuation on a field K to $K(X)$. - II

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On the Extension of a Valuation on a Field K to $K(X)$. - II.

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SUMMARY - Let K be a field and v a valuation on K. Denote by $K(X)$ the field of rational functions of one variable over K. In this paper we go further in the study of the extensions of v to $K(X)$. Now our aim is to characterize two types of composite valuations: r.a. extensions of first kind (Theorem 2.1) and the composite of two r.t. extension (Theorem 3.1). The results obtained are based on the fundamental theorem of characterization of r.t. extensions of a valuation (see [2], Theorem 1.2, and [6]) and on the theorem of irreducibility of lifting polynomials (see [7], Corollary 4.7 and [9], Theorem 2.1). The result of this work can be utilised, for example, to describe all valuations on $K(X_1, \ldots, X_n)$ (the field of rational functions of n independent variables) and elsewhere. A first account of this application is given in [10].

1. - Notations. General results.

1) By a valued field (K, v) we mean a field K and a valuation v on it. We shall utilise the notations given in [8, § 1] for notions like: residue field, value group, etc. Denote by $\overline{K}$ a fixed algebraic closure of K and denote by $\overline{v}$ a (fixed) extension of v to $\overline{K}$. Then $G_{\overline{v}}$ is just the rational closure of $G_v (G_{\overline{v}} = G_v \otimes_Z Q)$ and $k_{\overline{v}}$ is an algebraic closure of k_v. If $a \in \overline{K}$, the number $[K(a) : K]$ will be denoted by $\deg a$ (or $\deg_K a$ if there is danger of confusion). An element $(a, \delta) \in \overline{K} \times G_{\overline{v}}$ will be called a *minimal pair with respect to* (K, v) if for any $b \in \overline{K}$, the condition

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$\overline{v}(a-b) \geqslant \delta$ implies $\deg a \leqslant \deg b$. We shall say simply «minimal pair» if there are no doubts about (K, v).

Let $K(X)$ be the field of rational functions in an indeterminate X over K. If $r \in K(X)$, let $\deg r = [K(X) : K(r)]$. A valuation w on $K(X)$ will be called a r.t. (*residual transcendental*) *extension* of v to $K(X)$ if the (canonical) extension $k_v \subsetneq k_w$ is transcendental. The r.t. extensions of v to $K(X)$ are closely related to minimal pairs $(a, \delta) \in \overline{K} \times G_{\overline{v}}$.

Let (a, δ) be a minimal pair. Denote by: f the monic minimal polynomial of a over K and let $\gamma = \sum_{a'} \inf(\delta, \overline{v}(a-a'))$, where a' runs over all roots of f.

Moreover let v' the restriction of $\overline{v}$ to $K(a)$ (it may be proved that v' is the unique extension of v to $K(a)$).

Finally let e be the smallest non-zero positive integer such that $e\gamma \in G_{v'}$.

If $F \in K[X]$, let:

$$F = F_0 + F_1 f + \ldots + F_s f^s, \qquad \deg F_i < \deg f,$$

be the f-expansion of F. Let us put:

$$w_{(a,\delta)}(F) = \inf_{0 \leqslant i \leqslant s} (\overline{v}(F_i(a)) + i\gamma). \tag{1}$$

Then one has:

THEOREM 1.1 (see [2], [6]). *The assignment* (1) *defines a valuation on* $K[X]$ *which has a unique extension to* $K(X)$. *This valuation, denoted by* $w_{(a,\delta)}$ *is an r.t. extension of* v *to* $K(X)$. *Moreover one has:*

a) $G_{w(a,\delta)} = G_{v'} + Z\gamma \subseteq G_{\overline{v}}$.

b) Let $h \in K[X]$ *be such that* $\deg h < \deg f$ *and that* $v'(h(a)) = e\gamma$. *Then* $r = f^e/h$ *is an element of* $K(X)$ *of smallest degree such that* $w_{(a,\delta)}(r) = 0$, *and such that* r^* *the image of* r *in the residue field, is transcendental over* k_v. *One also has:* $k_{w(a,\delta)} = k_{v'}(r^*)$.

c) If (a, δ), (a', δ') *are two minimal pairs, then* $w_{(a,\delta)} = w_{(a',\delta')}$ *whenever* $\delta = \delta'$ *and* $\overline{v}(a-a') \geqslant \delta$.

d) If w *is a r.t. extension of* v *to* $K(X)$, *there exists a minimal pair* (a, δ) *(with respect to* (K, v)*) such that* $w = w_{(a,\delta)}$.

