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Conjugacy Class Lengths of Metanilpotent Groups.

CARLO CASOLO - SILVIO DOLFI (*)

ABSTRACT - We study prime divisors of the lengths of conjugacy classes of a finite group G, with emphasis on the metanilpotent case. In particular, we compare the number of prime divisors of $|G/Z(G)|$ with the maximum number of distinct prime divisors of the length of a single conjugacy class of G.

If G is a finite group, we write $\pi(G)$ for the set of all primes dividing the order of G. If $g \in G$, we denote by $\sigma_G(g)$ the set of all prime divisors of $|G : C_G(g)|$, the length of the conjugacy class of g; then we put

$$\sigma^*(G) = \max\{|\sigma_G(g)| : g \in G\} \quad \text{and} \quad \varrho^*(G) = \bigcup_{g \in G} \sigma_G(g).$$

Thus $\varrho^*(G)$ is the set of all primes dividing the lengths of conjugacy classes of G. It is an elementary fact that $\varrho^*(G) = \pi(G/Z(G))$.

Similarly, in character theory, one defines the set $\varrho(G)$ of all primes dividing the degrees of the irreducible characters of the group G, and the maximum number $\sigma(G)$ of distinct primes dividing the degree of a single irreducible character of G. It is conjectured that, for soluble groups, $|\varrho(G)| \leq 2\sigma(G)$. This has been verified by O. Manz [13] for $\sigma(G) = 1$, and by D. Gluck [11] for $\sigma(G) = 2$. To

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date, the best general result to be known is: $|\varrho(G)| \leq 3\sigma(G)+2$ (Manz and Wolf [14], see also [15]).

On the conjugacy class side, B. Huppert has asked whether, for soluble groups

$$|\varrho^*(G)| \leq 2\sigma^*(G).$$

This inequality has been verified by Chillag and Herzog [5] for $\sigma^*(G)=1$, by P. Ferguson [7] for $\sigma^*(G)=2$, and by Casolo [4] for $\sigma^*(G)=3$; also, it showed in [3] that it holds for every perfect group. Several results about the connection between $\varrho^*(G)$ and $\sigma^*(G)$ have been published by P. Ferguson. In particular she proves in [10] that $|\varrho^*(G)| \leq 4\sigma^*(G)+6$ for every finite soluble group G, and that $|\varrho^*(G)| \leq 2\sigma^*(G)$ for supersoluble groups in which G/G' is cyclic [8].

In this paper we collect some observations on Huppert's and related questions. In particular we prove that, even for metabelian groups, the answer to it is negative; we show that, for metanilpotent groups, $|\varrho^*(G)| < 3\sigma^*(G)$ and construct in Example 2 a family $\{G_n\}$ of supersoluble metabelian groups such that $\lim_{n\to\infty} |\varrho^*(G_n)|/(\sigma^*(G_n)) = 3$. All groups considered are finite.

Throughout the paper, we will use without reference the following elementary and well known facts (see, for instance [5, 6]).

Let G be a finite group.

1) *If x and y are commuting elements of coprime order of G, then*

$$\sigma_G(xy) \supseteq \sigma_G(x) \cup \sigma_G(y).$$

2) *Let $N \trianglelefteq G$, $x \in N$, $y \in G$. Then:*

(i) $\sigma_N(x) \subseteq \sigma_G(x)$ *and* $\sigma^*(N) \leq \sigma^*(G)$,

(ii) $\sigma_{G/N}(yN) \subseteq \sigma_G(y)$ *and*$\sigma^*(G/N) \leq \sigma^*(G)$.

3) *Let $p \in \pi(G)$. Then $p \notin \varrho^*(G)$ if and only if G has a central Sylow p-subgroup.*

If G is a finite group, we denote by $F(G)$ the Fitting subgroup of G, by $Z(G)$ the centre of G and by $A(G)$ the subgroup of G generated by all the normal abelian Sylow subgroups of G. By the Ito-Michler Theorem, a prime $p \in \pi(G)$ does not divide the degree of any irreducible character of G if and only if G has a normal abelian Sylow p-subgroup. Thus

$\varrho(G) = \pi(G/A(G))$ (which clearly coincides with $\pi(G/Z(F(G)))$. Our first proposition is a conjugacy class analogue of [11; Lemma 1.1].

PROPOSITION 1. *Let G be a finite metanilpotent group. Then there exists $x \in F(G)$ such that $\sigma_G(x) = \varrho(G)$.*

PROOF. Let G be a metanilpotent group, let $F = F(G)$, and argue by induction on $|G|$.

