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History-dependent scalar conservation laws

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History-Dependent Scalar Conservation Laws.

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ABSTRACT - We study the global existence of weak solutions in L^p for a scalar conservation law with memory as a model equation in the theory of viscoelasticity. The key idea is to relate the theory of scalar conservation laws with memory and the linearly degenerate hyperbolic systems.

1. - Introduction.

The purpose of the present paper is to study the global existence of weak solutions in the framework provided by the Compensated Compactness in L^p spaces, to some equations modeling the theory of viscoelastic materials.

In particular we are interested to investigate those mathematical models where the stress-strain relation depends on the past history of the solution. Several degrees of complications could be taken under consideration but along this paper we shall bound ourselves to take under consideration only a simple, but significant problem, which has been previously considered by several different authors.

We study a scalar conservation laws with memory, which was investigated by Dafermos [6] with a method completely different from ours, under more severe technical assumption that are completely removed by our method. He used this example as a test model for the techniques of Compensated Compactness and shows some very interesting relations with the theory of relaxation for nonlinear hyperbolic systems (see [11, 2, 13]) and with some models of singular perturbation investigated in [15] and [14].

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The equation that we will investigate is given by the following history-dependent conservation law

$$
(1.1)\qquad \begin{cases} \partial_t u(x, t) + \partial_x m(x, t) = 0\,, \\ m(x, t) = f(u(x, t)) + \int_0^t K(t-\tau)\, g(u(x, \tau))\, d\tau\,, \end{cases}
$$

with the Cauchy datum

$$
(1.2)\qquad u(x, 0) = u_0(x)\,.
$$

Let us conclude this introduction with definition of weak solutions to (1.1).

DEFINITION 1.1. *Let us a measurable function; we say that it is a weak solution to* (1.1) *if and only if* $u \in L^1_{\mathrm{loc}}$, $f \circ u \in L^1_{\mathrm{loc}}$, $g \circ u \in L^1_{\mathrm{loc}}$ *and for any test function* $\zeta \in C^\infty$, $\operatorname{supp} \zeta \subset \mathrm{R} \times [0, +\infty)$, *one has*

$$
(1.3)\qquad \int_{-\infty}^{+\infty}\int_0^{+\infty} \{u(x, t)\,\zeta_t + m(x, t)\,\zeta_x\}\, dx\, dt + \int_0^{+\infty} u_0(x)\,\zeta(x, 0)\, dx = 0\,.
$$

For the general theory of viscoelastic materials we refer to the book of Renardy, Hrusa and Nohel [17] and the references therein.

In particular we recall that for *small* initial data (1.1) possess globally defined smooth classical solutions which decay to equilibrium as t goes to $+\infty$ (see for instance [7, 8, 9, 12, 5]). When the initial data are *large* enough, smooth solution should develop singularities in finite time, see for instance [4, 16].

We use a completely new idea which shows the connection between the theory of scalar conservation laws with memory and the linearly degenerate hyperbolic systems.

The paper is organized as follows: in section 2 we prove an *a priori* estimate on solutions and discuss the hypotheses of the main result (Theorem 2.1), while in the final section we conclude with the Proof of Theorem 2.1.

2. - Global existence.

We consider in this section the problem of global existence in L^2 for the equation (1.1).

Let us rewrite (1.1) as an hyperbolic system with artificial viscosity, by setting $z_t = g(u)_x$ and $z(x, 0) = 0$ and integrating by parts, namely we will study

$$(2.1)\qquad \begin{cases} \partial_t u + \partial_x f(u) + K(0)z + \int_0^t K'(t-\tau)z(x,\tau)\,d\tau = \varepsilon\partial_x^2 u\,, \\ \partial_t - \partial_x g(u) = \varepsilon\partial_x^2 z\,, \end{cases}$$

with the initial condition

$$(2.2)\qquad \begin{cases} z(x,0) = 0\,, \\ u(x,0) = u_0(x)\,, \end{cases}$$

as $\varepsilon \to 0^+$.