If $w = w_{(a,\delta)}$, *we shall say that* w is defined by the minimal pair (a, δ) and v.

Let $w = w_{(a,\delta)}$ *be an r.t. extension of* v *to* $K(X)$. *We keep the notations of the previous theorem. Let* g *be a monic polynomial in* $k_{v'}[r^*]$, *(with respect to the «indeterminate»* r^**), i.e.:*

$$g(r^*) = r^{*m} + A_1 r^{*m-1} + \dots + A_m\,, \qquad A_i \in k_{v'}\,,\ \ 1 \leq i \leq m\,.$$

By a lifting *of* g *to* $K[X]$ with respect to w *we mean (see* [9]) *a polynomial* $G \in K[X]$ *such that:*

i) $\deg G = me$,

ii) $w(G) = me\gamma$,

iii) $(G/h^m)^* = g$.

It is clear that there are many liftings of g to $K[X]$ with respect to w. However one has the following result:

THEOREM 1.2 ([9]). *Let* g *be an irreducible polynomial of* $k_{v'}[r^*]$ *with non-zero free term. Then any lifting* G *of* g *to* $K[X]$ *(with respect to* w*) is also an irreducible polynomial.*

2) The reader can refer to [11] for the notion of composite valuations appearing in the next result.

THEOREM 1.3. *Let* $w = w_{(a,\,\delta)}$ *be a r.t. extension of* v *to* $K(X)$*. Let* $g \in k_{v'}[r^*]$ *be an irreducible polynomial with non-zero free term and let* G *be a lifting of* g *to* $K[X]$ *(with respect to* w*). Let* u' *be the valuation on* $k_{v'}(r^*)$*, trivial on* $k_{v'}$*, defined by irreducible polynomial* g*. Denote by* u *the valuation on* $K(X)$ *composite with* w *and* u'*. Then:*

i) G_u *(the value group of* u*) is isomorphic to the direct product* $G_w \times G_{u'}$*, ordered lexicografically.*

ii) *Let* $F \in K[X]$ *and let*

$$F = F_0 + F_1 G + \dots + F_q G^q\,, \qquad \deg F_j < \deg G,\ \ 0 \leq j \leq q$$

be the G-expansion of F*. Then one has:*

$$u(G) = (me\gamma,\, 1)$$

$$u(F) = \inf_{0 \leq j \leq q} (w(F_j) + mj\gamma,\, j)\,.$$

PROOF. It is well know that $G_{u'} \simeq Z$. We shall divide the proof in two steps.

A) At this point we shall prove that $G_u \simeq G_w \times Z$, this last group being ordered lexicografically. According to the general theory of composite valuations (see (11) or (5)) there exists the exact sequence of groups:

$$0 \to G_{u'} \xrightarrow{\varepsilon} G_u \xrightarrow{p} G_w \to 0$$

where ε and p are defined in a canonical way. Now look at the Theorem 1.1. Let $\alpha \in K(X)$. Since $G_w = G_{v'} + Z\gamma$, and $e\gamma \in G_{v'}$, one has $w(\alpha) = q + t\gamma$, where $q \in G_{v'}$, and $0 \leqslant t < e$. Let us denote:

$$A = \{Hf^t, H \in K[X], \ \deg H < n, \ 0 \leqslant t < e\}.$$

For any $\alpha \in K(X)$ there exists $\alpha' \in A$ such that $w(\alpha) = w(\alpha')$. Thus one has $w(\alpha/\alpha') = 0$ and $u(\alpha/\alpha') = \varepsilon(u'((\alpha/\alpha')^*))$. Hence

(2) $$u(\alpha) = u(\alpha') + \varepsilon(u'((\alpha/\alpha')^*)).$$

Now we shall prove that the subset:

$$B = \{u(\alpha) \,|\, \alpha \in A\}$$

is a subgroup of G and $B \cap \varepsilon(G_{u'}) = 0$. Indeed, let $b = u(Hf^t) \in B$. Then $p(b) = w(Hf^t) = v'(H(a)) + t\gamma$. If $b = \varepsilon(c)$, then $p(b) = 0$, and so $v'(H(a)) = 0$, and $t = 0$. But then $c = u'(H(a)^*) = 0$, since $H(a)^* \in k_{v'}$, and u' is trivial over $k_{u'}$. Hence $B \cap \varepsilon(G_{u'}) = 0$, as claimed.