We first observe that we may assume that G/F is abelian. In fact, if G_1/F is the centre of G/F, then $F(G_1) = F$ and, because G/F is nilpotent, $\pi(G/F) = \pi(G_1/F)$.

We show that, for each $p \in \pi(F)$, if F_p is the Sylow p-subgroup of F, there exists $x_p \in F_p$ such that $\sigma_G(x) = \pi(|G : C_G(F_p)|)$. By taking $x = \prod_{p \in \pi(F)} x_p$, we will have

$$\sigma_G(x) = \bigcup_{p \in \pi(F)} \sigma_G(x_p) = \bigcup_{p \in \pi(F)} \pi(|G : C_G(F_p)|) = \pi(|G : C_G(F)|)$$

and therefore the conclusion will follow, because $C_G(F) = Z(F)$.

Thus, let $K = F_p$ for some $p \in \pi(F)$. Suppose that $K' \neq 1$. Then $F(G/K') = F/K'$ so, if p divides $|G/F|$, then $\varrho(G) = \varrho(G/K')$ and we are done by the inductive hypothesis. Otherwise K is a Sylow p-subgroup of F and $G = KH$ where H is a Hall p'-subgroup of G. Let $C = C_H(K)$; as H/C is abelian, by [4; Lemma 2.4] we have $K = \langle x \in K;\ C_G(x) \leq KC \rangle$. Since K is not abelian, there exists $x \in K \setminus Z(K)$ such that $C_G(x) \leq KC$. Since $C_G(K) = Z(K)\, C_H(K)$, for such an element x we get $\sigma_G(x) = \pi(|G : C_G(x)|) = \pi(|G : C_G(K)|)$.

Suppose now that K is abelian. Then $C_G(K) \geq F$ and so $G/C_G(K)$ is an abelian group, the direct product of its p'-component T and its p-component P. Observe that $C_K(P)$ is a normal subgroup of G and that it is a proper subgroup of K if $P \neq 1$. As above, we can apply [4; Lemma 2.4] to get an element $x \in K$ whose orbit under T is regular, with the additional property that $x \notin C_K(P)$ if $P \neq 1$. For such an x, we obtain $\sigma_G(x) = \pi(|G : C_G(K)|)$. This completes the proof. ■

For each prime divisor p of the order of the group G, let G_p be a Sylow p-subgroup of G and let $n(p) = |N_G(G_p) : C_G(G_p)|$. It is shown in [3] that

$$\sigma^*(G) > \sum_{p \in \pi(G)} \frac{n(p) - 1}{n(p)} \tag{1}$$

Observing that, by the Burnside's criterion for p-nilpotency, $n(p) = 1$ if and only if $p \in \pi(G) \setminus \pi(G')$, it follows that the inequality $|\pi(G')| <$

$< 2\sigma^*(G)$ holds for any finite group G. More precisely, the following is true:

PROPOSITION 2. *Let G be a finite group and let p be the smallest prime divisor of $|G|$. Then*

$$|\pi(G')| < \frac{p}{p-1}\sigma^*(G).$$

For soluble groups, we can say slightly more.

PROPOSITION 3. *Let G be a finite soluble group and let p be the smallest prime divisor of $|G:G'Z(G)|$. Then*

$$|\pi(G')| < \frac{p}{p-1}\sigma^*(G).$$

That this inequality is best possible it is shown by Example 1 below. For the proof of it we need the following elementary observation.

LEMMA 1. *Let G be a finite soluble group, P a Sylow p-subgroup of G, and suppose that $G = G'C_G(P)$. Then G is p-nilpotent and P is abelian (in particular $n(p) = 1$).*

PROOF. By induction on $|G|$. Let N be a normal p'-subgroup of G. Write $\overline{G} = G/N$ and adopt the bar convention. Then, clearly, $\overline{G} = \overline{G}'C_{\overline{G}}(\overline{P})$. If $N \neq 1$, by the inductive hypothesis, we have that $\overline{G}$ is p-nilpotent and $\overline{P}$ is abelian; as $P \cong \overline{P}$ we get the conclusion. Otherwise $O_{p'}(G) = 1$, whence $O_p(G) = F(G)$ and so

