RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

J. A. AGUILAR CRESPO
I. PERAL ALONSO

On an elliptic equation with exponential growth

Rendiconti del Seminario Matematico della Università di Padova, tome 96 (1996), p. 143-175
<http://www.numdam.org/item?id=RSMUP_1996__96__143_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 96 (1996)

On an Elliptic Equation with Exponential Growth.

J. A. AGUILAR CRESPO - I. PERAL ALONSO (*)(**)

ABSTRACT - In this paper we deal with the following nonlinear degenerate elliptic problem

$$(P)\qquad \begin{cases} -\Delta_p u \equiv -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda V(x)e^u & \text{in } \Omega \subset \mathbf{R}^N, \quad N \geq 2, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where $p > 1$, $\lambda > 0$, Ω is a bounded domain in $\mathbf{R}^N$ and $V(x)$ is a given function in $L^q(\Omega)$ (q depending on the relationship between N and p). In particular, we study the existence of solutions in $W_0^{1,p}(\Omega)$, considering the cases: 1) Existence of solution for λ small and V possibly changing sign in Ω. 2) Conditions for positivity of solutions, with V changing sign in Ω. 3) Existence and behavior of the minimal solution for $V(x) \geq 0$ in Ω and $p < N$. 4) Existence of solution for V possibly changing sign in Ω and $p \geq N$. 5) Full analysis of the radial solutions for $V = r^{-\alpha}$, $\alpha < p$, $|x| = r$. It has to be remarked that these results are new even for the semilinear case, $p = 2$.

Introduction.

We study the following problem,

$$(P)\qquad \begin{cases} -\Delta_p u \equiv -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda V(x)e^u & \text{in } \Omega \subset \mathbf{R}^N, \quad N \geq 2, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where $p > 1$, $\lambda > 0$, $x \in \Omega$ a bounded domain, and $V(x)$ is a given function which may change sign in Ω.

The case $V \equiv$ constant has been studied in [GP] and [GPP] for gen-

(*) Indirizzo degli AA.: Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain.

(**) Both authors supported by project PB 94-0187 CICYT MEC Spain.

eral p. The semilinear case $p = 2$, with constant V too, has been extensively studied; see for instance the papers [Bd], [F], [Ge], [GMP], [JL], [MP1] and [MP2]. Motivation for the model in this case $p = 2$ can be found in [Ch], [FK] and [KW].

On the other hand, the case $p = 2$ and V a given function has a few precedents, see [BM] and [KW] for $N = 2$; the general case is new at all.

The aim of this paper is to show the following results for $V \in L^q(\Omega)$, where $q \geq 1$ for $p > N$, $q > 1$ for $p = N$ or $q > N/p > 1$ for $1 < p < N$.

(1) If λ is small enough, then there exists at least one solution to problem (P) for $q > N/p > 1$.

(2) If λ is small enough, then there exist at least one solution to problem (P) for $p \geq N$.

(3) If λ is large enough and $V \geq 0$, the problem (P) has no solution.

(Obviously, if $V < 0$ then the maximum principle implies that problem (P) has one negative solution).

In addition, we find a sufficient condition related to the existence of positive solutions to (P) with V changing sign in Ω, a result new even for the semilinear case $p = 2$.

We also prove that the first eigenvalue for the p-Laplacian with weight $V \in L^q(\Omega)$, $V(x) \geq 0$, is simple and isolated. This result is an extension to our context of those by [An], [B] and [L]; it will be used in the study of the nonexistence of solutions to (P).

The paper is organized as follows: first, we show the existence of solution to (P) for $V \in L^q(\Omega)$, in case V may change sign in Ω and a sufficient condition about the positivity of the solution when V changes sign is obtained. Next section is devoted to study the behavior of the minimal solution for $1 < p < N$, $V \geq 0$. Variational methods are used for the case $p \geq N$, V possibly with non constant sign. Finally, after dealing with the nonexistence of solutions, we analyze the radial solutions on the unit ball for $V(r) = r^{-a}$, $a < p$, $r = |x|$.

Before studying the existence of solutions, we give some definitions for solutions to the problem (P).

DEFINITION. *We say that* $u \in W_0^{1,p}(\Omega)$ *is a regular solution* (P) *if and only if* $e^u \in L^\infty(\Omega)$, *and the equation holds in the sense of* $W^{-1,p'}(\Omega)$. *If* $V(x)e^u \in L^1(\Omega)$, *we say that* u *is a singular solution of* (P), *and the equation holds in the sense of* $\mathcal{D}'(\Omega)$.

Obviously, if $u \in W_0^{1,p}(\Omega)$ with $p > N$, Morrey's Theorem [GT, Ch. 7] implies that $e^u \in L^\infty(\Omega)$. So, we just need $V(x) \in L^1(\Omega)$. On the other hand, by using the Stampacchia's lemma [S] and Trudinger's inequality [GT, p. 162], we get that a solution of (P) for $p = N$, $u \in W_0^{1,N}(\Omega)$, verifies $u \in L^\infty(\Omega)$ whenever the function V belongs to $L^q(\Omega)$, $q > 1$. In other words

PROPOSITION. *If $V \in L^q(\Omega)$, with either $q \geq 1$, $p > N$ or $q > 1$, $p = N$, then any singular solution of* (P) *$u \in W_0^{1,p}(\Omega)$ is a regular solution.*

1. - Existence of solution for λ small.

We show in this section the existence of solution to the problem

$$\text{(P)} \qquad \begin{cases} -\Delta_p u = \lambda V(x) e^u & \text{in } \Omega \subset \mathbf{R}^N, \quad N \geq 2, \\ u|_{\partial\Omega} = 0, \end{cases}$$

by means of a fixed point argument, where $\lambda > 0$, $x \in \Omega$, a bounded domain, and $V(x)$ is a given function in $L^q(\Omega)$, $q \geq 1$ if $p > N$, $q > 1$ if $p = N$ and $q > N/p$ otherwise.

It has to be noted that V may change sign in Ω.

LEMMA 1.1. *Let $B_\delta = \{\varphi \in \mathcal{C}(\Omega)\colon |\varphi| < \delta,\ \varphi|_{\partial\Omega} = 0\}$, $\delta > 0$. Let $F_\lambda\colon B_\delta \to L^\infty(\Omega)$ defined by $\varphi \to F_\lambda(\varphi) = \psi$, where ψ verifies the following problem*

$$\begin{cases} -\Delta_p \psi = \lambda V(x) e^\varphi & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

Then,

$$\|\psi\|_\infty \leq C(\lambda e^\delta \|V\|_q)^\gamma$$

where $C = C(p, N, \Omega)$ and $\gamma > 0$.

PROOF. *Case $p > N$.* In this case, $W_0^{1,p}(\Omega) \subset L^\infty(\Omega)$ and by Sobolev inequality we obtain the following estimate

$$\|\psi\|_\infty \leq C(p, N, \Omega)(\lambda e^\delta \|V\|_q)^{1/(p-1)}$$

Case $1 < p < N$. Since φ is bounded, we get that $\lambda V(x) e^{\varphi}$ belongs to $L^q(\Omega)$, $q > N/p$. Hence, $\lambda V(x) e^{\varphi}$ belongs to $W^{-1,r}(\Omega)$ for $r > N/(p-1)$. Then, there exist $f_1, f_2, \dots, f_N$ in $L^r(\Omega)$ such that, $\forall \eta \in W_0^{1,p}(\Omega)$,

$$\int_\Omega |\nabla \psi|^{p-2} \langle \nabla \psi, \nabla \eta \rangle \, dx = \int_\Omega \langle f, \nabla \eta \rangle \, dx$$

where $f = (f_1, f_2, \dots, f_N)$ and $\lambda V(x) e^{\varphi} = -\operatorname{div} f$ (see [Br, Prop. IX.20]). For $k > 0$, if we take as test

$$\eta = \operatorname{sign}(\psi)(|\psi| - k)^+ = \begin{cases} \psi - k & \psi \geqslant k, \\ \psi + k & \psi \leqslant -k, \\ 0 & \text{ortherwise}, \end{cases}$$

then $\nabla \eta = \nabla \psi$ in $A(k) = \{x \in \Omega : |\psi(x)| > k\}$ and $\eta = 0$ in $\Omega \backslash A(k)$. Then

$$\int_{A(k)} |\nabla \psi|^p \, dx = \int_{A(k)} \langle f, \nabla \eta \rangle \, dx \leqslant \left(\int_{A(k)} |\nabla \psi|^p \, dx \right)^{1/p} \|f\|_r \, |A(k)|^{1 - 1/p - 1/r}.$$

That is

$$\left(\int_{A(k)} |\nabla \psi|^p \, dx \right)^{(p-1)/p} \leqslant \|f\|_r \, |A(k)|^{1 - 1/p - 1/r}.$$

For $p < N$, by the Sobolev's inequality ($S^{1/p}$ being the best constant for this inequality, $p^* = Np/(N-p)$)

$$S^{1/p} \left(\int_{A(k)} |\psi|^{p^*} \, dx \right)^{p/p^*} \leqslant \left(\int_{A(k)} |\nabla \psi|^p \, dx \right)^{p/p}.$$

The case $p = N$ is reduced to the case $p < N$ because of the embedding $W_0^{1,N}(\Omega) \subset W_0^{1,p}(\Omega)$ for $1 < p < N$ (Ω is bounded). Therefore

$$S^{(p-1)/p} \left(\int_{A(k)} |\psi|^{p^*} \, dx \right)^{(p-1)/p^*} \leqslant \|f\|_r \, |A(k)|^{1 - 1/p - 1/r}.$$

If $0 < k < h$, $A(h) \subset A(k)$. Then

$$|A(k)|^{1/p^*}(h-k) = \left(\int_{A(k)} (h-k)^{p^*} dx \right)^{1/p^*} \leq$$

$$\leq \left(\int_{A(h)} |\psi|^{p^*} dx \right)^{1/p^*} \leq \left(\int_{A(k)} |\psi|^{p^*} dx \right)^{1/p^*}.$$