Let $u(Hf^t)$, $u(H'f^{t'})$ be two elements of B. In order to prove that B is a subgroup, one must show that their difference: $b = u((H/H')f^{t-t'})$ also belongs to B. First, let us assume that $t - t' \geqslant 0$. Let $H'' \in K[X]$ be such that $\deg H'' < n$ and that $w(H'') = v'(H''(a)) = w(H/H')$. Then $b = u(H''f^{t-t'})$. Indeed, one has $w((H/H')H'') = 0$ and so, according to ([7], Corollary 1.4), $((H/H')H'')^* \in k_{u'}$. Therefore, $u'(((H/H')f^{t-t'}/H''f^{t-t'})^*) = 0$, and so $u((H/H')f^{t-t'}) = b = u(H''f^{t-t'}) \in B$.

Now consider the case $t - t' < 0$. Then $(H/H')f^{t-t'} = (H/(H'f^e))f^{e+t-t'}$. Let $H'' \in K[X]$, $\deg H'' < n$, be such that $w(H'') = w(H/(H'f^e))$. As above, one has: $u((H/H')f^{t-t'}) = u(H''f^{e+t-t'}) \in B$. Therefore B is a subgroup of G_u, and by (2) it follows that there exists an isomorphism of groups:

$$G_u \xrightarrow{j} B \times \varepsilon(G_{u'}).$$

If $B \times \varepsilon(G_{u'})$ is ordered lexicografically, then j is an isomorphism of ordered groups. Indeed, let $\alpha, \beta \in K(X)$ be such that $u(\alpha) \leqslant u(\beta)$. Let $\alpha', \beta' \in A$ be such that $w(\alpha) = w(\alpha')$ and $w(\beta) = w(\beta')$. Then $u(\beta) = u(\beta') + \varepsilon(u'((\beta/\beta')^*))$. Since $u(\alpha) \leqslant u(\beta)$, it follows that $w(\alpha) \leqslant w(\beta)$ and so $w(\beta/\alpha') \geqslant 0$. Since the restriction of p to B defines an isomorphism of ordered groups to B onto G_w, it follows that $u(\beta') \geqslant \geqslant u(\alpha')$.

Let us assume that $u(\alpha) < u(\beta)$ and $u(\alpha') = u(\beta')$. Then by (2), it follows: $\varepsilon(u'((\beta/\beta')^*)) > \varepsilon(u'((\alpha/\alpha')^*))$. Hence $j(u(\beta)) > j(u(\alpha))$, as claimed.

We have already noticed that $B \simeq G_w$ and since $G_{u'} \simeq Z$ we may assume that

$$G_u \simeq G_w \times Z$$

where the right hand side is ordered lexicografically. Moreover, if $\alpha \in \in K(X)$ and $\alpha' \in A$ is such that $w(\alpha) = w(\alpha')$, then, by (2), one has: $u(\alpha) = (w(\alpha'), u'((\alpha/\alpha')^*)) \in G_w \times Z$.

B) Let G be a lifting of g (with respect to w). Now we shall determine u using G and w. Since $w(G) = me\gamma$ then we may choose $H \in A$ be such that $w(H) = me\gamma$. Then $u(G) = (w(H), u'((G/H)^*))$. But $(G/H)^* = (G/h^m)^*(h^m/H)^* = ag$, where $a \in k_v'$ (see [7] Corollary 1.4). Hence $u'((G/H)^*) = 1$. Therefore, one has:

$$u(G) = (w(H), 1) = (me\gamma, 1).$$

Now let $F \in K[X]$ be such that $\deg F < \deg G$. We assert that:

$$u(F) = (w(F), 0). \tag{3}$$

Indeed, let $\alpha \in A$, $\alpha = Hf^t$ be such that $w(\alpha) = w(F)$. Also, let $F = F_0 + + F_1 f + \ldots + F_s f^s$ be the f-expansion of F. Since $w(F) = w(\alpha) = = v'(H(\alpha)) + t\gamma$, then the smallest index i such that $w(F) = w(F_i) + i\gamma$ (see (1)) is necessary bigger than t, and thus

$$(F/\alpha)^* = \sum_{j=1}^{s} \left(\frac{F_j}{H} f^{j-t} \right)^* . \tag{4}$$

It is clear that if $j - t \not\equiv 0 (\mathrm{mod}\, e)$, then $w(F_j/Hf^{j-t}) > 0$ and so we may assume that only terms with $j - t \equiv 0 (\mathrm{mod}\, e)$ appear in (4). If we write for a such term:

$$\left(\frac{F_j}{H} f^{j-t} \right)^* = \left(\frac{F_j h^{(j-t)/e}}{H} \right)^* \left(\frac{f^{j-t}}{h^{(j-t)/e}} \right)^*$$

then, according to ([7], Corollary 1.4), it follows (4) is an element of $k_{v'}[r^*]$ whose degree (relatively to the variable r^*) is smaller than $m = = \deg g$. Hence $u'((F/\alpha)^*) = 0$, and so (3) holds, as claimed.