We assume that (2.1) possesses globally defined solutions $(u^\varepsilon, z^\varepsilon)$ with enough regularity and decay at infinity, and we shall concentrate only in the study of the behavior as $\varepsilon \to 0^+$.

We obtain uniform L^2 bounds on solutions of (2.1) by using a suitable entropy inequality and the hypothesis:

There exists $c_1 > 0$ such that, for all $u \in \mathbb{R}$

$$|g'(u)| \leqslant c_1\,|f'(u)|\,.$$

For the weak formulation of (1.1) to make sense one needs a quadratic growth bound on f and g.

Additionally, to avoid possible concentration effects and to use the L^2-Young measure representation of weak limits we require slight more:

$$(2.4)\qquad \lim_{|u|\to 0} \frac{|f(u)| + |g(u)|}{u^2} = 0\,.$$

With these two hypotheses, it is possible to prove the weak continuity of f with respect to the subsequence associated with the Young measure.

However the hypothesis (2.3) is too weak to insure that g is also weakly continuous.

One needs either additional compactness by making appropriate hypotheses on the kernel K of the memory term or additional coupling between f and g. We choose the latter alternative to study this problem:

(2.5) If f is affine on an open interval, then g is affine on that interval

A stronger hypothesis implies compactness of the sequence in the strong topology of L^q_{loc}, $q < 2$:

$$f \text{ is never affine on any open interval}. \tag{2.6}$$

In some sense this hypothesis represents the genuine nonlinearity in its weak-est form. Let us assume the kernel $K(t)$ satisfies the following

$$K \in \mathcal{AC}(\mathbb{R}^+) \quad \text{and} \quad K' \in L^\infty \cap L^1(\mathbb{R}^+). \tag{2.7}$$

THEOREM 2.1. *The following results hold:*

Global existence. Assume the hypotheses (2.3), (2.4), (2.5). *Then for any initial datum* $u_0 \in L^2(\mathbb{R})$ *and* $T > 0$ *there exists a weak solution of* (1.1)

$$u \in L^\infty([0, T]; L^2(\mathbb{R})) \cap \operatorname{Lip}([0, T]; H^{-1}(\mathbb{R})). \tag{2.28}$$

Compactness. With the hypotheses (2.3), (2.4) *and* (2.6) *there exists a subsequence of viscous solutions* $\{u^\varepsilon\}$ *of* (2.1) *strongly converging in* $L^q([0, T]; L^p_{\text{loc}}(\mathbb{R}))$, $q < 2$, $1 \leq p < 2$, *to a weak solution of* (1.1).

We postpone the proof to the last section.

REMARK 2.2. The system

$$\begin{cases} u_t + f(u)_x + K(0)z + \displaystyle\int_0^t K'(t-s)\, z(x, s)\, ds = 0\,, \\ z_t - g(u)_x = 0\,, \end{cases} \tag{2.9}$$

with the initial condition

$$\begin{cases} z(x, 0) = 0\,, \\ u(x, 0) = u_0(x)\,, \end{cases}$$

is equivalent to (1.1). Indeed, let us considered a test function ζ, with $\operatorname{supp} \zeta \subset \{(x, t) \colon t > 0\}$, then $\langle \zeta_x, g(u) \rangle = \langle \zeta_t, z \rangle$, where $\langle , \rangle$ denotes the inner product in the x, t variables. In then follows:

$$\langle \zeta, (K' * z) + K(0)z \rangle = \langle \zeta, (K*z)_t \rangle =$$

$$= -\langle \zeta_t, K*z \rangle = -\int_0^{+\infty} K(s) \int_s^{+\infty} \int_{-\infty}^{+\infty} \zeta_t(t)\, z(t-s, x)\, dx\, dt\, ds =$$