$$C_G(P) \leqslant C_G(O_p(g)) = C_G(F(G)) = Z(O_p(G)).$$

Thus $G = G'Z(O_p(G))$. As G is soluble, this yields $G = Z(O_p(G))$. ■

PROOF OF PROPOSITION 3. By formula (1), it is enough to show that $n(q) \geqslant p$ for every $q \in \pi(G')$. Thus, let $q \in \pi(G)$, let G_q be a Sylow q-subgroup of G, and let $N = N_G(G_q)$ and $C = C_G(G_q)$. Now, the number $v = |NG'Z(G) : CG'Z(G)| = |N : C(N \cap G'Z(G))|$ is a divisor of $n(q) = |N:C|$. By our choice of p, we have either $n(q) \geqslant v \geqslant p$ or $v = 1$. Also, clearly $CG'Z(G) = CG'$ and, by the Frattinit argument, $NG'Z(G) = G$. However, if $v = 1$, we have $G = CG'$ and hence $n(q) = 1$ by Lemma 1; therefore $q \notin \pi(G')$ in this case. ■

We define the *class covering number* $\operatorname{cln}(G)$ of a finite group G as the smallest integer k for which there exist k elements $g_1, g_2, \dots, g_k$ of

G such that $\sigma_G(g_1) \cup \ldots \cup \sigma_G(g_k) = \varrho^*(G)$. Similarly, the *character covering number* $\mathrm{ccn}(G)$ is the smallest integer k for which there exist k irreducible characters $\chi_1, \chi_2, \ldots, \chi_k$ of G such that $\pi(\chi_1(1)) \cup \ldots \cup \pi(\chi_k(1)) = \varrho(G)$. Alvis and Barry [1,2] prove that, for any finite simple group G, $\mathrm{cln}(G) \leq 2$ and $\mathrm{ccn}(G) \leq 3$. We will observe in Example 1 that the class covering number of a metabelian group can be arbitrarily high. It is interesting to note that, in this respect, the parallelism between conjugacy classes and characters fails. The proof of the results in Section 1 of Manz and Wolf [14] can be easily modified to give the following statement:

Let G be a soluble group, then there exist irreducible characters χ_1, χ_2, χ_3 of G such that $\pi(\chi_1(1)) \cup \pi(\chi_2(1)) \cup \pi(\chi_3(1)) \supseteq \varrho(G)\backslash\{2, 3\}$.

Thus $\mathrm{ccn}(G) \leq 5$ for any soluble group G.

EXAMPLE 1. Let A be an elementary abelian p-group of rank $n \geq 1$, and $\mathfrak{M}$ be the set of all maximal subgroups of A. Thus $|\mathfrak{M}| = (p^n - 1)/(p - 1) =: k$. Let $\{q_M;\, M \in \mathfrak{M}\}$ be a set of k distinct primes such that $p \mid q_M - 1$ for all $M \in \mathfrak{M}$. For each $M \in \mathfrak{M}$ let C_M be a cyclic group of order q_M and let A act on C_M with kernel M. Set $N = \operatorname{Dir}_{M \in \mathfrak{M}} C_M$ with the action of A induced by the action on each C_M. Then each non identity element of A centralizes exactly $(p^{n-1} - 1)/(p - 1)$ primary components of N, as any such element is contained in that number of maximal subgroups of A. Let $G = G_n$ be the semidirect product $N \rtimes A$, and let g be an element of it. Then we can write $g = xa$, with $x \in N$, $a \in A$ and $[x, a] = 1$. Now, $\sigma_G(g) = \sigma_G(x) \cup \sigma_G(a)$; clearly $\sigma_G(x) = \{p\}$ and, by what we have observed, $|\sigma_G(a)| \leq (p^n - 1)/(p - 1) - (p^{n-1} - 1)/(p - 1) = p^{n-1}$. Thus $\sigma^*(G) = p^{n-1} + 1$. Now $N = G'$, whence $|\pi(G')| = (p^n - 1)/(p - 1)$ and the factor $|\pi(G')|/\sigma^*(G)$ tends to $p/(p - 1)$ as n goes to infinity. In particular, by taking $p = 2$, we get a family of metabelian groups G_n such that $\lim_{n \to \infty} |\pi(G_n')|/(\sigma^*(G_n)) = 2$.

These same classes of examples show that the class covering number of a metabelian group cannot be bounded by any positive integer. Having fixed a prime number p, let G_n be the group constructed above. We claim that $\mathrm{cln}(G_n) = n$.