Furthermore, let $F \in K[X]$, and let $F = F_0 + F_1 G + \ldots + F_q G^q$ be the G-expansion of F. Let i be the smallest index such that $w(F_i) +$

$+ iw(G) \leqslant w(F_j) + jw(G)$ for all j, $0 \leqslant j \leqslant q$, and such that $w(F_i) +$ $+ iw(G) < w(F_j) + jw(G)$ for all $j < i$. We assert that one has:

$$w(F) = w(F_i) + iw(G)\,. \tag{5}$$

For that, we shall prove that an inequality in (5) (necessarily $>$) leads to a contradiction. Indeed, since $w(F_j G^j / F_i G_i) \geqslant 0$ for all j, $0 \leqslant j \leqslant q$, by the choice of i one has:

$$1 + \sum_{t=1}^{q-i} \left(\frac{F_{i+t}}{F_i} G^t \right)^* = 0 \tag{6}$$

or equivalently, since $(G/h^m)^* = g$,

$$1 + \sum_{t=1}^{s-i} \left(\frac{F_{i+t}}{F_i} h^{tm} \right)^* g^t = 0\,. \tag{7}$$

At this stage it is easy to see (according to the above considerations) that for all t, the non-zero coefficients of g^t in (6) are of the form U/V where $U, V \in k[r^*]$ and that $\deg U < m$, $\deg V < m$ (the degrees with respect to r^*). This shows that (6) is impossible, and so (5) holds, as claimed.

Furthermore by (7) it follows that $u'((F/F_i G^i)^*) = 0$ so that $u(F/F_i G^i) = \varepsilon(u'((F/F_i G)^*)) = 0$. Since $\deg F_i < \deg G$ we then have

$$u(F) = u(F_i G^i) + u(F/F_i G^i) = u(F_i) + iu(G) = (w(F_i), 0) + i(w(G), 1) =$$
$$= (w(F), i) = \inf_{0 \leqslant j \leqslant q} (w(F_j G^j), j) = \inf_{0 \leqslant j \leqslant q} (w(F_j) + mej\gamma, j)\,.$$

The proof of Theorem 1.3 is now complete.

2. – Extensions of the first kind in general setting.

We shall freely use the notations and definitions given in the previous section.

Let (K, v) be a valued field. A valuation u on $K(X)$ will be called an r.a. (residual algebraic) extension of v if u is an extension of v and the extension $k_v \subseteq k_u$ is algebraic. The r.a. extension u is called of the *first kind* if there exists an r.t. extension w of v to $K(X)$ such that $u \leqslant w$. Theorem 4.4 in [8] describes all r.a. extensions of the first kind of v when K is algebraically closed. Now we shall describe these extensions in the general setting (i.e. K is not necessarily algebraically closed). The results of this section generalise the results given in ([8], Section 3). Moreover, we give a simplified proof.

THEOREM 2.1. *Let (K, v) be a valued field. Let u be an r.a. extension of the first kind of v to $K(X)$. Let w be an r.t. extension of v to $K(X)$ such that $u \leqslant w$. Let u' be the valuation induced by u on k_w such that u is the composite with valuations w and u'. Then one has:*

1) *There exists an isomorphism of ordered groups $G_u \simeq G_w \times Z$, the direct product being ordered lexicografically.*

2) *Let u' be defined by the monic irreducible polynomial $g \in k_{v'}[r^*]$, whose free term is not zero (i.e. $g \not\equiv r^*$). Let G be a lifting of g to $K[X]$ with respect to w. If $F \in K[X]$ and $F = F_0 + F_1 G + \ldots + F_q G^q$ is the G-expansion of F, then one has:*

$$u(F) = \inf_{0 \leqslant j \leqslant q} (w(F_j G^j), j).$$

3) *Let u' be defined by r^*. If $F \in K[X]$ and $F = F_0 + F_1 f + \ldots + F_q f^q$ is the f-expansion of F, then one has:*

$$u'(F) = \inf_{0 \leqslant j \leqslant q} (w(F_j f^j), [j/e]).$$

4) *If u' is the valuation at the infinity (i.e. defined by r^{*-1}) then:*

$$u(F) = \inf_{0 \leqslant j \leqslant q} (u(F_j f^j), -[j/e]).$$

(Here $[j/e]$ means the integral part of a real number).

PROOF. The points 1) and 2) have been proved in Theorem 1.3, so we have to prove only 3) and 4).