$$= -\int_0^{+\infty} K(s) \int_s^{+\infty} \int_{-\infty}^{+\infty} \zeta_x((t-s)+s)\, g(u(t-s, x))\, dx\, ds\, dt =$$

$$= -\langle \zeta_x, K*g(u) \rangle . \quad \blacksquare$$

REMARK 2.3. The system (2.9) is hyperbolic with eigenvalues $f'(u)$, 0 and eigenvectors $(f'(u), -g'(u))$, $(0, 1)$ and although (1.1) is a scalar equation, there is the possibility of resonance between the two nonlinear terms. As noted by Dafermos [6] this can occur when $f'(u) = 0$ and can also be seen in the loss of strict hyperbolicity in (2.9). $\blacksquare$

However in this case, enough of the scalar structure remains and we prove global existence without the hypothesis of strict hyperbolicity.

The two hypotheses (2.5), (2.6) and the latter implication of compactness helps partially clarify and settle the issue raised by Dafermos of weakening his assumption of *linear nondegeneracy*:

$$\text{(2.11)} \qquad \text{For all } u \in \mathbb{R}\text{: } \sum_{n=2}^{N} |f^{(n)}(u)| \neq 0 , \qquad \text{for } f \in C^N .$$

This generalizes the Lax definition of genuine nonlinearity and essentially states that the conservation law is nonlinear at all points in the state space. Typically some versions of linear nondegeneracy will imply compactness of the solution operator. So it is be reasonable to get compactness from the assumption (2.6).

We conclude this section by proving the *a priori* estimates on solutions of (2.1).

PROPOSITION 2.4. *Let* (u, z) *a solution for the Cauchy problem* (2.1)-(2.2). *For all* $T > 0$ *there exists a constant* $c_T > 0$, *independent of* ε, *such that*

$$\text{(2.12)} \qquad |u(\cdot, t)|_{L^2(\mathbb{R})} + |z(\cdot, t)|_{L^2(\mathbb{R})} \leq c_T$$

for all $t \in [0, T]$ *and*

$$\varepsilon \int_0^T \int_{-\infty}^{+\infty} |u_x|^2 + |z_x|^2 \, dx \, dt \leqslant c_T . \tag{2.13}$$

PROOF. Set

$$G(u) = \int_0^u \frac{g'(\xi)}{f'(\xi)} d\xi ,$$

and note that from (2.3)

$$|G(u)| \leqslant c_1 |u| .$$

A calculation shows that an entropy-entropy flux pair (η, q) for the system (2.1) is given by

$$\begin{aligned} \eta(u, z) &= z^2 + 2zG(u) + 2c_1^2 u^2 , \\ q(u) &= \int^u (4c_1^2 \xi f'(\xi) - 2g'(\xi) G(\xi)) \, d\xi . \end{aligned} \tag{2.2}$$

There is a constant c such that

$$c^1 (z^2 + u^2) \leqslant \eta(u, z) \leqslant c(z^2 + u^2) .$$

Integrating the entropy inequality, we obtain

$$\begin{aligned} \frac{d}{dt} \int_{-\infty}^{+\infty} \eta(u, z) \, dx &= - \int_{-\infty}^{+\infty} \eta_u (u, z)[K(0) z + (K' * z)] \, dx - \\ &\quad - \varepsilon \int_{-\infty}^{+\infty} \nabla^2 \eta((u_x, z_x), (u_x, z_x)) \, dx \leqslant \\ &\leqslant c_0 \int_{-\infty}^{+\infty} (u^2 + z^2) \, dx + c_0 \int_0^t \int_{-\infty}^{+\infty} (u^2 + z^2) \, dx \, ds . \end{aligned} \tag{2.3}$$

Then by the comparison principle, it follows there exists $c_T > 0$ such that

$$\sup_{t \in [0, T]} \int_{-\infty}^{+\infty} \eta(u, z)\, dz \leq c_T . \qquad \blacksquare$$

3. – Proof of the Theorem 2.1.