Finally

$$|A(k)|^{1/p^*} \leq \frac{1}{S^{1/p}} \frac{1}{h-k} \|f\|_r^{1/(p-1)} (|A(k)|^{1-1/p-1/r})^{1/(p-1)}.$$

In other words

$$|A(h)| \leq \frac{1}{S^{p^*/p}} \frac{1}{(h-k)^{p^*}} \|f\|_r^{p^*/(p-1)} |A(k)|^{p^*(1/p-1/(r(p-1)))}.$$

Since $r > N/(p-1)$, the exponent for $|A(k)|$ is greater than 1. So, we can apply Stampacchia's Lemma, [S], to conclude that there exists some h for which $|A(h)| = 0$, that is, $\psi \in L^\infty(\Omega)$ and

$$\|\psi\|_\infty \leq C(p, r, N, \Omega)(\lambda e^\delta \|V\|_q)^{p'/p}. \qquad \blacksquare$$

In addition, the inequalities in $\mathbf{R}^N$

$$\langle |x|^{p-2}x - |y|^{p-2}y, x-y \rangle \geq \begin{cases} C_p |x-y|^p & \text{if } p \geq 2, \\ C_p \dfrac{|x-y|^2}{(|x|+|y|)^{2-p}} & \text{if } p \leq 2, \end{cases}$$

imply the following result

LEMMA 1.2. *Given* $f_1, f_2 \in W^{-1,p'}(\Omega)$, *consider* $u_1, u_2 \in W_0^{1,p}(\Omega)$ *such that* $-\Delta_p u_i = f_i$, $i = 1, 2$. *Then:*

$$\int_\Omega (f_1 - f_2)(u_1 - u_2)\, dx = \int_\Omega \langle |\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2, \nabla u_1 - \nabla u_2 \rangle\, dx \geq$$

$$\geq \begin{cases} C_p \displaystyle\int_\Omega |\nabla(u_1 - u_2)|^p dx & \textit{if } p \geq 2, \\ C_p \displaystyle\int_\Omega \frac{|\nabla(u_1 - u_2)|^2}{(|\nabla u_1| + |\nabla u_2|)^{2-p}}\, dx & \textit{if } p \leq 2. \end{cases}$$

As a consequence

$$C_p \int_\Omega |\nabla(u_1 - u_2)|^p dx \leqslant \|u_1 - u_2\|_{W_0^{1,p}(\Omega)} \|f_1 - f_2\|_{W^{-1,p'}(\Omega)} \qquad \text{if } p \geqslant 2,$$

$$\|u_1 - u_2\|_{W_0^{1,p}(\Omega)} \leqslant$$

$$\leqslant C_p (\|\nabla u_1\|^p_{W_0^{1,p}(\Omega)} + \|\nabla u_2\|^p_{W_0^{1,p}(\Omega)})^{(2-p)/p} \|f_1 - f_2\|_{W^{-1,p'}(\Omega)} \qquad \text{if } p < 2.$$

Now we can state the existence theorem

THEOREM 1.3. *If λ is small enough, then there exists one solution to the problem* (P).

PROOF. Lemma 1.1 implies that, for λ small enough, F_λ applies the ball of radius δ in $L^\infty(\Omega)$ to itself. On the other hand, if $\psi_1 = F_\lambda \varphi_1$, $\psi_2 = F_\lambda \varphi_2$, where $\varphi_1, \varphi_2 \in B_\delta$, Lemma 1.2 implies

$$\gamma_0 \int_\Omega |\nabla\psi_1 - \nabla\psi_2|^p dx \leqslant$$

$$\leqslant \lambda e^\delta \|\varphi_1 - \varphi_2\|_\infty \|\psi_1 - \psi_2\|_\infty \|V\|_q |\Omega|^{1/q'} \qquad \gamma_0(p) > 0, \ p \geqslant 2,$$

$$\gamma_1 \int_\Omega |\nabla\psi_1 - \nabla\psi_2|^p dx \leqslant \lambda e^\delta \|\varphi_1 - \varphi_2\|_\infty^{p/2} \|\psi_1 - \psi_2\|_\infty^{p/2} \cdot$$

$$\cdot (\|\psi_1\|_\infty + \|\psi_2\|_\infty)^{(2-p)/2} \|V\|_q |\Omega|^{1/q'} \qquad \gamma_1(p) > 0, \ p < 2,$$

by means of the mean value theorem. Therefore by either Sobolev inclusion in the case $p > N$, or by Stampacchia method in the general case,

$$\|F_\lambda \varphi_1 - F_\lambda \varphi_2\|_\infty = \|\psi_1 - \psi_2\|_\infty \leqslant C(p, N, \Omega, \|V\|_q)(\lambda e^\delta)^{\gamma(p)} \|\varphi_1 - \varphi_2\|_\infty$$

where $\gamma(p) > 0$. So we have proved that F_λ is contractive if λ is small enough; therefore, the classical *Banach-Picard fixed point theorem* allows us to conclude the proof. ■

REMARK. *In the case $p = N$, the potential can be considered with less regularity. Precisely $V \in L^1(\log L)^\beta(\Omega)$, the usual Zygmund space with $\beta > N - 1$, gives that each iteration in the proof of Theorem* 1.3 *verifies $u \in L^\infty(\Omega)$.* (*See* [BPV]).

2. – A sufficient condition of existence of positive solution.

When the sign of the potential V is constant, it is easy to know the sign of the corresponding solutions. In this section we will give a sufficient condition related to positive solutions with V changing sign. This result is new even for the semilinear case, $p=2$, which is treated following.

Let us consider the problem,

$$(\mathrm{P}')\qquad \begin{cases} -\Delta w = V(x) & \text{in } \Omega \subset \mathbf{R}^N, \quad N \geq 1, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where $x \in \Omega$, a bounded domain, and $V(x)$ is a given function in $L^q(\Omega)$ changing sign in Ω ($q \geq 1$ if $N > 2$, $q > N/2$ otherwise). We assume that Ω verifies the classical *interior ball condition.*

Let us assume that $w > 0$ in Ω, that is, if $G(x, \xi)$ is the Green's function for Ω,

$$w(x) = \int_\Omega G(x, \xi)\, V(\xi)\, d\xi > 0 .$$

Since $V = V^+ - V^-$, we get ($V^+ = \max(V, 0)$, $V^- = \max(-V, 0)$)

$$w_1(x) = \int_\Omega G(x, \xi)\, V^+(\xi)\, d\xi > \int_\Omega G(x, \xi)\, V^-(\xi)\, d\xi = w_2(x)$$

where w_1, w_2 are the solutions of the problems

$$\begin{cases} -\Delta w_1 = V^+(x) & \text{in } \Omega, \\ w_1|_{\partial\Omega} = 0, \end{cases}$$

$$\begin{cases} -\Delta w_2 = V^-(x) & \text{in } \Omega, \\ w_2|_{\partial\Omega} = 0. \end{cases}$$

Let $y(x)$ be the function defined in $\bar{\Omega}$ by

$$y(x) = \begin{cases} \dfrac{w_1(x)}{w_2(x)} & \text{if } x \in \Omega, \\[2ex] \dfrac{\partial_\nu w_1(x)}{\partial_\nu w_2(x)} & \text{if } x \in \partial\Omega, \end{cases}$$

where ν is the exterior unit normal to $\partial\Omega$, This function is well defined, positive and continuous in $\bar{\Omega}$ by the Hopf's Lemma. Let us suppose that

$\min_{x \in \Omega} y(x) = 1 + m$ for some $m > 0$. That implies $w_1 > (1 + m) w_2$. Let M be such that $M = \|y\|_\infty$.

Now, if we take a function φ such that $0 \leqslant \varphi \leqslant \delta \log(y(x))$, with $\delta > 0$ to be determined, we can define for $\lambda > 0$ the application T_λ as follows

$$\psi = T_\lambda \varphi = \lambda \int_\Omega G(x, \xi) V(\xi) e^{\varphi(\xi)} d\xi .$$

Thus

$$T_\lambda \varphi \geqslant \lambda \int_\Omega G(x, \xi) V^+ (\xi) d\xi -$$

$$-\lambda \int_\Omega G(x, \xi) V^- (\xi)(y(\xi))^\delta d\xi \geqslant \lambda(w_1(x) - M^\delta w_2(x)),$$

then we can take δ small enough to get

$$T_\lambda \varphi \geqslant \lambda(w_1(x) - M^\delta w_2(x)) > \lambda(w_1(x) - (1 + m) w_2(x)) > 0 .$$

Then, $T_\lambda \varphi$ is positive if δ is small enough. By fixing a δ in tyhese hypothesis, T_λ sends the ball of radius M^δ in $L^\infty(\Omega)$ to itself and is contractive for λ small enough, by Theorem 1.3. In this way the existence of one positive solution to the problem

$$\text{(P'')} \qquad \begin{cases} -\Delta u = \lambda V(x) e^u & \text{in } \Omega \subset \boldsymbol{R}^N, \\ u|_{\partial\Omega} = 0, \end{cases}$$

can be shown by means of a fixed point argument ($u = T_\lambda u$), as a consequence of the positivity of the solution to (P′).

Then we conclude with the following result.

PROPOSITION 2.1. *Let* $p = 2$, w_1, w_2 *as above. If* $w_1(x) = (1 + \varrho(x)) w_2(x)$ *bounded function and* $\varrho(x) \geqslant m > 0$ *in* $\bar{\Omega}$, *then problem* (P″) *has at least one positive solution for* λ *small enough.*

Now we take p general. The corresponding result is the following

THEOREM 2.2. *Let w be the solution of*

$$\begin{cases} -\Delta_p w = V^+(x) - (1+\mu(x))V^-(x) & \text{in } \Omega \subset \mathbf{R}^N, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where $V \in L^q(\Omega)$ changes sign in Ω ($q \geq 1$ if $p > N$, $q > 1$ if $p = N$, $q > N/p$ otherwise) and $\mu(x)$ verifies that there exists a positive constant m such that $\mu(x) \geq m > 0$. Suppose that $w > 0$ in Ω; then, if λ is small enough, there exists one positive solution to (P).