Consider again the set A defined in the proof of Theorem 1.3. Let $\alpha \in A$ be such that $w(\alpha) = w(F)$. Let i be the smallest index j, such that, according to (1), one has:

$$w(F) = w(F_i f^i) = w(\alpha). \tag{7}$$

By this equality it follows that $i \geqslant t$. Hence:

$$\left(\frac{F}{\alpha}\right)^* = \sum_{j=i}^{q} \left(\frac{F_j}{H} f^{j-t}\right)^*.$$

By (7) it follows that for any j such that $j - i \not\equiv 0 (\bmod e)$, one has $((F_j/H) f^{j-t})^* = 0$. Therefore, we may assume that in the last equality only terms with $j - t \equiv 0 (\bmod e)$ appear. Since every term in the right hand side of the last equality may be written as:

$$\left(\frac{F_j}{H} f^{j-t}\right)^* = \left(\frac{F_j}{H} h^{(j-t)/e}\right)^* \left(\frac{f^{j-t}}{h^{(j-t)/e}}\right)^* = a_j r^{*(j-t)/e}$$

where $a_j \in k_{v'}$ (see [7], Corollary 1.4), and since $a_i \neq 0$, then one has:

$$u'((F/a)^*) = \frac{i-t}{e}$$

if u' is defined by r^*, and

$$u'((F/a)^*) = -\frac{i'-t}{e},$$

if u' is the valuation at infinity. (Here i' is the smallest index j such that $w(F) = w(F_j f^j)$.)

The proof of 3) and 4) follows by these two last equalities and (2).

3. - Composite of r.t. extensions.

Now, we are considering the Theorem 4.3 of [8] in the general setting.

Let (K, v) be a valued field (K is not necessarily algebraically closed) and let w be an r.t. extension of v to $K(X)$. As always, we preserve the notation and hypotheses given in Theorem 1.1. Let z' be a valuation on k_v and u' an extension of z' to $k_w = k_{v'}(r^*)$. Let z be the valuation on K composite with the valuations v and z' and let u be the valuation on $K(X)$ composite with the valuations w and u'. It is easy to see that u is an extension of z to $K(X)$. Moreover, according to ([8], Section 4.2), it follows that u is an r.t. extension of z to $K(X)$ if and only if u' is an r.t. extension of z' to $k_{v'}(r^*)$.

In this section we shall describe u by means of z', z, u', v and w. We shall use also Theorem 4.3 of [8].

Let $\overline{K}$ be an algebraic closure of K and let $\bar{z}$ be an extension of z to $\overline{K}$. Let $\bar{u}$ be a common extension of u and $\bar{z}$ to $\overline{K}(X)$ (see [3], Section 2). Let $s: G_u \to G_w$ be the canonical homomorphism of ordered groups for which $su = w$. Let $\overline{G}_w = G_w \otimes_Z Q$ and let $\bar{s}: G_{\bar{u}} \to \overline{G}_w$ be the unique homomorphism of ordered groups which naturally extends s. Let $\overline{w} = \bar{s}\bar{u}$. It is easy to see that $\overline{w}$ is a valuation on $\overline{K}(X)$ which extends w. Let $\bar{v}$ be the restriction of $\overline{w}$ to $\overline{K}$. It is clear that $\bar{v}$ is an extension of v to $\overline{K}$ and that $\overline{w}$ is a common extension of $\bar{v}$ and w to $\overline{K}(X)$. Also it is easy to see that (under the notation in [8]) one has: $\bar{z} \leqslant \bar{v}$ and $\bar{u} \leqslant \overline{w}$. Denote by $\bar{u}'$

the valuation induced by $\bar{u}$ on $k_{\bar{w}}$ and denote by $\bar{z}'$ the valuation induced by $\bar{z}$ on $k_{\bar{v}}$. It is clear that $\bar{u}'$ is an r.t. extension of $\bar{z}'$ and $\bar{z}'$ is an extension of z' to $k_{\bar{v}}$. Moreover, $\bar{u}'$ is a common extension of u' and $\bar{z}'$ to $k_{\bar{w}}$. One should note that $k_{\bar{w}} = k_{\bar{v}}(t)$, where t is a suitable element of $k_{\bar{w}}$, and t is transcendental over $k_{\bar{v}}$ (t will be defined later).