We now prove the Theorem 2.1 by modifying the single entropy argument for scalar equations due to Chen and Lu [3].

We assume the reader is familiar with the standard methods of compensated compactness and we refer to [10] for the basic L^p theory (see also [18]).

PROOF. We assume we have an L^2-Young measure $\widetilde{\nu}_{(x, t)}(du\, dz)$ associated to a weakly converging subsequence $\{(u^\varepsilon, z^\varepsilon)\}$ of solutions to (2.1).

The variable z will play no future role so we integrate it out of the Young measure and define

$$\nu = \nu_{(x, t)}(\cdot) = \int_{\mathbb{R}_z} \widetilde{\nu}_{(x, t)}(\cdot\, dz) \tag{3.2}$$

namely $\nu_{(x, t)} = \operatorname{proj} \widetilde{\nu}_{(x, t)}$.

In the argument of Chen and Lu, one applies an L^∞-Young measure to the function $(f(u))^2$. But in this case, $(f(u))^2$ may have superquadratic growth and not be integrable with respect to an L^2-Young measure. To handle this we use a truncation procedure and first establish the weak continuity of the truncated L^∞ function f_N. Letting the truncation parameter N tend to infinity, we obtain weak continuity for f. Weak continuity for g is shown similarly.

Step 1 (Weak continuity for f). $\bar{f}(x, t) = f(\bar{u}(x, t))$ for a.e. $(x, t) \in \mathbb{R} \times [0, T]$ where $\bar{f} = \langle \nu, f(\cdot) \rangle$ and $\bar{u} = \langle \nu, u \rangle$.

Define the Lipschitz continuous functions

$$u_N = I_N(u) = \begin{cases} u & \text{for } |u| \leq N, \\ \pm N & \text{for } \pm u > N, \end{cases} \tag{3.3}$$

$$f_N(u) = \int_0^u I_N'(s) f'(s)\, ds = \begin{cases} f(u) & \text{for } |u| \leq N, \\ f(\pm N) & \text{for } \pm u > N. \end{cases} \tag{3.4}$$

Using the first equation of (2.1) and the estimates from Proposition 2.4 and since $\{u_N^\varepsilon\}$, $\{f_N(u^\varepsilon)\}$ are L^∞ functions, the standard argument shows that for fixed N and k a constant

$$(u_N^\varepsilon - k)_t + (f_N(u^\varepsilon) - f_N(k))_x \tag{3.5}$$

and

$$(f_N(u^\varepsilon) - f_N(k))_t + \left(\int_k^{u^\varepsilon} |f_N'(s)|^2 ds \right)_x \tag{3.6}$$

are both in a compact subset of H_{loc}^{-1}.

Define the function

$$H_N(u, k) = (u-k) \int_k^u |f_N'(s)|^2 ds - \left(\int_k^u f_N'(s)\, ds \right)^2 \tag{3.7}$$

then Tartar's commutative relation becomes

$$\langle \nu_{(x,t)}, H_N(u, \overline{u}_N) \rangle = - \left(\left\langle \nu_{(x,t)}, \int_{\overline{u}_N}^u f_N'(s)\, ds \right\rangle \right)^2 \tag{3.8}$$

where we have replaced k by $\overline{u}_N = \langle \nu_{(x,t)}, I_N(\cdot) \rangle$. By Cauchy Schwarz inequality, $H_N \geqslant 0$, hence

$$\bar{f}_N = \langle \nu, f_N(\cdot) \rangle = f_N(\overline{u}_N). \tag{3.9}$$

Note also that $\nu_{(x,t)}$ is supported on the zero set of $H_N(\cdot, \overline{u}_N)$.