PROOF. Let $\delta = \log(1+m)^{1/2}$ (hence, $e^{2\delta} = 1 + m$ and $e^{-\delta} = (1+m)^{-1/2}$). Theorem 1.3 now implies that there exists one solution to (P) belonging to the ball B_δ. Let ψ such a solution. then

$$-\Delta_p \psi = \lambda(V^+(x)e^\psi - V^-(x)e^\psi) \geq$$

$$\geq \lambda(V^+(x)e^{-\delta} - V^-(x)e^{\delta}) = \lambda e^{-\delta}(V^+(x) - V^-(x)e^{2\delta}).$$

That is

$$-\Delta_p \psi \geq \lambda(1+m)^{-1/2}(V^+(x) - (1+m)V^-(x)) \geq \lambda(1+m)^{-1/2}(-\Delta_p w).$$

The weak comparison principle allows us to conclude that

$$\psi \geq C(\lambda, m, N)w > 0 \quad \text{in } \Omega \quad \blacksquare$$

3. - $V \geq 0$ and $1 < p < N$. Minimal solution.

We show in this section the existence of a solution to the problem (P) for the case $1 < p < N$, $V \geq 0$ by comparison arguments. The following results are extensions to the variable coefficients case of those in [GPP]. We give the proofs of the results that need some changes.

DEFINITION. *We say that $u \in W_0^{1,p}(\Omega)$ ($W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ for $1 < p \leq N$) is a regular supersolution of problem* (P) *if*

$$\begin{cases} -\Delta_p u \geq \lambda V(x)e^u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

We say that u_m, a solution of (P) *is minimal if, for each supersolution u of* (P), *we have $u_m \leq u$.*

With this definition at hand, we study the existence and behavior of minimal solution of (P).

LEMMA 3.1. *Let u_0 be a regular supersolution of* (P). *Then, there exists $0 \leqslant u_m \leqslant u_0$, u_m being a minimal regular solution of* (P).

COROLLARY 3.2. *If there exists regular solution of* (P) *for $\lambda_0 > 0$, then there exists regular solution for all $\lambda \leqslant \lambda_0$.*

THEOREM 3.3. *If $V \in L^q(\Omega)$, $q > N/p > 1$, there exists a constant λ^* such that if $\lambda < \lambda^*$, problem* (P) *has one positive solution.*

THEOREM 3.4. *If $u_0 \in W_0^{1,p}(\Omega)$ is a singular solution of*

$$(\mathrm{P}_{\lambda^*}) \qquad \begin{cases} -\Delta_p u_0 = \lambda^* V(x) e^{u_0} & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where $V \in L^q(\Omega)$, $q > N/p > 1$, then, for all $\lambda \in (0, \lambda^*)$ the problem

$$(\mathrm{P}_{\lambda}) \qquad \begin{cases} -\Delta_p u = \lambda V(x) e^{u} & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

has one positive minimal regular solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

PROOF. If u_0 is a singular solution, then $V(x) e^{u_0} \in L^1(\Omega)$ and we can consider $V(x) e^{u_0} \in W^{-1,p'}(\Omega)$. The function

$$u_1 = \left(\frac{\lambda}{\lambda^*}\right)^{1/(p-1)} u_0$$

is a solution of the problem

$$\begin{cases} -\Delta_p u_1 = \lambda V(x) e^{u_0} & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

and it verifies $V(x)^{(\lambda/\lambda^*)^{1/(p-1)}} e^{u_1} \in L^{(\lambda^*/\lambda)^{1/(p-1)}}$, $0 < u_1 < u_0$, and $V(x) e^{u_1} \in W^{-1,p'}(\Omega)$; moreover

$$\int_\Omega V(x) u_1^r \, dx < \int_\Omega V(x) u_0^r \, dx < \int_\Omega V(x) e^{u_0} \, dx < \infty \qquad \forall r \in (1, \infty).$$

If we consider the problem

$$\begin{cases} -\Delta_p u_2 = \lambda V(x)\, e^{u_1} & \text{in } \Omega\,, \\ u_2\,|_{\partial\Omega} = 0\,; \end{cases}$$

then $u_2 \in W_0^{1,p}(\Omega)$; by using the weak comparison principle, we have $0 < u_2 \leqslant u_1 < u_0$ and

$$\int_\Omega V(x)\, u_2^r\, dx < \infty\,.$$

By the convexity of $f(t) = e^{tx_0}$ for $0 < t < 1$, we get

$$e^{tx_0} + (1-t)\, x_0\, e^{tx_0} \leqslant e^{x_0}\,.$$

Then, if $t = (\lambda/\lambda^*)^{1/(p-1)}$ and $x_0 = u_0$,

$$e^{u_1} + (1-t)\, u_0\, e^{u_1} \leqslant e^{u_0}\,.$$

Since $u_2 \leqslant u_1 < u_0$,

$$(*) \qquad \lambda e^{u_1} \leqslant e^{u_0} - \lambda(1-t)\, u_2\, e^{u_1}\,.$$

In addition,

$$-\Delta_p\left(\frac{p-1}{p}\, v^{p/(p-1)}\right) = -v\Delta_p v - |\nabla v|^p\,.$$

By replacing v for u_2 in the last equality, we arrive to

$$-\Delta_p\left(\frac{p-1}{p}\, u_2^{p/(p-1)}\right) = -u_2\Delta_p u_2 - |\nabla u_2|^p \leqslant \lambda u_2\, V(x)\, e^{u_1}\,.$$

Now, by the homogeneity and $(*)$

$$-\Delta_p\left((1-t)^{1/(p-1)}\, \frac{p-1}{p}\, u_2^{p/(p-1)}\right) \leqslant \lambda(1-t)\, u_2\, V(x)\, e^{u_1} \leqslant$$

$$\leqslant \lambda V(x)\, e^{u_1} + \lambda(1-t)\, u_2\, V(x)\, e^{u_1} \leqslant \lambda V(x)\, e^{u_0} = -\Delta_p u_1\,.$$

If we assume that $u_2^{p/(p-1)} \in W_0^{1,p}(\Omega)$, by applying the weak comparison principle,

$$(1-t)^{1/(p-1)}\, \frac{p-1}{p}\, u_2^{p/(p-1)} \leqslant u_1$$

and

$$V(x)\, e^{(1-t)^{1/(p-1)}((p-1/p))u_2^{p/(p-1)}} \in L^1(\Omega).$$

Therefore, if $V(x) \in L^q(\Omega)$, with $q > N/p > 1$, then $V(x)\, e^{u_2} \in L^r(\Omega)$, for $r > N/p$ (by the Hölder inequality). So, the Stampacchia's lemma [S] implies that $u_3 \in L^\infty(\Omega)$, u_3 being the third iteration. In this way, we have obtained a regular supersolution of (P_λ). Lemma 3.1 states that there exists one positive minimal regular solution of (P_λ).

It remains to show that $u_2^{p/(p-1)} \in W_0^{1,p}(\Omega)$. We observe that

$$(**) \qquad -\Delta_p\left(\frac{p-1}{p}\, u_2^{p/(p-1)}\right) = \lambda u_2 V(x)\, e^{u_1} - |\nabla u_2|^p \in L^1(\Omega)$$

since $V(x)\, e^{u_1} \in W^{-1,p'}(\Omega)$ and

$$\lambda \int_\Omega u_2 V(x)\, e^{u_1} dx = \int_\Omega |\nabla u_2|^p dx = \|\nabla u_2\|_p^p < \infty .$$

It we define w_k as

$$w_k(x) \equiv \begin{cases} \dfrac{p-1}{p}\, u_2^{p/(p-1)} & \text{if } \dfrac{p-1}{p}\, u_2^{p/(p-1)} \leqslant k, \\ k & \text{if } \dfrac{p-1}{p}\, u_2^{p/(p-1)} \geqslant k, \end{cases}$$

By multiplying $(**)$ by w_k, the Hölder inequality gives

$$\int_\Omega |\nabla w_k|^p dx \leqslant \lambda \int_\Omega w_k u_2 V(x)\, e^{u_1} dx \leqslant \lambda\, \frac{p-1}{p} \int_\Omega u_2^{p/(p-1)+1} V(x)\, e^{u_1} dx =$$

$$= \lambda\, \frac{p-1}{p} \int_\Omega [V(x)^{1-(\lambda/\lambda^*)^{1/(p-1)}} u_2^{p/(p-1)+1}][V(x)^{(\lambda/\lambda^*)^{1/(p-1)}} e^{u_1}]\, dx \leqslant$$

$$\leqslant \lambda\, \frac{p-1}{p} \left(\int_\Omega V(x)\, u_2^r\, dx\right)^{1-(\lambda/\lambda^*)^{1/(p-1)}} \left(\int_\Omega V(x)\, e^{u_0} dx\right)^{(\lambda/\lambda^*)^{1/(p-1)}} < \infty$$

Then $\{w_k\}$ is uniformly bounded in $W_0^{1,p}(\Omega)$ and therefore the limit $(p-1)/p\, u_2^{p/(p-1)} \in W_0^{1,p}(\Omega)$. ■

We need the following result.

Lemma 3.5. *Let* $\underline{u} = \underline{u}(\lambda)$ *be a minimal regular solution of* (P). *If* *we define the set* $\mathcal{K}$ *as*

$$\mathcal{K} = \{v \in W_0^{1,p}(\Omega) \,|\, 0 \leqslant v \leqslant \underline{u}\}$$

the following functional

$$J(u)\frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \lambda\int_\Omega V(x)\,e^u\,dx$$

is well defined on $\mathcal{K}$. *Then, the minimizer* $u \in \mathcal{K}$ *for* J *is the minimal solution* $\underline{u}$. *Moreover,* $\underline{u}$ *satisfies the estimate*

$$\lambda\int_\Omega V(x)\,e^{\underline{u}}w^2\,dx \leqslant (p-1)\int_\Omega |\nabla \underline{u}|^{p-2}|\nabla w|^2\,dx \qquad \forall w \in W_0^{1,p}(\Omega).$$

(See [GPP] for a proof).