Let $w = w_{(a,\,\delta)}$ (see Theorem 1.1). Then $\bar{w}$ is also defined by the minimal pair (a, δ) (with respect to valuation $\bar{v}$). One has the following commutative diagram, whose rows are exact sequences:

$$\begin{array}{ccccccccc} 0 & \rightarrow & G_{u'} & \xrightarrow{\varepsilon} & G_u & \xrightarrow{s} & G_w & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & G_{\bar{u}'} & \xrightarrow{\bar{\varepsilon}} & G_{\bar{u}} & \xrightarrow{\bar{s}} & G_{\bar{w}} & \rightarrow & 0. \end{array}$$

In this diagram s and $\bar{s}$ are defined above and ε, $\bar{\varepsilon}$ are the natural inclusions. Since $G_{\bar{w}} = G_{\bar{v}}$, then (see [8], Theorem 3.3) we may assume that $G_{\bar{u}}$ is canonically isomorphic to the direct product $G_{\bar{w}} \times G_{\bar{u}}$ ordered lexicografically.

Let $(a', \delta') \in k_{\bar{v}} \times G_{\bar{z}}$, be a minimal pair with respect $k_{v'}$ such that $\bar{u}'$ is defined by this minimal pair and $\bar{z}'$. Denote by g the monic minimal polynomial of a' over $k_{v'}$. Because r^* is transcendental over $k_{v'}$ and $k_{v'}$ is a finite extension of k_v, then we may assume that $g \in k_{v'}[r^*]$. Let us assume that $g \neq r^*$ or, equivalently, $a' \neq 0$. Let G be a lifting of g in $K[X]$ with respect to w. Set $\lambda = (\delta, \delta') \in G_{\bar{u}}$. One has the fundamental result:

THEOREM 3.1. *There exists a root c of G in $\bar{K}$ such that (c, λ) is a minimal pair with respect to (K, z), and that u is defined by (c, λ) and z (i.e. one has: $u = w_{(c,\,\lambda)}$).*

PROOF. Denote by m the degree of the polynomial g with respect to variable r^*. According to the definition of a lifting polynomial, one has in k_w: $g = (G/h^m)^*$. Now we shall determine $(G/h^m)^{**}$, the image of G/h^m in $k_{\bar{w}}$. For that, we know that g is transcendental over k_v. Then, according to ([3], Proposition 1.1), there exist the roots $c_1, \dots, c_p$ of $G(X)$ such that (c_i, δ) is a pair of definition of $\bar{w}$ and $v(a - c_i) \geq \delta$, for all $1 \leq$ $\leq i \leq p$. Moreover, for other roots c' of G, which do not belong to $\{c_1, \dots, c_p\}$ one has: $v(a - c') < \delta$. Therefore, in $\bar{K}(X)$, we may write: $G(X) = \prod_{i=1}^{p} (X - c_i) G_1$, where $G_1 \in \bar{K}[X]$. It is clear that $\bar{w}(G_1(X)) =$

$=\bar{v}(G_1(a))$. Let $d \in \bar{K}$ be such that $\bar{v}(d)=\delta$. We may write:

$$G(X)=\prod_{i=1}^{p}(X-c_i)G_1(X)=\prod_{i=1}^{p}\left(\frac{X-a}{d}-\frac{(c_i-a)}{d}\right)G_1(X)d^p$$

and thus:

$$\left(\frac{G(X)}{G_1(a)d^p}\right)^*=\prod_{i=1}^{p}\left(\left(\frac{X-a}{d}\right)^*-\left(\frac{c_i-a}{d}\right)^*\right)\left(\frac{G_1(X)}{G_1(a)}\right)^*=$$

$$=b\prod_{i=1}^{p}\left(t-\left(\frac{c_i-a}{d}\right)^*\right)=\psi\ (t)$$

where

$$t=\left(\frac{X-a}{d}\right)^*,\qquad b=\left(\frac{G_1(X)}{G_1(a)}\right)^*\in k_{\bar{v}}\,.$$

Therefore, in the field $k_{\bar{w}}$, one has:

$$(G/h^m)^{**}=\left(\frac{G}{G_1(a)d^p}\right)^*\left(\frac{G_1(a)d^p}{h^m(a)}\right)^*=b_1\psi(t)\,.$$