Let $g_1, g_2, \ldots, g_t \in G_n$. Then $\sigma_G(g_1) \cup \ldots \cup \sigma_G(g_t) \subseteq \{p\} \cup \sigma_G(a_1) \cup \ldots \cup \sigma_G(a_t)$, where, for each $i = 1, 2, \ldots, t$, a_i is the p-component of g_i. By replacing g_i with a conjugate if necessary, we may suppose without loss of generality that all a_i's belong to A. Now, the subgroup of A generated by $\{a_1, \ldots, a_t\}$ is elementary abelian of rank at most t, and is therefore contained in at least $\nu = (p^{n-t} - 1)/(p-1)$ maximal subgroups, of A. Hence $\langle a_1, \ldots, a_t \rangle$, and so $\langle g_1, \ldots, g_t \rangle$, centralizes at least ν primary components of G_n. It follows that $\sigma_G(g_1) \cup \ldots \cup \sigma_G(g_t) = \pi(G_n) = \varrho^*(G_n)$ implies $t \geq n$. ■

THEOREM 1. *Let G be a finite metanilpotent group, and let p be the smallest prime divisor of $|G : G' Z(G)|$. Then*

$$|\varrho^*(G)| < \frac{2p-1}{p-1} \sigma^*(G).$$

PROOF. We first observe that for any finite group G, $\varrho^*(G) = \pi(G/F) \cup \pi(G')$, where F is the Fitting subgroup of G. In fact, let $p \notin \pi(G/F) \cup \pi(G')$, and let P be a Sylow p-subgroup of G. Then $P \leq F$ and so P is normal in G. Also, $P \cap G' = 1$, whence $[G, P] \leq P \cap G' = 1$; thus $P \leq Z(G)$ and $p \notin \varrho^*(G)$. The converse is clear.

Let now G be a metanilpotent group; then $(p/(p-1))\sigma^*(G) > |\pi(G')|$ by Proposition 3, and $\sigma^*(G) \geq |\pi(G/F)|$ by Proposition 1. Thus

$$\frac{2p-1}{p-1} \sigma^*(G) > |\pi(G/F)| + |\pi(G')| \geq$$

$$\geq |\pi(G/F) \cup \pi(G')| = |\varrho^*(G)|. \quad ■$$

That the inequality in Theorem 1 is best possible it is shown by Example 2; in particular, by letting $p = 2$, we construct an infinite family of supersoluble metabelian groups G_n such that

$$\lim_{n \to \infty} \frac{|\varrho^*(G_n)|}{(\sigma^*(G_n))} = 3.$$

EXAMPLE 2. Let p be a fixed prime number. For any pair (n, k) of positive integers set

$$v_{n,k} = \frac{p^{nk} - p^{(n-1)k}}{p^{n-1}(p-1)}.$$

We fix a positive integer n, and write $\beta = (v_{n,n} - v_{n,n-1})/p^{n-1}$. For simplicity, we take $n \geq 3$, in which case β is an integer.

Let $\alpha = (p^n - 1)/(p-1)$, and choose a family $(p_{ij};\, i = 1, \ldots, \alpha, j = 1, \ldots, \beta)$ of $\alpha\beta$ distinct primes such that p divides $p_{ij} - 1$ for all i, j. For each $i = 1, \ldots, \alpha$, let R_i be a cyclic group of order $p_{i1} p_{i2} \ldots p_{i\beta}$.

Let A be an abelian group which is the direct product of n cyclic groups of order p^n. Then A has exactly α maximal subgroups $M_1, \dots, M_\alpha$. For each $i = 1, \dots, \alpha$ there is a fixed-point-free action of A/M_i on R_i. These actions determine an action of A on the direct product $R = R_1 \times \dots \times R_\alpha$, with kernel $\Phi(A) = A^p$. In the semidirect product RA, we have (in a way similar to that in Example 1), for any $a \in A$:

$$(2) \qquad |\pi([R, a])| = \begin{cases} 0 & \text{if } |a| \leq p^{n-1}, \\ \beta p^{n-1} & \text{if } |a| = p^n. \end{cases}$$

Write now $\gamma = v_{n,n}$, and let $q, q_1, \dots, q_\gamma$ be distinct primes (and all distinct from p and the p_{ij}), chosen in such a way (use Dirichlet's Theorem) that $q_1 q_2 \dots q_\gamma$ and p^n divide $q - 1$. Let C be a cyclic group of order $q_1 q_2 \dots q_\gamma$ and $N = N_1 \times \dots \times N_\gamma$ an elementary abelian group of order q^γ. It is not hard to check that $\gamma = v_{n,n}$ is exactly the number of subgroups K of A such that A/K is cyclic of order p^n; we list as $A_1, \dots, A_\gamma$ all such subgroups.