We now let $N \to \infty$. Clearly, $I_N(u) \to u$ pointwise in u and $|I_N(u)| \leqslant |u|$. From the theory of L^p-Young measures (see Lin [10] or Ball [1]) $|u|/|u|^2 \to 0$ as $|u| \to \infty$ implies $|\overline{u}| = \langle \nu, |u| \rangle$ is in $L^1(\mathbb{R} \times \times [0, T])$, hence $|\overline{u}|(x, t) < \infty$ for a.e. $(x, t) \in \mathbb{R} \times [0, T]$. By Lebesgue dominated convergence, $\overline{u}_N = \langle \nu, I_N(u) \rangle \to \langle \nu, u \rangle = \overline{u}$ for a.e. (x, t). Hence also $f_N(\overline{u}_N(x, t))$ converges to $f(\overline{u}(x, t))$ pointwise a.e. in (x, t).

Similarly, $f_N(u_N) \to f(u)$ pointwise in u and $|f_N(u)| \leqslant f^*(u)$, where $f^*(u) = \sup_{|v| \leqslant |u|} |f(v)|$.

Since $f^*(u)/|u|^2 \to 0$ as $|u| \to \infty$, we also have $\bar{f}^* \in L_{\mathrm{loc}}^1$, $\bar{f}^* < \infty$ a.e. (x, t) and $\bar{f}_N \to \bar{f}$ pointwise a.e. (x, t). Since $\bar{f}_N = f_N(\overline{u}_N)$, we conclude $f(\overline{u}) = \bar{f}$ for a.e. (x, t).

Step 2. $\overline{g}_N = g(\overline{u}_N)$ and $g = g(\overline{u})$ a.e. (x, t). Recall from the proof of step 1 that ν is supported in the set

$$Z_N = \{u\colon\ H_N(u) = 0\}.$$

If $Z_N = \{\overline{u}_N\}$, then ν is a point mass and $\overline{g}_N = g(\overline{u}_N)$. If $u_0 > \overline{u}_N$ and $H_N(u_0) = 0$, then by considering the Cauchy Schwarz inequality, we have $f' \equiv$ const on the interval $[\overline{u}_N, u_0]$. Hence $[\overline{u}_N, u_0] \subset Z_N$ and Z_N is an interval on which f is affine. By (2.5) g_N is also affine and by linearity $\overline{g}_N = g(\overline{u}_N)$. Hence $\overline{g}_N = g(\overline{u}_N)$. For a.e. (x, t) and the same argument as in step 2 shows that $\overline{g} = g(\overline{u})$.

Step 3. $\overline{u}$ is a weak solution of (1.1). If $\{u^\varepsilon\}$, $\{z^\varepsilon\}$ are solutions of (2.1), then for any smooth function φ, compactly supported in $(0, T) \times \times \mathbb{R}$ we have:

$$\int_0^T \int_{-\infty}^{+\infty} \{\varphi_t u^\varepsilon + \varphi_x f(u^\varepsilon) + \varphi_x (K * g(u^\varepsilon))\}\, dx\, dt =$$

$$= \varepsilon \int_0^T \int_{-\infty}^{+\infty} \varphi_{xx} \{(K * z^\varepsilon) + u^\varepsilon\}\, dx\, dt.$$

Letting $\varepsilon \to 0^+$ we can use the representation of weak limits and the equalities $f(\overline{u}) = \bar{f}$ and $g(\overline{u}) = \overline{g}$ to get

$$\int_0^T \int_{-\infty}^{+\infty} \{\varphi_t \overline{u} + \varphi_x (f(\overline{u}) + K * g(\overline{u})\}\, dx\, dt = 0.$$

Step 4 (*Compactness*). Assuming the hypothesis (2.6) that f' is never constant on any interval, we conclude that $\nu_{(x,t)}$ is supported at a point for a.e. (x, t). Indeed, from step 1, $\nu_{(x,t)}$ is supported on the zero set of $H_N(\cdot, \overline{u}_N(x, t))$. Strict convexity and (2.6) in Jensen's inequality imply that the zero set of H_N is a point for a.e. (x, t). From the L^2 theory of Young measures, u^ε converges strongly to $\overline{u}$ in $L^p(\mathbb{R} \times [0, T])$ for any $p < 2$. Since u^ε is also converging weakly in $L^\infty([0, T]; L^2)$, we have by interpolation that u^ε converges strongly in $L^q([0, T]; L^p([0, T] \times \mathbb{R}))$ for any $p < 2$, $q < \infty$. ■

REFERENCES

[1] J. M. Ball, *A version of the fundamental theorem for Young measures*, in D. Serre, editor, *Proc. Conf. on Partial Differential Equation and Continuum Models of Phase Transitions*, Springer-Verlag (1988).