Now, since $\bar{w}$ is an extension of w to $\bar{K}(X)$, there exists the natural inclusion $k_w=k_v'(r^*)\rightarrow k_{\bar{w}}=k_{\bar{v}}(t)$. That inclusion is defined by the canonical inclusion $k_v'\subseteq k_{\bar{v}}$ and by the assignment:

$$r^*\mapsto\varphi(t)$$

where $\varphi(t)$ is a polynomial defined as follows: Let $f(X)=\prod_{i=1}^{q}(X-a_i)f_1$, where $a_1=a, a_2,\dots,a_q$ are all the roots of f such that $\bar{v}(a_i-a)\geqslant\delta$, and $f_1\in\bar{K}[X]$. One has: $\bar{w}(f_1(X))=\bar{v}(f_1(a))$, and $w(h(X))=\bar{v}(h(a))$. We may write:

$$(8)\qquad \left(\frac{f^e}{h}\right)^{**}=r^*=\left(\frac{f^e}{h(a)}\right)^{**}=\left(\left(\prod_{i=1}^{q}(X-a_i)\right)^e\frac{f_1^e(a)}{h(a)}\right)^{**}=$$

$$=\prod_{i=1}^{q}\left(\left(\frac{X-a}{d}\right)-\left(\frac{a_i-a}{d}\right)\right)^{**}\left(\frac{d^{eq}f_1^e(a)}{h(a)}\right)^{**}=$$

$$=\prod_{i=1}^{q}\left(t-\left(\frac{a_i-a}{d}\right)^{**}\right)^e b'=\varphi(t),\qquad b'\in k_{\bar{v}}\,.$$

Therefore, one has:

$$(G/h)^{**} = (G/h)^{*}(\varphi(t)) = g(\varphi(t)).$$

On the other hand, if $a_1', \dots, a_m'$ are all the roots of $g(r^*)$ in $k_{\bar{v}}$, then, according to (8) the last equality becomes:

$$(9) \qquad (G/h)^{**} = \prod_{j=1}^{m} (\varphi(t) - a_j') = \prod_{j=1}^{m}\left(\prod_{i=1}^{q}\left(t - \left(\frac{a_i - a}{d}\right)^{**}\right)^{e} b' - a_j'\right) =$$

$$= b_1\,\varphi(t) = b_1 b \prod_{i=1}^{p}\left(t - \left(\frac{c_i - a}{d}\right)^{**}\right).$$

Denote by $a_1' = a'$. Then, by (9), there exists a root c of $G(X)$ such that $t - ((c-a)/d)^*$ is a root of $\varphi(t) - a'$, or, equivalently, $\varphi(((c-a)/d)^*) = a'$. This c is the root we looked for Theorem 3.1. We assert that:

$$(10) \qquad a' = \left(\frac{f^e(c)}{h(c)}\right)^{*} = \varphi\left(\left(\frac{c-a}{d}\right)^{*}\right)$$

i.e. a' is the image of $f^e(c)/h(c)$ in $k_{\bar{v}}$. Hence we must show that this last element has an image in $k_{\bar{v}}$ and this image is just a'. In order to do this, we notice that $\bar{w}(X - c) = \delta$, or equivalently, $\bar{v}(a - c) \geqslant \delta$. Therefore, for any $A \in K[X]$ with $\deg A < n$, one has: $\bar{v}(A(a)) = w(A(X)) = \bar{v}(A(c))$. Also, one has:

$$\bar{v}(f(c)) = \bar{v}\left(\prod_{i=1}(c - a_i)\right) = \sum_i \bar{v}(c - a_i).$$

But $\bar{v}(c - a_i) \geqslant \inf(\delta, \bar{v}(a - a_i)) = \bar{w}(X - a_i)$, and thus $\bar{v}(f(c))) \geqslant w(f(X)) = \gamma$. In conclusion, $\bar{v}(f^e(c)) \geqslant e\gamma = w(h) = \bar{v}(h(c))$ and thus $\bar{v}(f^e(c)/h(c)) \geqslant 0$ i.e. there exists $(f^e(c)/h(c))^*$. On the other hand we can write:

$$\frac{f^e(c)}{h(c)} = \frac{\prod_{i=1}^{n}(c - a_i)^e}{h(c)} =$$

$$= \prod_{i=1}^{q} \frac{(c - a_i)^e f_1^e(c)}{h(c)} = \prod_{i=1}^{q}\left(\frac{c-a}{d} - \frac{a - a_i}{d}\right)^{e} \frac{d^{eq} f_1^e(c)}{h(c)} =$$

$$= \prod_{i=1}^{q}\left(\frac{c-a}{d} - \frac{(a - a_i)}{d}\right)^{e} \frac{d^{eq}}{h(a)} f_1^e(a) \cdot \frac{h(a)}{h(c)} \cdot \frac{f_1^e(c)}{f_1^e(a)}$$

and thus:

$$\left(\frac{f^e(c)}{h(c)}\right)^* = \varphi\left(\left(\frac{c-a}{d}\right)^*\right)\cdot\left(\frac{h(a)}{h(c)}\cdot\frac{f_1^e(c)}{f_1^e(a)}\right)^*.$$

In proving (10) we must show that the second factor in the right hand side of the last equality is 1. This will result by the following statement:

($\triangle$). – Let $B(X) \in \overline{K}[X]$ and let $b_1, \dots, b_t$ be the roots of B in $\overline{K}$. Assume that, for any $1 \leqslant i \leqslant t$, one has: $\overline{v}(a - b_i) < \delta$. Then $\overline{v}(c - b_i) < \delta$, $1 \leqslant i \leqslant t$, $\overline{v}(B(a)) = \overline{v}(B(c))$ and $(B(a)/B(c))^* = 1$.