We define an action of $A \times C$ on $R \times N$ in the following way. We let C centralize R and induce a non trivial automorphism of order q_k on each N_k, $k = 1, \dots, \gamma$, and let A act on R as described above, and on N by stabilizing each N_k and inducing on N_k a group of automorphisms of order p^n with kernel A_k. Now, if y is an element of order p^n in A then $A/\langle y \rangle$ is the direct product of $n - 1$ cyclic groups of order p^n. Thus y is contained in exactly $v_{n.n-1}$ subgroups of A whose corresponding factor group is cyclic of order p^n. Each of such subgroups is one of the A_i's, i.e. the centralizer in A of one of the components N_k of N. Hence, we have that the number of components N_k contained in $C_N(y)$ is precisely $v_{n,n-1}$; also, clearly, $C_N(y)$ is the direct product of such components.

Let $G = G_n$ be the semidirect product $(R \times N) \rtimes (A \times C)$.

Then $|\varrho^*(G)| = |\pi(R)|\,|\pi(C)| + |\{p, q\}| = \alpha\beta + \gamma + 2$.

We compute $\sigma^*(G)$. Let $g \in G$; by replacing g with a conjugate if necessary, we can write $g = xa$, where $x \in (R \times N)C$, $a \in A$ and $[x, a] = 1$. If $\sigma_R(a) = \sigma_G(a) \cap \pi(R)$ then, by formula (2),

$$|\sigma_R(a)| = \begin{cases} 0 & \text{if } |a| \leq p^{n-1}, \\ \beta p^{n-1} & \text{if } |a| = p^n. \end{cases}$$

Now, by construction of G, x centralizes R, so $\sigma_G(x) \subseteq \pi(C) \cup \{p, q\}$. In fact, if $h \in N$ is the q-component of x, $\sigma_G(x) \subseteq \sigma_C(h) \cup \{p, q\}$, where $\sigma_C(x) \subseteq \sigma_G(h) \cap \pi(C)$. Also, $h \in C_N(a)$, so, if $|a| = p^n$ then h is contained in the product of at most $v_{n,n-1}$ of the N_k's. It follows that, $|\sigma_C(h)| \leq v_{n,n-1}$, if $|a| = p^n$.

In any case, we have $\sigma_G(g) \subseteq \sigma_G(x) \cup \sigma_G(a) \subseteq \sigma_C(h) \cup \sigma_R(a) \cup \{p,q\}$. Hence,

i) if $|a| = p^n$, then $|\sigma_G(g)| \leq |\sigma_C(h)| + |\sigma_R(a)| + 2 \leq \leq v_{n,n-1} + \beta p^{n-1} + 2 = v_{n,n} + 2$.

ii) if $|a| \leq p^{n-1}$, then $|\sigma_G(g)| \leq |\sigma_C(h)| + |\sigma_R(a)| + 2 \leq \leq v_{n,n} + 2$.

Thus $\sigma^*(G) \leq v_{n,n} + 2$; in fact it is easy to see that $\sigma^*(G) = v_{n,n} + + 2 = \gamma + 2$. Hence:

$$\frac{|\varrho^*(G)|}{\sigma^*(G)} = \frac{\alpha\beta + \gamma + 2}{\gamma + 2} = \frac{p^n - 1}{(p-1)p^{n-1}} \left(\frac{v_{n,n} - v_{n,n-1}}{v_{n,n} + 2} \right) + \frac{v_{n,n} + 2}{v_{n,n} + 2}.$$

Finally, we have:

$$\lim_{n \to \infty} \frac{|\varrho^*(G_n)|}{\sigma^*(G_n)} = \frac{p}{p-1} \left(1 - \lim_{n \to \infty} \frac{v_{n,n-1}}{v_{n,n} + 2} \right) + $$
$$+ 1 = \frac{p}{p-1} + 1 = \frac{2p-1}{p-1},$$

as

$$\lim_{n \to \infty} \frac{v_{n,n-1}}{v_{n,n}} = \lim_{n \to \infty} \frac{p^{n(n-1)} - p^{(n-1)(n-1)}}{p^{nn} - p^{(n-1)n}} = \lim_{n \to \infty} \frac{p^{n-1} - 1}{p^{n-1}(p^n - 1)} = 0.$$

Appendix.