THEOREM 3.6. *Let* $\{\lambda_n\}_{n\in N}$ *be an increasing sequence such that*

$$\lambda_n \to \lambda^* \equiv \sup\{\lambda \,|\, (\mathrm{P}_\lambda) \text{ has solution}\}$$

If $V \in L^q(\Omega)$, $q > N/p > 1$, *and* $\underline{u}_n = \underline{u}_n(\lambda)$ *is the corresponding minimal solution of* (P_{λ_n}), *then* $\underline{u}_n \to u^*$ *strongly in* $W_0^{1,p}(\Omega)$, $V(x)\,e^{\underline{u}_n} \to V(x)\,e^{u^*}$ *in* $L^{p^*/(p^*-1)}(\Omega)$, *and* u^* *is a singular solution of* (P_{λ^*}).

PROOF. If $\underline{u}_n$ is the minimal solution of (P_{λ_n}) we get, taking $w = \underline{u}_n$ and using Lemma 3.5,

$$\lambda_n\int_\Omega V(x)\,e^{\underline{u}_n}\underline{u}_n^2\,dx \leqslant (p-1)\int_\Omega |\nabla u_n|^p\,dx = (p-1)\lambda_n\int_\Omega V(x)\,e^{\underline{u}_n}u_n\,dx$$

Let us introduce the sets $\mathcal{E}_n = \{x \in \Omega \,|\, \underline{u}_n > 2(p-1)\}$. Then, in $\Omega - \mathcal{E}_n$, $0 < \underline{u}_n < 2(p-1)$ and

$$\lambda_n\int_\Omega V(x)\,e^{\underline{u}_n}\underline{u}_n^2\,dx \leqslant (p-1)\lambda_n\int_{\Omega\setminus\mathcal{E}_n} V(x)\,e^{\underline{u}_n}\underline{u}_n\,dx +$$

$$+(p-1)\lambda_n\int_{\mathcal{E}_n} V(x)\,e^{\underline{u}_n}\underline{u}_n\,dx \leqslant 2(p-1)^2\lambda_n e^{2(p-1)}\int_\Omega V(x)\,dx +$$

$$+\frac{\lambda_n}{2}\int_{\mathcal{E}_n} V(x)\,e^{\underline{u}_n}\underline{u}_n^2\,dx\,.$$

Therefore

$$\int_\Omega V(x)\, e^{\underline{u}_n} \underline{u}_n^2\, dx \leq 4(p-1)^2 e^2 (p-1) \int_\Omega V(x)\, dx$$

and we get

$$\int_\Omega V(x)\, e^{\underline{u}_n} \underline{u}_n\, dx \leq C\,, \qquad \int_\Omega |\nabla \underline{u}_n|^p\, dx \leq C\,.$$

Then, if we take a subsequence $\{\underline{u}_n\}$

(1) $\underline{u}_{n_k} \rightharpoonup u^*$ weakly in $W_0^{1,p}(\Omega)$.

(2) By monotone convergence $e^{\underline{u}_{nk}} \to e^*$ in $L^1(\Omega)$.

Besides, $\{\underline{u}_n\}$ is monotone (remember that λ_n is increasing). Hence the limit u^* is unique and the whole sequence converges. In order to prove that u^* is a singular solution of (P_{λ^*}), we consider the following inequalities:

$$\int_\Omega |\nabla \underline{u}_n|^{p-2} \langle \nabla \underline{u}_n, \nabla \varphi \rangle\, dx = \lambda \int_\Omega V(x)\, e^{\underline{u}_n} \varphi\, dx \qquad \forall \varphi \in W_0^{1,p}(\Omega)\,,$$

$$\lambda \int_\Omega V(x)\, e^{\underline{u}_n} \psi^2\, dx \leq (p-1) \int_\Omega |\nabla \underline{u}_n|^{p-2} |\nabla \psi|^2\, dx \qquad \forall \psi \in W_0^{1,p}(\Omega)\,.$$

If we take $\varphi = (1/2\alpha)(e^{2\alpha \underline{u}_n} - 1)$, $\psi = e^{\alpha \underline{u}_n} - 1$ in the above inequalities, we arrive to

$$\left(\frac{1}{(p-1)\alpha} - \frac{1}{2} \right) \int_\Omega V(x)\, e^{(2\alpha+1)\underline{u}_n}\, dx \leq \frac{2}{(p-1)\alpha} \int_\Omega V(x)\, e^{(\alpha+1)\underline{u}_n}\, dx\,.$$

Taking α such that

$$\frac{1}{(p-1)\alpha} > \frac{1}{2} \qquad \text{i.e.} \qquad 2\alpha + 1 < \frac{3+p}{p-1}$$

we get, using the Young's inequality,

$$C(\alpha) \int_\Omega V(x)\, e^{(2\alpha+1)\underline{u}_n}\, dx \leq \int_\Omega V(x)\, e^{\underline{u}_n}\, dx \leq \int_\Omega V(x)\, e^{u^*}\, dx\,.$$

If we also assume that

$$2\alpha + 1 > \frac{p^*}{p^* - 1} = \frac{Np}{N(p-1) + p}$$

then we obtain by Hölder inequality

$$\int_\Omega (V(x)\,e^{\underline{u}_n})^{p^*/(p^*-1)}\,dx =$$

$$= \int_\Omega (V(x)^{1/2\alpha}\,e^{\underline{u}_n})^{p^*/(p^*-1)}\,(V(x)^{2\alpha/(2\alpha+1)})^{p^*/(p^*-1)}\,dx \leqslant$$

$$\leqslant \left(\int_\Omega V(x)\,e^{(2\alpha+1)\underline{u}_n}\,dx\right)^{p^*/((p^*-1)(2\alpha+1))} \cdot$$

$$\cdot \left(\int_\Omega V(x)^{2\alpha p^*/((2\alpha+1)(p^*-1)-p^*)}\,dx\right)^{1/((p^*-1)(2\alpha+1)/p^*)'} .$$

As $V \in L^q(\Omega)$, $q > N/p > 1$, this quantity is finite if

$$\frac{2\alpha p^*}{(2\alpha+1)(p^*-1)-p^*} \leqslant \frac{N}{p}$$

that is

$$\alpha > \frac{1}{2}\,\frac{1}{N+p-1}.$$

Since the following is always true

$$\frac{Np}{N(p-1)+p} < \frac{Np}{N(p-1)} < \frac{3+p}{p-1}; \qquad \frac{1}{2}\,\frac{1}{N+p-1} < \frac{2}{p-1}$$

all the requirements about the value of α hold: there always exists some α verifying them.

Then, we have proven that $V(x)\,e^{\underline{u}_n} \in L^{p^*/(p^*-1)}(\Omega) \subset W_0^{1,p}(\Omega)$, and $V(x)\,e^{\underline{u}_n}$ converges in $W^{-1,p'}(\Omega)$ by the monotone convergence theorem. The continuity of $(-\Delta_p)$: $W^{-1,p'}(\Omega) \to W_0^{1,p}(\Omega)$ implies that the sequence $\{V(x)\,e^{\underline{u}_n}\}$ converges strongly in $W_0^{1,p}(\Omega)$. Therefore, if we take $\varphi \in W_0^{1,p}(\Omega)$

$$\int_\Omega \langle |\nabla u^*|^{p-2}\nabla u^*, \nabla\varphi\rangle\,dx = \lim_{n\to\infty} \int_\Omega \langle |\nabla \underline{u}_n|^{p-2}\nabla \underline{u}_n, \nabla\varphi\rangle\,dx =$$

$$= \lim_{n\to\infty} \int_\Omega V(x)\,e^{\underline{u}_n}\varphi\,dx = \int_\Omega V(x)\,e^{u^*}\varphi\,dx . \qquad \blacksquare$$

The next result gives the conditions in which the limit minimal solution is regular or singular.

THEOREM 3.7. *If V, λ^* and u^* are as in Theorem 3.6 and the dimension satisfies*

$$N < \frac{pq(3+p)}{4+q(p-1)} = \frac{p(3+p)}{4/q+p-1}$$

then $u^* \in L^\infty(\Omega)$ and it is regular solution for (P_{λ^*}).

PROOF. We have to show that $V(x)e^{u_n} \in L^r(\Omega)$ with $r > N/p$, since the Stampacchia's lemma [S] implies that $\|u_n\|_\infty \leq C$ uniformly in λ, and so the limit u^* is regular. If we apply Hölder inequality

$$\int_\Omega (V(x)e^{u_n})^r dx = \int_\Omega (V(x)^{1/(2a+1)} e^{u_n})^r (V(x)^{2a/(2a+1)})^r dx \leq$$

$$\leq \left(\int_\Omega V(x)e^{(2a+1)u_n} dx\right)^{r/(2a+1)} \left(\int_\Omega V(x)^{2ar/(2a+1-r)} dx\right)^{(2a+1-r)/(2a+1)}.$$

We are assuming that $V \in L^q(\Omega)$, $q > N/p$. then, the above quantity is finite if

$$\frac{2ar}{2a+1-r} < q$$

i.e. $(r > N/p)$

$$\frac{2aN/p}{2a+1-N/p} = \frac{2aN}{(2a+1)p-N} < q$$

or

$$a > \frac{(N-p)}{2(pq-N)}.$$

But we are also assuming that

$$a < \frac{2}{p-1}.$$

Then

$$\frac{(N-p)q}{2(pq-N)} < \frac{2}{p-1} \quad \text{i.e.} \quad N < \frac{pq(3+p)}{4+q(p-1)}. \qquad \blacksquare$$

REMARK. *If we take* $V \in L^\infty$, *the last relationship transforms in*

$$N < p + \frac{4p}{p-1}$$

This is the relationship appearing in [GPP], *where* $V \equiv 1$.

