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A. Ballester-Bolinches

M. C. Pedraza-Aguilera

M. D. Pérez-Ramos

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On Π-Normally Embedded Subgroups of Finite Soluble Groups.

A. BALLESTER-BOLINCHES (*) - M. C. PEDRAZA-AGUILERA (*)
M. D. PÉREZ-RAMOS (*)

1. - Introduction and statement of results.

All groups considered in this paper are finite and soluble.

Let π be a set of primes. A subgroup H of a group G is said to be π-normally embedded in G if a Hall π-subgroup of H is also a Hall π-subgroup of some normal subgroup of G. A Hall π-subgroup of a normal subgroup of G is an obvious example of a π-normally embedded subgroup of G. This embedding property was studied in [1] and [2]. The present paper represents an attempt to carry this study further. In fact we analyze what properties related to p-normal embedding property (p a prime number) can be extended to a set of primes π.

Our first Theorem concerns $\mathfrak{F}$-normalizers associated to a saturated formation $\mathfrak{F}$. Chambers [3] proved that in a group G with abelian Sylow p-subgroups, the $\mathfrak{F}$-normalizers of G are p-normally embedded in G, where $\mathfrak{F}$ is saturated formation. We prove:

THEOREM 1. *Let $\mathfrak{F}$ be a saturated formation. If G is a group with abelian Hall π-subgroups, then the $\mathfrak{F}$-normalizers of G are π-normally embedded in G.*

Let π be a set of primes. A subgroup U of a group G is said to be π-pronormal in G if U and U^g are conjugate in $O^\pi(\langle U, U^g\rangle)$ for all $g \in G$. Note that the π-pronormality is just the $\mathfrak{F}$-pronormality introduced in [6] when $\mathfrak{F}$ is the saturated formation of all soluble

(*) Indirizzo degli AA.: Departament d'Algebra, Universitat de València, C/Doctor Moliner 50, 46100 Burjassot (València), Spain.

π-groups. This embedding property is closely related to π-normal embedding one as it is shown in the following Theorem.

THEOREM 2. *For a π-subgroup U of a group G, the following statements are equivalent:*

(i) *U is π-normally embedded in G.*

(ii) *U is π-pronormal in G and a CAP subgroup of G.*

(iii) *U permutes with every Hall π'-subgroup of G and is normalized by each Hall π-subgroup of G containing it.*

Wood [5] proves the following result:

THEOREM (Wood). Let p be a prime and let G be a group. The following statements are pairwise equivalent:

(a) All the maximal subgroups of G are p-normally embedded in G.

(b) Every Sylow p-subgroup of every maximal subgroup of G is pronormal in G.

(c) G has p-length at most one.

The obvious extension of Wood's Theorem to a set of primes π does not hold. Let H be the symmetric group of degree 3. It is known that H has an irreducible and faithful module over GF(5), the finite field of 5 elements, V say. Let $\pi = \{2, 5\}$ and $G = [V]H$. Then the Hall π-subgroups of the maximal subgroups of G are pronormal in G. However the π-length of G is bigger than 1.

THEOREM 3. *Let π be a set of primes. Let G be a group. The following statements are equivalent:*

(i) *All maximal subgroups of G are π-normally embedded in G.*

(ii) *Every Hall π-subgroup of a maximal subgroup of G is π-pronormal in G.*

If either (i) *or* (ii) *hold, then G has π-length at most one.*

The symmetric group of degree 3 has π-length one for $\pi = \{2, 3\}$. However the maximal subgroup $\langle (12) \rangle$ of G is not π-normally embedded in G. So the assertion in Theorem 3: G has π-length at most one, does not imply (i) and (ii) in general.

2. – Preliminaries.

In this section we collect some results which are needed in proving our Theorems. All the known results concerning finite soluble groups which we will need appear in [4]. This book is the main reference for the notation and terminology.

From now on π will be a set of primes.