We add some remarks on the general soluble case, refering to a recent paper [14] of Manz and Wolf. Firstly, we observe that the arguments in their Section 1, involving irreducible characters, can be easily rephrased in terms of conjugacy classes. In particular, modifications in Lemma 1.3 of [14] lead to the following parallel statement.

LEMMA 2. *Suppose that $M = C_G(M)$ is a normal elementary abelian subgroup of a soluble group G and a completely reducible G-module (possibly of mixed characteristic). Assume that G splits over M with complement H. Then there exist $x \in M$, $y \in H$ such that*

$$\sigma_G(x) \cup \sigma_G(y) \supseteq \pi(G/M) \setminus \{2, 3\}.$$

From this it follows, as in [14; Theorem 1.4] that, given a soluble group G, then $|\pi(G/F(G))| \leq 2\sigma^*(G) + 2$ (see Theorem 3 below for an easy improvement). This fact can be used to give an alternative proof of a (slightly improved) result of P. Ferguson.

THEOREM 2. *Let G be a soluble group. Then $|\varrho^*(G)| < 4\sigma^*(G) + 2$. More precisely, let p be the smallest prime divisor of $|G:G'Z(G)|$. Then*

$$|\varrho^*(G)| < \frac{3p-2}{p-1}\sigma^*(G) + 2\,.$$

PROOF. Simply apply to the above observation Proposition 3, as in the proof of Theorem 1.

Indeed, the constant term $+2$ in Manz and Wolf's inequality is attached to the occurrence of the primes $\{2, 3\}$ among the divisors of $|G/F(G)|$; observing that if the prime 2 or 3 occurs as divisor of $|G'|$ then it is controlled by Proposition 3, we can remove the additive constant 2 in Theorem 2 if $p \geqslant 5$. In particular, if G is a soluble group such that all prime divisors of $|G/G'|$ are greater or equal to 5, then $|\varrho^*(G)| < 3.25\sigma^*(G)$ (another slight improvement of a result of P. Ferguson's).

Also, the arguments of Manz and Wolf can be exploited to get a result for arbitrary soluble groups, comparable to Proposition 1, namely

THEOREM 3. *Let G be a soluble group, F the Fitting subgroup of G. Then there exist $x, y \in G$ such that $\sigma_G(x) \cup \sigma_G(y) \supseteq \varrho(G)\setminus\{2, 3\}$.*

PROOF. Let γ be the set of all primes p such that G has a normal non-abelian Sylow p-subgroup. By Ito's Theorem $\varrho(G) = \gamma \cup \pi(G/F)$. Let $N = \Phi(G)$, $\overline{G} = G/N$ and adopt the bar convention. Thus $\overline{F} = C_{\overline{G}}(\overline{F})$ and, by a Theorem of Gaschültz, $\overline{F}$ is a completely reducible $\overline{G}$-module and it is complemented in $\overline{G}$. If $\overline{H}$ is a complement of $\overline{F}$ in $\overline{G}$, we can choose H to be a γ'-subgroup of G. By Lemma 2, there exist $x \in F$ and $y \in H$ such that $\sigma_{\overline{G}}(\overline{x}) \cup \sigma_{\overline{G}}(\overline{y}) \supseteq \pi(G/F)\setminus\{2, 3\}$. Clearly then, $\sigma_G(x) \cup \sigma_G(y) \supseteq \pi(G/F)\setminus\{2, 3\}$. Let β be the set of all $p \in \gamma$ such that y centralizes the normal Sylow p-subgroup of G; that is $\beta = \gamma\setminus\sigma_G(y)$. For each $q \in \beta$, we may take a non-central element g_q of $O_q(G)$ and set $g = \prod_{q\in\beta} g_q$. Then $\sigma_G(g) \supseteq \beta$. Also, g and y are two commuting elements of G of coprime order so: $\sigma_G(gy) \supseteq \sigma_G(g) \cup \sigma_G(y) \supseteq \beta \cup \sigma_G(y) = \gamma \cup \sigma_G(y)$.

Finally, $\sigma_G(x) \cup \sigma_G(gy) \supseteq (\gamma \cup \pi(G/F))\setminus\{2, 3\} = \varrho(G)\setminus\{2, 3\}$.

COROLLARY *Let G be a soluble group. Then $|\varrho(G)| \leqslant 2\sigma^*(G) + 2$.*

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