[2] G.-Q. CHEN - T.-P. LIU, *Zero relaxation and dissipation limits for hyperbolic conservation laws*, Comm. Pure Appl. Math., **46** (1993), pp. 755-781.
[3] G.-Q. CHEN - Y. LU, *The study on application way of the compensated compactness theory*, Clin. Sci. Bull., **34** (1989), pp. 15-19.
[4] C. M. DAFERMOS, *Development of singularities in the motion of materials with fading memory*, Arch. Rational Mech. Anal., **91** (1986), pp. 193-205.
[5] C. M. DAFERMOS, *Estimates for conservation laws with little viscosity*, SIAM J. Math. Anal., **18** (1987), pp. 409-421.
[6] C. M. DAFERMOS, *Solutions in L^∞ for a conservation law with memory*, Analyse Mathématique et Applications, (1988), pp. 117-128.
[7] C. M. DAFERMOS - J. A. NOHEL, *Energy methods for nonlinear hyperbolic Volterra integrodifferential equations*, Comm. Partial Differential Equations, **4** (1979), pp. 219-278.
[8] C. M. DAFERMOS - J. A. NOHEL, *A nonlinear hyperbolic Volterra equation in viscoelasticity*, Amer. J. Math., suppl. dedicated to P. Hartman, (1982), pp. 87-116.
[9] W. J. HRUSA - J. A. NOHEL, *The Cauchy problem in one-dimensional nonlinear viscoelasticity*, J. Differential Equations, **59** (1985), pp. 388-412.
[10] P. LIN, *Young measure and application of compensated compactness to one-dimensional nonlinear elastodynamics*, Trans. Amer. Math. Soc., **329** (1992), pp. 377-413.
[11] T.-P. LIU, *Nonlinear waves for viscoelasticity with fading memory*, J. Differential Equations, **76** (1988), pp. 26-46.
[12] R. C. MACCAMY, *A model for one-dimensional nonlinear viscoelasticity*, Quart. Appl. Math., **35** (1977), pp. 21-33.
[13] P. MARCATI, *Relaxation and singular convergence for quasilinear hyperbolic systems*, in L. BOCCARDO, M. A. HERRERO and A. TESEI, editors, *First Italian-Spain Meeting on Nonlinear Analisis*, pp. 140-148, Rome (1989), Quaderno IAC-CNR.
[14] P. MARCATI - A. MILANI, *The one-dimensional Darcy's law as the limit of a compressible Euler flow*, J. Differential Equations, **13** (1990), pp. 129-147.
[15] P. MARCATI - A. MILANI - P. SECCHI, *Singular convergence of weak solutions for a quasilinear nonhomogeneous hyperbolic system*, Manuscripta Math., **60** (1988), pp. 49-69.
[16] J. A. NOHEL - R. C. ROGERS, *Development of singularities in nonlinear viscoelasticity*, in *Amorphous Polymers and Non-Newtonian Fluids*, vol. 6, pp. 139-152, Springer Lecture Notes in Mathematics (1987), IMA Vol. Math. Appl.
[17] M. RENARDY - W. J. HRUSA - J. A. NOHEL, *Mathematical Problems in Viscoelasticity*, John Wiley and Longmann Press (1987).
[18] M. E. SCHONBEK, *Convergence of solutions to nonlinear dispersive equations*, Comm. Partial Differential Equations, **7** (1982), pp. 959-1000.

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