LEMMA 1 [2]. *Let U be a π-normally embedded subgroup of a group G, K a normal subgroup of G, and L a subgroup of G. Then:*

(i) *If U is a subgroup of L, then U is π-normally embedded in L.*

(ii) *UK is π-normally embedded in G and UK/K is π-normally embedded in G/K.*

(iii) *If K is a subgroup of L and L/K is π-normally embedded in G/K, then L is π-normally embedded in G.*

LEMMA 2. *Assume that τ is a set of primes with $\pi \cap \tau = \emptyset$. Let P be a π-subgroup of a group G and let Q be a τ-subgroup of G. Suppose that P is π-normally embedded in G and Q is τ-normally embedded in G. If $\langle P, Q\rangle$ is a $(\pi \cup \tau)$-group, then $PQ = QP$.*

PROOF. By virtue of Lemma 1, we have that P is a π-normally embedded subgroup in $T = \langle P, Q\rangle = P[P, Q]Q$. So P is a Hall π-subgroup of its normal closure $\langle P^T\rangle = P[P, Q]$. Since Q is a τ-group and $\tau \cap \pi = \emptyset$, it follows that P is a Hall π-subgroup of T. Analogously we have that Q is a Hall τ-subgroup of T. This implies that $T = PQ = QP$ because T is a $(\pi \cup \tau)$-group.

LEMMA 3. *If a group G has an abelian Hall π-subgroup, then G has π-length at most one.*

PROOF. Consider the upper $\pi'\pi$-series of G:

$$1 \trianglelefteq P_0 \trianglelefteq N_0 \trianglelefteq P_1 \trianglelefteq \ldots \trianglelefteq G$$

where $N_0 = O_{\pi'}(G)$, $P_1 = O_{\pi'\pi}(G)$ and $N_1/P_1 = O_{\pi'}(G/P_1)$. Let H be a Hall π-subgroup of G. By ([4] [I; (3.2)]), we know that HN_0/N_0 is a Hall π-subgroup of G/N_0. Since $P_1/N_0 = O_\pi(G/N_0)$ is a normal subgroup of G/N_0, it follows that $P_1/N_0 \leq HN_0/N_0$. Now HN_0/N_0 is abelian. This implies that $HN_0/N_0 \leq C_{G/N_0}(P_1/N_0) \leq P_1/N_0$ by virtue of [7]. In particular $H \leq P_1$ and G/P_1 is then a π'-group. This means that $N_1 = G$ and G has π-length at most one.

3. – Proofs of the Theorems.

PROOF OF THEOREM 1. We argue by induction on $|G|$. Let D be an $\mathcal{F}$-normalizer of G associated to the Hall system Σ of G. It is clear that we can assume that $D < G$. Let N be a minimal normal subgroup of G. Then DN/N is an $\mathcal{F}$-normalizer of G associated to the Hall system $\Sigma N/N$ of G/N ([4] [V; (3.2)]). Since G/N has abelian Hall π-subgroups, it follows that DN/N is π-normally embedded in G/N and so DN is π-normally embedded in G. Assume that N_1 and N_2 are two distinct minimal normal subgroups of G. Then DN_i is π-normally embedded in G for $i \in \{1, 2\}$. Since Σ reduces in DN_1 and DN_2, we have that $DN_1 \cap DN_2$ is π-normally embedded in G by ([2] [Th. 1]). Now, by ([4] [V; (3.2)]), D either covers or avoids N_i. If $N_i \leq D$ for some i, it follows that D is π-normally embedded in G and we are done. So D avoids N_1 and N_2. This implies that $D = DN_1 \cap DN_2$ is π-normally embedded in G and we are done. Consequently G has a unique minimal normal subgroup, N say. Then $F(G)$ is a p-group for some prime p. Since DN is π-normally embedded in G, we have that $p \in \pi$. In particular, $O_{\pi'}(G) = 1$ and G has a normal Hall π-subgroup H because G has π-length at most one by Lemma 3. Since H is abelian and $F(G) \leq H$, it follows that $H \leq C_G(F(G)) \leq$ $\leq F(G)$. This means that $F(G)$ is an abelian Hall π-subgroup of G. By ([4] [V; (3.6) and (3.7)]), there exists a maximal subgroup M of G such that $G = F(G)M$ and D is an $\mathcal{F}$-normalizer of M. By induction, D is π-normally embedded in M. Let A be a Hall π-subgroup of D. Then A is a Hall π-subgroup of $\langle A^M \rangle$. Since $D \leq F(G)$ and $F(G)$ is abelian, we have that $\langle A^M \rangle = \langle A^G \rangle$ and so $\langle A^M \rangle$ is a normal subgroup of G. Therefore D is π-normally embedded in G and the Theorem is proved.