The previous results show that, under the regularity hypothesis above cited about $V \geq 0$, there exists at least one positive regular solution of (P), for $1 < p < N$. However, for the subcritical case $p \geq N$, we can do a variational argument.

4. – V changing sign and $p \geq N$.

We will assume in this section the following hypotheses:

(1) $p \geq N$.

(2) $V(x) \in L^q(\Omega)$, $q > 1$ for $p = N$, $q \geq 1$ for $p > N$.

(3) There exists an open ball $B \subset \Omega$ such that $V(x) > 0$ for $x \in B$.

In these hypotheses the comparison argument don't work in general. But the condition $p \geq N$ allow us to state a result by critical points methods. More precisely we have the theorem:

THEOREM 4.1. *There exists a constant* $\lambda_0 > 0$ *such that if* $\lambda < \lambda_0$, *problem* (P) *has two regular solutions at least.*

Hypothesis (3) imply that $V^+ \not\equiv 0$: it plays a fundamental role in the existence of two solutions: notice that for $V < 0$ (i.e. $V^+ \equiv 0$) there is only one negative solution. The proof of 4.1 follows the argument used in [GP] for the case of constant potential.

The energy functional corresponding to our problem is

$$J(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \lambda \int_\Omega V(x) e^u \, dx .$$

It satisfies the following inequality

$$J(u) \geq \frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \lambda \|V\|_q C \exp\left\{ D \left(\int_\Omega |\nabla u|^p \, dx \right)^{N/p} \right\}$$

where $C = (k_1|\Omega|)^{1/q'}$ and $D = k_2(q')^{N-1}|\Omega|^{(p-1)/N}$ (k_1, k_2 are the constants that appear in Trudinger's inequality [GT, p. 162]).

LEMMA 4.2. *The functional J verifies the Palais-Smale condition.*

PROOF. Let $\{u_j\} \subset W_0^{1,p}(\Omega)$ be a Palais-Smale sequence for J; i.e.

$$J(u_j) \to C,$$

$$J'(u_j) \to 0 \quad \text{in} \quad W^{-1,p'}(\Omega).$$

It is necessary to show that any Palais-Smale sequence contains a subsequence which converges strongly in $W_0^{1,p}(\Omega)$. If $\varepsilon_j = J'(u_j)$ then $\varepsilon_j \to 0$ in $W^{-1,p'}(\Omega)$; therefore, we can assume that $\|\varepsilon_j\|_{W^{-1,p'}(\Omega)} \leqslant 1$, and

$$C = \lim_{j\to\infty} \left\{ J(u_j) - \frac{1}{2p} < \varepsilon_j, u_j > + \frac{1}{2p} \langle \varepsilon_j, u_j > \right\} \geqslant$$

$$\geqslant \lim_{j\to\infty} \left\{ \frac{1}{2p} \int_\Omega |\nabla u_j|^p dx + \lambda \int_\Omega V(x) e^{u_j} \left(\frac{u_j}{2p} - 1 \right) dx - \frac{1}{2p} \left(\int_\Omega |\nabla u_j|^p dx \right)^{1/p} \right\} \geqslant$$

$$\geqslant \lim_{j\to\infty} \left\{ \frac{1}{2p} \int_\Omega |\nabla u_j|^p dx + \lambda \|V\|_q C_0 |\Omega|^{1/q'} - \frac{1}{2p} \left(\int_\Omega |\nabla u_j|^p dx \right)^{1/p} \right\}$$

where

$$-C_0 = \min_{x\in R} \left\{ e^x \left(\frac{x}{2p} - 1 \right) \right\} < 0$$

since

$$g(x) = e^x \left(\frac{x}{2p} - 1 \right)$$

is a continuous function defined in the whole $\boldsymbol{R}$ that verifies $g(x) \to 0^-$ as $x \to -\infty$ and $g(x) \to \infty$ as $x \to \infty$. Hence g attains a minimum value $-C_0 < 0$.

Thus, the sequence $\{u_j\}$ is bounded in $W_0^{1,p}(\Omega)$; the rest of the proof of Lemma 4.2 is identical to the one appearing in [GP] for $V \equiv 1$. ■

As in [GP], it is easy to show the following properties of the functional J for $0 < \lambda < \lambda_0$:

(1) If $J(0) = -\lambda \int_\Omega V(x)\,dx \leq 0$, i.e. if $\int_\Omega V(x)\,dx \geq 0$, the functional J verifies that, for λ small enough there exist $R_1 > 0$, $\varrho \in \boldsymbol{R}$ such that if $\|\nabla u\|_p = R_1$, then $J(u) > \varrho > J(0)$: by taking $R_1 = 1$, $\varrho = 1/2p$, $\lambda < (2p\|V\|_q Ce^D)^{-1}$, we get

$$J(u) \geq \frac{1}{p} - \lambda \|V\|_q\, Ce^D > \frac{1}{2p} = \varrho > J(0).$$

(2) If $\int_\Omega V(x)\,dx = -\alpha > 0$, then $J(0) = \lambda\alpha$. If we take $\|\nabla u\|_p = R_1$, we have

$$J(u) \geq \frac{R_1^p}{p} - \lambda\|V\|_q Ce^{DR_1^N} > \frac{\alpha}{p} > \lambda\alpha = J(0)$$

whenever

$$\lambda < \min\left(\frac{R_1^p - \alpha}{p\,\|V\|_q Ce^{DR_1^N}},\ \frac{1}{p}\right).$$

(3) There exits $w_0 \in W_0^{1,p}(\Omega)$, with $\|\nabla w_0\|_p = R_2 > R_1$ and $J(w_0) < J(0)$; for any $w \in C_0^\infty(B) \subset W_0^{1,p}(\Omega)$, $w \geq 0$, where $B \subset \Omega$ is the ball where V is positive, we have

$$J(tw) = \frac{t^p}{p} \int_\Omega |\nabla w|^p\,dx - \lambda \int_\Omega V(x)\, e^{tw}\,dx =$$

$$= \frac{t^p}{p} \int_\Omega |\nabla w|^p\,dx - \lambda \int_\Omega V^+(x)\, e^{tw}\,dx + \lambda \int_\Omega V^-(x)\,dx \leq$$

$$\leq \frac{t^p}{p} \int_\Omega |\nabla u|^p\,dx - \lambda t^{2p} \int_\Omega V^+(x)\, w^{2p}\,dx + \lambda \int_\Omega V^-(x)\,dx \to -\infty \text{ as } t \to \infty$$

since $e^{tw} > (tw)^{2p}$ and $\int_\Omega V^-(x)\,dx$ and $\int_\Omega V^+(x)\,w^{2p}\,dx$ are bounded.

The graph of J is then contained in the region above the graph of

$$f(\|\nabla u\|) = \frac{1}{p}\,\|\nabla u\|_p^p - \lambda\,\|V\|_q\, Ce^{D\|\nabla u\|_p^N}$$

f has two critical points: a local minimum near 0 and a local maximum i.e., the geometrical conditions required for the existence of critical points of J are fulfilled. Therefore, we can use the Mountain Pass Lemma (see [AR]) to get

LEMMA 4.3. *There exists a constant $\lambda_0 > 0$ such that if $0 < \lambda < \lambda_0$, problem* (P) *admits a solution corresponding to a critical point of the functional J with critical value*

$$c = \inf_{\varphi \in \mathcal{C}} \max_{t \in [0,1]} J(\varphi(t))$$

where $\mathcal{C} = \{\varphi \in (C([0,1]), W_0^{1,p}(\Omega)): \varphi(0) = 0, \varphi(1) = w_0\}$ for some $w_0 \in W_0^{1,p}(\Omega)$ such that $J(w_0) \leq J(0)$. Moreover, $c > J(0)$.

To obtain the other critical point, we make the truncation in the functional appearing in [GP]. Let us consider a cut-off function $\tau \in \mathcal{C}^\infty(\Omega)$ such that for the previously defined R_1 and R_2

$$\tau(x) \equiv \begin{cases} 1 & \text{if } x \leq R_1 , \\ 0 & \text{if } x \geq R_2 , \end{cases}$$

and τ nonincreasing. Thus, we obtain the truncated functional

$$F(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \lambda \int_\Omega V(x)\, \tau(\|\nabla u\|_p)\, e^u \, dx$$

It has to be noted that

(1) J and F are the same if $\|\nabla u\|_p \leq R_1$.

(2) If $\|\nabla u\|_p \geq R_2$, then $F(u) = (1/p) \int_\Omega |\nabla u|^p \, dx$.

(3) $F(u) < J(0) = -\lambda \int_\Omega V(x)\, dx$, then $\|\nabla u\|_p \geq R_1$, since F is increasing for u with $\|\nabla u\|_p = R_1$. Hence, J and F are the same in a neighborhood of u and the Palais-Smale condition for J implies the subsequent Palais-Smale condition for F.

LEMMA 4.4. *Let $\{u_j\} \in W_0^{1,p}(\Omega)$ be such that*

$$\begin{cases} F(u_j) \to C < J(0), & \\ F'(u_j) \to 0 & \text{in } W^{-1,p'}(\Omega). \end{cases}$$

Then there exists a subsequence which converges strongly to $u \in W_0^{1,p}(\Omega)$, LEMMA 4.5.

$$\inf_{u \in W_0^{1,p}(\Omega)} F(u) < J(0) = -\lambda \int_\Omega V(x)\,dx\,.$$

PROOF. Let w be a function in $\mathcal{C}_0^\infty(B) \in W_0^{1,p}(\Omega)$, $w \geq 0$, where B is again the open ball where $V > 0$. If $\|\nabla w\|_p = 1$, and $\varrho < R_1$, then

$$F(\varrho w) = \frac{\varrho^p}{p} - \lambda \int_\Omega V^+(x)\,e^{\varrho w}\,dx + \lambda \int_\Omega V^-(x)\,dx \leq$$

$$\leq \frac{\varrho^p}{p} - \lambda \int_\Omega V^+(x)(1+\varrho w)\,dx + \lambda \int_\Omega V^-(x)\,dx =$$

$$= \varrho \left\{ \frac{\varrho^{p-1}}{p} - \lambda \int_\Omega V^+(x)\,w\,dx \right\} - \lambda \int_\Omega V^+(x)\,dx + \lambda \int_\Omega V^-(x)\,dx < J(0)$$

whenever ϱ is small enough $\left(\int_\Omega V^+(x)\,w\,dx \text{ is bounded} \right)$. ■

By the same argument as in [GP] we have the following result.