Proof of $\triangle$. Since $\overline{v}(a - c) \geqslant \delta$, then, by hypothesis, it follows that $\overline{v}(B(a)) = \overline{v}(B(c))$. Furthermore, we may write:

$$\frac{B(a)}{B(c)} = \prod_{i=1}^{t}\left(\frac{a-b_i}{c-a_i}\right) = \prod_i\left(1 + \frac{a-c}{c-a_i}\right)$$

and so, since $\overline{v}(a - c) > v(c - a_i)$, $1 \leqslant i \leqslant t$, it follows that: $(B(a)/B(c))^* = 1$, as claimed.

Now we are proving that (c, λ) is a minimal pair with respect to (K, z). In order to do this let $c' \in \overline{K}$ be such that $\overline{z}(c - c') \geqslant \lambda$. We must show that $[K(c):K] \leqslant [K(c'):K]$. According to the definition of $\overline{z}$, one has: $\overline{v}(c - c') \geqslant \delta$ whence (c', δ) is also a pair of definition of $\overline{w}$. Hence we may write:

$$\left(\frac{X-c'}{d}\right)^* = \left(\frac{X-a}{d}\right)^* - \left(\frac{c'-a}{d}\right)^* = t - \left(\frac{c'-a}{d}\right)^*.$$

By the hypothesis $\overline{z}(c - c') \geqslant \lambda$, the following holds:

$$\overline{z}'\left(\varphi\left(\left(\frac{c-a}{d}\right)^*\right) - \varphi\left(\left(\frac{c'-a}{d}\right)^*\right)\right) \geqslant \delta'. \tag{11}$$

Now, since (a', δ') is a minimal pair with respect to $(k_{v'}, z')$, by (10) and (11) it follows that the minimal polynomial of $\varphi(((c' - a)/d)^*)$ over $k_{v'}$, has the degree at least m.

Suppose that $[K(c):K] > [K(c'):K]$. Let G_1 be the monic minimal polynomial of c' over K and let

$$G_1 = A_0 + A_1 f + \ldots + A_q f^q$$

be the f-expansion of G_1. By hypothesis, one has: $q \leqslant (me-1)n$. Let $H \in K[X]$, $\deg H < n$ and let $0 \leqslant t < e$ be such that $w(G_1) = w(Hf^t)$. Let i be the smallest index j such that $w(G_1) = w(A_i f^i)$ (see (1)). Then, necessarily, $i \geqslant t$ and for any $j \geqslant i$, $w((A_j/H)f^{j-t}) > 0$ if $j - t \not\equiv 0$ $(\mathrm{mod}\, e)$. Hence $g_1(r^*) = (G_1/Hf^t)$ belongs to $k_v'(r^*)$ and its degree (with respect to r^*) is at most $m-1$. As above (see $(\triangle)$) it is easy to see that $(f^e(c')/h(c'))^* = \varphi(((c'-a)/d))^*)$ is a root of $g_1(r^*)$. But this is a contradiction to (11) and to the result which claims that $(\varphi(((c-a)/d))^*), \delta')$ is a minimal pair (with respect to $(k_{v'}, z')$). In conclusion (c, λ) is a minimal pair, as claimed.

To finish the proof we must show that $\bar{u}$ is defined by (c, λ). In order to do this let $\bar{u}_1$ be the r.t. extension of $\bar{z}$ to $\bar{K}(X)$ defined by the pair (c, λ) (see Theorem 1.1). Since $\bar{s}(\lambda) = \delta$, and $\bar{v}(c-a) \geqslant \delta$ it follows that (c, δ) is a pair of definition of $\bar{w}$, hence one has: $\bar{u}_1 \leqslant \bar{w}$. According to ([8], Proposition 3.2) one has necessarily that $\bar{u}_1 = \bar{u}$ and so, the restriction of $\bar{u}_1$ to $K(X)$ is just u. Hence u is defined by the minimal pair (c, λ), as claimed.

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