PROOF OF THEOREM 2. (i) implies (ii). Since U is a Hall subgroup of a normal subgroup of G, it follows that U is pronormal in G. Then, there exists $x \in J = \langle U, U^g \rangle$ with $U^g = U^x$. By Lemma 1, U is π-normally embedded in J and $UO^{\pi}(J)/O^{\pi}(J)$ is π-normally embedded in $J/O^{\pi}(J)$, which is a π-group. This implies that $J = UO^{\pi}(J)$. In particular $x = uz$, with $u \in U$ and $z \in O^{\pi}(J)$. So $U^g = U^x = U^z$ and U is π-pronormal in G.

Let H/K be a chief factor of G. If H/K is a π'-group, then U avoids H/K. Assume that H/K is a π-group. Let Σ be a Hall system of G reducing into U. Then Σ reduces into UK. By Lemma 1, UK is π-normally embedded in G. Since Σ also reduces into H, we have that $UK \cap H$ is π-normally embedded in G by ([2] [Th. 1]). In particular, $UK \cap H$ is also subnormal in G, we have that $UK \cap H \trianglelefteq G$, and U either covers or avoids H/K. Therefore U is a CAP subgroup of G.

(ii) implies (i). Assume that U is π-pronormal in G and a CAP subgroup of G. We prove that U is π-normally embedded in G by induction on $|G|$. Let N be a minimal normal subgroup of G. Since UN/N is π-pronormal in G/N and UN/N is a CAP subgroup of G, we have that UN is a π-normally embedded subgroup of G by induction. If N is a π'-group, then U is π-normally embedded in G and we are done. Hence we may assume $O_{\pi'}(G) = 1$. Now U is π-pronormal in UN and UN is π-group. This means that U is a normal subgroup of UN. Moreover, U either covers or avoids N. If $N \leqslant U$, then U is π-normally embedded in G and we are done. Thus $U \cap N = 1$ and $U \leqslant C_G(N)$.

Therefore we may assume that U centralizes every minimal normal subgroup of G. With the same arguments to those used in ([4] [I; (7.12)]), we conclude that U is normal in G and so U is π-normally embedded in G.

(i) implies (iii). Assume that U is π-normally embedded in G and let $G_{\pi'}$ be a Hall π'-subgroup of G. By Lemma 2, U permutes with $G_{\pi'}$, because $G_{\pi'}$ is π'-normally embedded in G.

Let G_π be a Hall π-subgroup of G such that $U \leqslant G_\pi$. Since U is π-normally embedded in G, we have that U is π-normally embedded in G_π. This means that U is normal in G_π. Hence G_π normalizes U.

· (iii) implies (i). Suppose that $G_{\pi'}$ is a Hall π'-subgroup of G such that $G_{\pi'}$ permutes with U and let G_π be a Hall π-subgroup of G such that $U \leqslant G_\pi$. Then U is normalized by G_π. Moreover $G = G_\pi G_{\pi'}$ and hence $\langle U^G \rangle = \langle U^{G_{\pi'}} \rangle \leqslant UG_{\pi'}$. Since U is a Hall π-subgroup of $UG_{\pi'}$ and $U \leqslant \langle U^G \rangle$, it follows that U is a Hall π'subgroup of $\langle U^G \rangle$ and U is π-normally embedded in G.

PROOF OF THEOREM 3. Since the maximal subgroups of G are CAP subgroups of G, it follows that every Hall π-subgroup of every maximal subgroup is a CAP subgroup of G. So the equivalence between (i) and (ii) in Theorem 3 follows from the equivalence between (i) and (ii) in Theorem 2. Assume that there exists a group G such that G has the property (i) but G does not have π-length at most one. Let us consider G of minimal order. Since the property (i) is invariant under epimorphic images, we have that G/N has π-length at most one for every minimal normal subgroup N of G. Since the class of all groups with π-length at most one is a saturated formation, it follows that G has a unique minimal normal subgroup, N say, such that $G = MN$ and $M \cap N = 1$ for some maximal subgroup M of G. By hypothesis M is π-normally embedded in G. If N is a π-group, then M should be a π'-group and then G has π-length at most one, a contradiction. So N must be a π'-group. But then G also has π-length at most one, a contradiction.

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