LEMMA 4.6. *There exists a positive constant $\lambda_0 > 0$ such that if $0 < \lambda < \lambda_0$, then problem* (P) *has a solution corresponding to a critical point of the functional J, with critical value $c' < J(0)$.*

With the lemmas we conclude the proof of Theorem 4.1 in an inmediate way.

5. – A remark on eigenvalues and nonexistence results.

Let Ω be a bounded domain, $\partial\Omega \in \mathcal{C}^{2,\beta}$ and now we assume $V(x) \geq 0$, $V(x) \in L^q(\Omega)$ ($q \geq 1$ if $p > N$, $q > 1$ if $p = N$, $q > N/p > 1$ otherwise), with $|\{x \in \Omega\colon V(x) > 0\}| \neq 0$. $|\cdot|$ means the Lebesgue measure.

Let us consider the problem ($1 < p < \infty$):

$$(\mathrm{P}_\lambda) \quad \begin{cases} u \in W_0^{1,p}(\Omega),\ u \not\equiv 0\,, \\ -\Delta_p u \equiv -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda V(x)|u|^{p-2}u\,,\ u|_{\partial\Omega} = 0\,. \end{cases}$$

DEFINITION 5.1. *We say that λ is an eigenvalue if* (P_λ) *admits* *a solution. such solution is said an eigenfunction corresponding to the eigenvalue* λ.

We now define the first eigenvalue, λ_1, as

$$\lambda_1 = \inf\left\{ \int_\Omega |\nabla w|^p dx \colon w \in W_0^{1,p}(\Omega), \int_\Omega V(x)\,|u|^p dx = 1 \right\}.$$

This problem when $V \in L^\infty$, was studied by [An], [Bl] and [L]. We need the case $V \in L^q$, with q in the hypothesis above.

The proof follows closely the one in [An], and then we concentrate our attention only in the points that need some change.

The two next results are well known.

LEMMA 5.2. *If u is a solution of* (P_λ), $u \in \mathcal{C}^{1,\alpha}(\Omega)$. *Moreover, if* $u \geqslant 0$ *then* $u > 0$ *in* Ω *and* $\partial u / \partial \nu < 0$ *on* $\partial\Omega$, *where* ν *denotes the unit exterior normal vector to* $\partial\Omega$. *(Hopf's lemma).*

LEMMA 5.3. λ_1 *is an eigenvalue and every eigenfunction* u_1 *corresponding to* λ_1 *does not change sign in* Ω: *either* $u_1 > 0$ *or* $u_1 < 0$.

We consider $I(u, v)$ defined as

$$I(u, v) = \left\langle -\Delta_p u, \frac{u^p - v^p}{u^{p-1}} \right\rangle + \left\langle -\Delta_p v, \frac{v^p - u^p}{v^{p-1}} \right\rangle$$

with

$$(u, v) \in D_I = \left\{ (u, v) \in (W_0^{1,p}(\Omega))^2 \colon u, v \geqslant 0 \text{ and } \frac{u}{v} \in L^\infty(\Omega) \right\}$$

we get the following results (see [An, Prop. 1 and Th. 1]):

PROPOSITION 5.4. $\forall (u, v) \in D_I$, $I(u, v) \geqslant 0$. *Moreover,* $I(u, v) = 0$ *if and only if there exists* $a \in (0, \infty)$ *such that* $u = av$.

The proof of this proposition consists of the calculation of

$$\left\langle -\Delta_p u, \frac{u^p - v^p}{u^{p-1}} \right\rangle \quad \text{and} \quad \left\langle -\Delta_p v, \frac{v^p - u^p}{v^{p-1}} \right\rangle$$

to show that $I(u, v)$ is the integral of a sum of two non-negative functions, hence $I(u, v) \geqslant 0$. Moreover, $I(u, v)$ vanishes if there exists some $a \in (0, \infty)$ such that $u = av$. As a consequence we get

PROPOSITION 5.5. *λ_1 is simple, i.e. if u, v are two eigenfunctions corresponding to the eigenvalue λ_1, then $u = av$ for some a.*

With respect to the isolation, we extend the results by Anane [An]:

PROPOSITION 5.6. *If w is an eigenfunction corresponding to the eigenvalue λ, $\lambda > 0$, $\lambda \not\equiv \lambda_1$, then w changes sign in Ω: $w^+ \not\equiv 0$, $w^- \not\equiv 0$ and*

$$|\Omega^-| \geq (\lambda \|V\|_q C^p)^\sigma$$

where $\Omega^- = \{x \in \Omega: w(x) < 0\}$, $\sigma = -2q'$ if $p \geq N$, $\sigma = -qN/(qp - N)$ if $1 < p < N$ and λ_1 is the first eigenvalue for the p-laplacian with weight V in Ω.

PROOF. Let u, w be two eigenfunctions corresponding to λ_1 and λ respectively, with $\|\nabla u\|_p = \|\nabla u\|_p = 1$. If w does not change sign, by applying Lemmas 5.2, 5.3 and Proposition 5.4 we get

$$(u, w) \in D_I, \qquad I(u, w) \geq 0 .$$

But

$$0 \leq I(u, w) = \int_\Omega (\lambda_1 - \lambda) V(x)(u^p - w^p)\, dx = (\lambda_1 - \lambda)\left(\frac{1}{\lambda_1} - \frac{1}{\lambda}\right) < 0$$

and we arrive to a contradiction.

If w^- replaces to w in (P_λ), we get

$$\|\nabla w^-\|_p^p = \lambda \int_\Omega V(w^-)^p dx \leq \lambda \|V\|_q \|(w^-)^q\|_\alpha |\Omega^-|^{1/\beta}$$

with $1/q + 1/\alpha + 1/\beta = 1$. Now we are considering two cases

(1) $p \geq N$. By Sobolev's inequality

$$\|(w^-)^p\|_\alpha = \|w^-\|_{\alpha p}^p \leq C\|\nabla w^-\|_p^p \qquad (\alpha > 1).$$

Thus, if we take $\alpha = \beta = 2q'$ then

$$|\Omega^-| \geq (\lambda \|V\|_q C^p)^{-2q'}$$

(2) $1 < p < N$. We take

$$\alpha = N/(N - p), \qquad \beta = qN/(qp - N)(\|(w^-)^p\|_\alpha = \|w^-\|_{p^*}^p,$$

where $p^* = Np/(N-p))$. by Sobolev's inequality

$$\|\nabla w^-\|_p^p \leqslant \lambda \|V\|_q \|w^-\|_{p^*}^p |\Omega^-|^{(qp-N)/qN} \leqslant$$
$$\leqslant \lambda \|V\|_q C^p \|\nabla w^-\|_p^p |\Omega^-|^{(qp-N)/qN}.$$

Hence

$$|\Omega^-| \geqslant (\lambda \|V\|_q C^p)^{-qN/(qp-N)}. \quad \blacksquare$$

REMARK 1. *If $q' \to 1$ ($q \to \infty$), we obtain the estimation by Anane* (*see* [An, Prop. 2]).

REMARK 2. *As Lindqvist pointed out, we can also extend these results to any bounded domain* (*see* [L]).

In the hypotheses of Proposition 5.6 we obtain the next theorem which proof follow [An]. However, we include it for the sake of completeness.

THEOREM 5.7. *λ_1 is isolated; that is, λ_1 is the unique eigenvalue in $[0, a]$ for some $a > \lambda_1$.*

PROOF. Let $\lambda \geqslant 0$ be an eigenvalue and v be the corresponding eigenfunction. By the definition of λ_1 (it is the infimum) we have $\lambda \geqslant \geqslant \lambda_1$. Then, λ_1 is left-isolated.

We are now arguing by contradiction. We assume there exists a sequence of eigenvalues (λ_k), $\lambda_k \not\equiv \lambda_1$ which converges to λ_1. Let (u_k) be the corresponding eigenfunctions with $\|\nabla u_k\|_p = 1$. we can therefore take a subsequence, denoted again by (u_k), converging weakly in $W^{1,p}$, strongly in $L^p(\Omega)$ and almost everywhere in Ω to a function $u \in W_0^{1,p}$. Since $u_k = -\Delta_p^{-1}(\lambda_k V|u_k|^{p-2}u_k)$, the subsequence (u_k) converges strongly in $W_0^{1,p}$, and subsequently u is the eigenfunction corresponding to the first eigenvalue λ_1 with norm equals to 1. Hence, by applying the Egorov's Theorem ([B, Th. IV.28]), (u_k) converges uniformly to u in the exterior of a set of arbitrarily small measure. Then, there exists a piece of Ω of arbitrarily small measure in which exterior u_k is positive for k large enough, obtaining a contradiction with the conclusion of Proposition 4.6. $\blacksquare$

Now we are ready to show a nonexistence result for the problem (P): if $V(x) \in L^q(\Omega)$, ($q \geqslant 1$ for $p > N$, $q > 1$ for $p = N$, $q > N/p > 1$ otherwise) $V \geqslant 0$, and λ is large enough then problem (P) does not have a solution.

In the end of the proof we use the isolation of the first eigenvalue for $-\Delta_p$ with weight $V(x)$.

THEOREM 5.8. *Problem* (P) *does not have a solution if*

$$\lambda > \max\left\{\lambda_1, \lambda_1\left(\frac{p-1}{e}\right)^{p-1}\right\}$$

where $V \in L^q(\Omega)$, ($q \geq 1$ *for* $p > N$, $q > 1$ *for* $p = N$, $q > N/p$ *otherwise*), $V \geq 0$ *and* λ_1 *is the first eigenvalue for the p-Laplacian with weight* $V(x)$.

PROOF. If λ_1 is the first eigenvalue for the p-Laplacian with weight $V(x)$, take $\lambda_\varepsilon = \lambda_1 + \varepsilon$, $\varepsilon > 0$, v_1 a positive eigenfunction corresponding to λ_1 with $\|v_1\|_\infty \leq 1$, and suppose that problem (P) has a solution $u \in W_0^{1,p}(\Omega)$ and

$$\lambda > \max\left(\lambda_1, \lambda_1\left(\frac{p-1}{e}\right)^{p-1}\right)$$

for small ε, we have ($\lambda_\varepsilon < \lambda$):

$$\lambda_\varepsilon x^{p-1} < \lambda e^x, \qquad \forall x \geq 0,$$

$$-\Delta_p v_1 = \lambda_1 V v_1^{p-1} < \lambda_\varepsilon V < \lambda V \leq \lambda V e^u = -\Delta_p u.$$

By using the weak comparison principle for the p-Laplacian we obtain

$$v_1 \leq u.$$

Let v_2 be the solution of

$$\begin{cases} -\Delta_p v_2 = \lambda_\varepsilon V(x) v_1^{p-1} & \text{in } \Omega, \\ v_2 |_{\partial\Omega} = 0. \end{cases}$$

We know by regularity results that $v_2 \in \mathcal{C}^{1,\alpha}(\Omega)$. In addition

$$-\Delta_p v_2 = \lambda_\varepsilon V v_1^{p-1} < \lambda V e^{v_1} \leq \lambda V e^u = -\Delta_p u,$$

$$-\Delta_p v_1 \leq \lambda_\varepsilon V v_1^{p-1} = -\Delta_p v_2.$$

By applying again the weak comparison principle, we get $v_1 \leq v_2 \leq u$.

Now, let us consider the problems

$$\begin{cases} -\Delta_p v_k = \lambda_\varepsilon V(x) v_{k-1}^{p-1} & \text{in } \Omega, \\ v_k \mid_{\partial\Omega} = 0. \end{cases}$$

The solutions of these problems form an increasing sequence $\{v_k\}$ such that

$$v_1 \leqslant v_k \leqslant v_{k+1} \leqslant u$$

Passing to the limit, we obtain a solution $w \in W_0^{1,p}(\Omega)$ to the problem

$$\begin{cases} -\Delta_p v = \lambda_\varepsilon V(x) w^{p-1} & \text{in } \Omega, \\ v_k \mid_{\partial\Omega} = 0. \end{cases}$$

But this is impossible for ε small enough, because the first eigenvalue for the p-Laplacian with weight in $L^q(\Omega)$ is isolated by Theorem 5.7. ■

This argument also shows the nonexistence of positive solutions to (P) for λ large enough when V changes sign in Ω:

COROLLARY 5.9. *Suppose that* $V \in L^q(\Omega)$ ($q \geqslant 1$ *for* $p > N$, $q > 1$ *for* $p = N$, $q > N/p$ *otherwise*), *V changes sign in Ω and there exists a ball $B \subset \Omega$ such that $V(x) > 0$ for $x \in B$. If*

$$\lambda > \max\left\{\lambda_1, \lambda_1\left(\frac{p-1}{e}\right)^{p-1}\right\}$$

then (P) *has no positive solutions, λ_1 being the first eigenvalue for the p-Laplacian with weight $V(x)$ in B.*

PROOF. Let w_1 a positive eigenfunction corresponding to λ_1 in B with $\|v_1\|_\infty \leqslant 1$, that is, w_1 verifies

$$\begin{cases} -\Delta_p w_1 = \lambda_1 V(x) w_1^{p-1} & \text{in } B, \\ w_1 \mid_{\partial B} = 0. \end{cases}$$

Take $\lambda_\varepsilon = \lambda_1 + \varepsilon$, $\varepsilon > 0$, and suppose that problem (P) has a positive solution u for

$$\lambda > \max\left(\lambda_1, \lambda_1\left(\frac{p-1}{e}\right)^{p-1}\right).$$

Then, for small ε, we have ($\lambda_\varepsilon < \lambda$):

$$\lambda_\varepsilon x^{p-1} < \lambda e^x, \qquad \forall x \geq 0,$$

$$-\Delta_p w_1 = \lambda_1 V w_1^{p-1} < \lambda_\varepsilon V < \lambda V \leq \lambda V e^u = -\Delta_p u \quad \text{in } B,$$

$$w_1 = 0 \leq u \quad \text{in } \partial B.$$

By using the weak comparison principle for the p-Laplacian we obtain

$$w_1 \leq u.$$

Using the argument in the proof of Theorem 5.8, we obtain a solution to the following problem

$$\begin{cases} -\Delta_p w = \lambda_\varepsilon V(x) w^{p-1} & \text{in } B, \\ w|_{\partial B} = 0. \end{cases}$$

But this is impossible for ε small enough, because the first eigenvalue for the p-Laplacian with weight in $L^q(B)$ is isolated by Theorem 5.7. ∎

6. – Analysis of the radial solutions in a ball.

In this section, we consider the problem

$$(\mathrm{P}_r) \qquad \begin{cases} -\Delta_p u = \lambda \dfrac{e^u}{r^\alpha} & \text{in } B_1(0) \subset \boldsymbol{R}^N, \ \alpha \in \boldsymbol{R}, \ r = |x|, \\ u|_{\partial B_1(0)} = 0. \end{cases}$$

where $B_1(0)$ denotes the unit ball in $\boldsymbol{R}^N$ and $\lambda > 0$.

In order to study the existence of singular solutions, we just consider the case $1 < p < N$ since for $p \geq N$ every solution is a regular solution. In this hypothesis, it is easy to prove the nonexistence of singular solutions for some α:

If we consider the problem

$$\begin{cases} -\Delta_p v = \dfrac{\lambda}{r^\alpha} & \text{in } B_1(0) \subset \boldsymbol{R}^N, \\ v|_{\partial B_1(0)} = 0 \end{cases}$$

with $p < \alpha < N$, $\lambda > 0$ and we try the solutions $v(r) = \beta r^{\gamma}$, we get that

$$\gamma = \frac{p-\alpha}{p-1}, \qquad \lambda = (-\beta\gamma)^{p-1}[(p-1)(\gamma-1)+N-1],$$

and therefore the solution v is not bounded for $p < \alpha < N$. For $\alpha = p$ we can take

$$v(r) = \beta \log r,$$

obtaining $\beta = -(\lambda/(N-p))^{1/(p-1)}$ and so v is also not bounded.

If we assume now that u is a positive singular solution of (P_r), i.e. $u \in W_0^{1,p}(B_1(0))$, $e^u r^{-\alpha} \in L^1(B_1(0))$, being $p < \alpha < N$, then

$$-\Delta_p u = \lambda \frac{e^u}{r^\alpha} \geqslant \frac{\lambda}{r^\alpha} = -\Delta_p v$$

that is, $u > v$. But this leads to a contradiction, since

$$\frac{e^u}{r^\alpha} \geqslant \frac{e^v}{r^\alpha} \notin L^1(B_1(0))$$

On the other hand, if $\alpha \geqslant N$ then the potential $V(r) = r^{-\alpha}$ does not belong to $L^1(B_1(0))$.

Then we directly assume that $\alpha < p$, independently on the dimension. Following the procedure carried out in [GPP], we introduce the new variables

$$s = \log r,$$

$$r(s) = |u_s|^{p-2}u_s,$$

$$u(s) = -\lambda e^{u+(p-\alpha)s}.$$

In the plane (v, w) the radial solutions of (P_r) satisfy the following autonomous system

$$\begin{cases} \dfrac{dv}{ds} = w - (N-p)v, \\ \dfrac{dw}{ds} = (p - \alpha + |v|^{1/(p-1)}\operatorname{sign}(v))\, w. \end{cases}$$

By the definition of the new variables, the region of interest is $v < 0$, $w < 0$ (a radial solution of (P_r) is positive and its radial derivatives is negative). In this region we find two stationnary points: $P_1(0, 0)$ and $P_2(-(p-\alpha)^{p-1}, -(p-\alpha)^{p-1}(N-p))$ $u \in W_0^{1,p}(\Omega)$.

The point P_1 is an unstable hyperbolic point. The v-axis is the stable manifold for this point, and the unstable manifold is tangent to the straightline $w = (N - \alpha)v$.

With respect to the point P_2, it is

(1) A stable nodus if $N \geqslant p + (4(p - \alpha))/(p - 1)$.

(2) A stable spiral point if $p < N < p + (4(p - \alpha))/(p - 1)$.

We can see also that a singular selfsimilar solution of (P_r) is

$$S(x) = \log\left(\frac{1}{|x|^{p-\alpha}}\right)$$

with λ being equal to $\tilde{\lambda} = (p - \alpha)^{p-1}(N - p)$. This singular solution corresponds to the critical point P_2 in the phase portrait, and it verifies the following interesting property:

$$\frac{1}{|x|^{\alpha}} e^{S(x)} = \frac{1}{|x|^{p}}$$

for every α.

We need some previous lemmas (their proofs are similar to those in [GPP]).

LEMMA 6.1. *Let u be a radial solution of (P_r) and (v, w) the corresponding trajectory of the autonomous system. Then, u is a regular solution of (P_r) $\left(\lim_{s \to -\infty} u(s) = A < \infty\right)$ if and only if $\lim_{s \to -\infty} (v(s), w(s)) = (0, 0)$.*

LEMMA 6.2. *The unique trajectory of the autonomous system corresponding to a solution of (P_r) such that $\lim_{s \to -\infty} u(s) = \infty$ is the critical point P_2.*

Let $\underline{u}(\lambda)$ be the minimal solution of (P_r), $\lambda^* = \sup\{\lambda\colon (P_r) \text{ has solution}\}$, and $\tilde{\lambda} = (p - \alpha)^{p-1}(N - p)$. In this way, we arrive to the

THEOREM 6.3. i) *If $N \geqslant p + (4(p - \alpha))/(p - 1)$ then $\Lambda^* = \tilde{\lambda}$, for each $\lambda < \lambda^*$ we have a unique radial regular solution, and $\lim_{\lambda \to \lambda^*} \underline{u}(\lambda) = u_*$ is a singular solution;*

ii) *If $p < N < p + (4(p - \alpha))/(p - 1)$ then $\tilde{\lambda} < \lambda^*$, and for $\lambda = \tilde{\lambda}$, there are infinitely many regular radial solutions, the values at the origin going to infinity.*

Moreover, in case ii), $\lim_{\lambda \to \lambda^*} \underline{u}(\lambda) = u_* \in L^\infty$, *and there exists a positive constant,* $\varepsilon_0 > 0$ *such that, if* $0 < |\lambda - \tilde{\lambda}| < \varepsilon_0$ *then the corresponding problem* (P_r) *has a finite family of radial solutions.*

PROOF. In case i) we show that the trajectory joining P_1 and P_2, denoted by ϕ, is a monotone curve contained in the region $-(p-\alpha)^{p-1} < v < 0$, $-(p-\alpha)^{p-1}(N-p) < w < 0$. Thus, there exists a unique point of intersection for each line $w = -\lambda$, i.e., there exists a unique regular radial solution for each $\lambda \in (0, (p-\alpha)^{p-1}(N-p))$.

First, it is easy to see that ϕ is below the line $w = (N-p)v$. We need a lower bound for ϕ; for that, we consider two different cases.

If $N \geqslant \max\{p + (4(p-\alpha))/(p-1), 3p - 2\alpha\}$, and R is the line

$$w = \frac{N-p}{2} v - (p-\alpha)^{p-1} \frac{N-p}{2}$$

we will show that $dw/dv < (N-p)/2$ along R, whenever

$$-(p-\alpha)^{p-1} < v < 0 .$$

In this way the trajectories (v, w) in the phase plane must cross R from below; this implies that ϕ cannot cut R, since it stars from above.

Then, it suffices to show that

$$\frac{dw}{ds} - \frac{N-p}{2} \frac{dv}{ds} > 0$$

when $(v, w) \in R$, $-(p-\alpha)^{p-1} < v < 0$ (it has to be noted that $dv/ds < 0$ in the region $-(p-\alpha)^{p-1} < v < 0$, $-(p-\alpha)^{p-1}(N-p) < w < (N-p)v$). So

$$\frac{dw}{ds} - \frac{N-p}{2} \frac{dv}{ds} = ((p-\alpha)^{p-1} - |v|) \left\{ \left(\frac{N-p}{2} \right)^2 + \right.$$

$$\left. \frac{N-p}{2} (p - \alpha - |v|^{1/(p-1)}) - (N-p)(p-\alpha)^{p-1} \frac{p - \alpha - |v|^{1/(p-1)}}{(p-\alpha)^{p-1} - |v|} \right\}.$$

The factor $((p-\alpha)^{p-1} - |v|)$ is positive; if we write $s = |v|^{1/(p-1)}/(p-\alpha)$, and we suppose $1 < p < 2$, the function

$(1-s)/(1-s^{p-1})$ is increasing in $(0,1)$. We obtain (remember that $N \geqslant p + (4(p-\alpha))/(p-1)$)

$$\frac{N-p}{2} + (p-\alpha)(1-s) - 2(p-\alpha)\frac{1-s}{1-s^{p-1}} >$$

$$> \frac{N-p}{2} - \frac{2(p-\alpha)}{p-1} > 0.$$

If $p > 2$, then

$$\frac{N-p}{2} + (p-\alpha)(1-s) - 2(p-\alpha)\frac{1-s}{1-s^{p-1}} = \frac{1}{1-s^{p-1}} f(s)$$

where $f(s)$ is

$$f(s) = (p-\alpha)s^p - \left(\frac{N+p-2\alpha}{2}\right)s^{p-1} + (p-\alpha)s + \frac{N-3p+2\alpha}{2}$$

for $s \in (0,1)$. This function verifies the following properties:

(1) $f(0) = (N-3p-2\alpha)/2 > 0$, $f'(0) = p-\alpha > 0$.

(2) $f(1) = 0$ and $f'(1) \leqslant 0$ since $N \geqslant p + (4(p-\alpha))/(p-1)$.

(3) f has two critical points, the first between 0 and 1, the second one greater or equal to 1.

This implies that

$$\frac{dw}{ds} - \frac{N-p}{2}\frac{dv}{ds} > 0$$

when $(u, v) \in R$, $-(p-\alpha)^{p-1} < v < 0$, and therefore the trajectory ϕ cannot cross R.

When $p + (4(p-\alpha))/(p-1) \leqslant N < 3p - 2\alpha$, we can do a different argument. We consider now the curve

$$f(v) = -(p-\alpha)^{(p-1)/2}(N-p)\,|v|^{1/2}$$

contained in the region $-(p-\alpha)^{p-1} < v < 0$. Then f verifies

(1) $f(0) = 0$, $f(-(p-\alpha)^{p-1}) = -(p-\alpha)^{p-1}(N-p)$, that is, f connects the two singular points in the phase plane.

(2) f is increasing and convex in $(-(p-\alpha)^{p-1}, 0)$.

(3) $dw/dv < f'(v)$ on $(v, f(v))$.

Then, it follows that f is a lower bound for the trajectory ϕ and we conclude the analysis for i).

In case ii) the line $w = -\lambda$ cross the manifold ϕ infinitely many times. Each point of intersection s_j corresponds to a radial solution of (P_r) by scaling s in such a way that for $s = 0$ we have as an initial value s_j. The rest is a consequence of the analysis carried out in this section. ■

REFERENCES

[AR] A. AMBROSETTI - P. H. RABINOWITZ, *Dual variational methods in critical point theory and applications*, J. Funct. Abal., **14** (1973), pp. 349-381.

[An] A. ANANE, *Simplicité et isolation de la première valeur propre du p-laplacien avec poids*, C. R. Acad. Sci. Paris, Série I, **305** (1987), pp. 725-728.

[Bd] C. BANDLE, *Existence theorems. Qualitative results and a priori bounds for a class of nonlinear Dirichlet problems*, Arch. Rational Mech. Anal., **49** (1973), pp. 241-269.

[Bl] G. BARLES, *Remarks on the uniqueness results of the first eigenvalue of the p-laplacian*, Ann. Fac. Science de Toulousse, Serie 5, **1** (1988), pp. 65-75.

[Br] H. BRÉZIS, *Analyse fonctionelle*, Masson (1983).

[BPV] L. BOCCARDO - I. PERAL - J. L. VÁZQUEZ, *Some remarks on the N-Laplacian*, preprint.

[BM] H. BRÉZIS - F. MERLE, *Uniform estimates and blow-up behavior for solutions of* $-\Delta u = V(x)e^u$ *in two dimensions*, Commu. in P.D.E., **16**, nn. 8-9 (1992), pp. 1223-1253.

[Ch] S. CHANDRASEKAR, *An Introduction ot the Study of Stellar Structure*, Dover Publ. Inc. (1985).

[DiB] E. DI BENEDETTO, $C^{1,\alpha}$ *local regularity of weak solutions of degenerate elliptic equations*, Nonlinear analysis. Theory, Methods and Applications, **7**, n. 8 (1983), pp. 827-850.

[FK] D. A. FRANK-KAMENETSKII, *Diffusion and Heat Transfer in Chemical Kinetics*, Plenum Press (1969).

[F] H. FUJITA, *On the nonlinear equation* $\Delta u + \exp u = 0$ *and* $v_t = \Delta u + \exp u$, Bull. A.M.S., **75** (1969), pp. 132-135.

[Ge] I. M. GUELFAND, *Some problems in the theory of quasilinear equations*, Amer. Math. Soc. Transl. (Ser. 2), **29** (1963), pp. 295-381.

[GMP] T. GALLOUET - F. MIGNOT - J. P. PUEL, *Quelques résultats sur le problème* $-\Delta u = \exp u$, C. R. Acad. Sci. Paris, **307** (1988), pp. 289-292.

[GP] J. GARCIA AZOZERO - I. PERAL ALONSO, *On a Emden-Fowler type equation*, Nonlinear Analysis T.M.A., **18**, n. 11 (1992), pp. 1085-1097.

[GPP] J. GARCIA AZOZERO - I. PERAL ALONSO - J. P. PUEL, *Quasilinear problems with exponential growth in the reaction term*, Nonlinear Analysis T.M.A., **22**, n. 4 (1994), pp. 481-498.

[GT] D. GILBARG - N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, New York (1983).

[JL] D. JOSEPH - T. S. LUNDGREN, *Quasilinear Dirichlet problems driven by positive sources*, Arch. Rational Mech. Anal., **49** (1973), pp. 241-269.

[KW] J. L. KAZDAN - J. L. W. WARNER, *Curvature functions for compact 2-manifolds*, Ann. Math., **99** (1974), pp. 14-47.

[L] P. LINDQVIST, *On the equation* $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda |u|^{p-2} = 0$, Proc. Amer. Math. Soc., **109**, n. 1 (1990), pp. 157-164.

[MP1] F. MIGNOT - J. P. PUEL, *Sur une classe de probleme non lineaire avec non linearite positive, crossante, convexe*, Communication in P.D.E., **5** (8) (1980), pp. 791-836.

[MP2] F. MIGNOT - J. P. PUEL, *Solution radiale singuliere de* $-\Delta u = \lambda e^u$, C. R. Acad. Sci. Paris, Série I, **307** (1988), pp. 379-382.

[S] G. STAMPACCHIA, *Equations Elliptiques du Second Ordre a Coefficients Discontinus*, Les Presses de l'Université de Montreal (1965).

Manoscritto pervenuto in redazione il 7 aprile 1995.