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**Multiplicity results for a class of semilinear elliptic equations on $\mathbb{R}^m$**

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# Multiplicity Results for a Class of Semilinear Elliptic Equations on $\mathbb{R}^m$.

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ABSTRACT - We prove the existence of an uncountable set of bounded solutions for a class of semilinear elliptic equations on $\mathbb{R}^m$.

## 1. - Introduction.

In this work we are concerned with the study of existence and multiplicity of solutions to the problem

$$(P) \qquad -\Delta u + u = f(x, u), \qquad u \in H^1(\mathbb{R}^m)$$

where $m \geq 1$ and $f$ satisfies the assumptions

(f1) $f \in C(\mathbb{R}^m \times \mathbb{R}, \mathbb{R})$.

(f2) $f(x, 0) = f_z(x, 0)$ for any $x \in \mathbb{R}^m$.

(f3) $\exists b_1, b_2 > 0$ such that $|f(x, z)| \leq b_1 + b_2 |z|^s$, $\forall (x, z) \in \mathbb{R}^m \times \mathbb{R}$, where $s \in (1, 2^* - 1)$ with $2^* = 2m/(m-2)$ if $m > 2$ and $s$ is not restricted if $m = 1, 2$.

The hypotheses (f1)-(f3) are the ones studied in [10] assuming also that $f(x, z)$ is periodic in $x$ and superquadratic in $z$. Here we consider the following more general case.

We say that a set $A \subset \mathbb{R}^m$ is large at infinity if $\forall R > 0$ $\exists x \in A$ such that $B_R(x) = \{y \in \mathbb{R}^m / |y - x| < R\} \subset A$. Clearly any cone in $\mathbb{R}^m$ is large at infinity. Another example is a cone minus the union of the annuli centered in zero and with radii $(2n)^2$, $(2n+1)^2$.

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We assume that there exists a function $f_\infty : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ verifying (f1)-(f3), and a set $A \subset \mathbb{R}^m$ large at infinity such that

(f4) $\exists \beta > 2$ and $\alpha \in [0, \mu/2 - 1)$ such that $\beta F_\infty(x, z) = \beta \int_0^z f_\infty(x, t)\, dt \leq f_\infty(x, z) z + \alpha |z|^2$, $\forall (x, z) \in \mathbb{R}^m \times \mathbb{R}$, and $F_\infty(x_0, z_0) > \alpha/(\beta - 2) z_0^2$ for an $(x_0, z_0) \in \mathbb{R}^m \times \mathbb{R}$.

(f5) $f_\infty(x + p, z) = f_\infty(x, z)$ for any $p \in \mathbb{Z}^m$, $(x, z) \in \mathbb{R}^m \times \mathbb{R}$.

(f6) $\forall \varepsilon > 0\ \exists R > 0$ such that $\sup_{x \in A \setminus B_R(0)} |f(x, z) - f_\infty(x, z)| \leq \varepsilon(|z| + |z|^s)\ \forall z \in \mathbb{R}$.

Putting $F(x, z) = \int_0^z f(x, t)\, dt$ we define on $X = H^1(\mathbb{R}^m)$ the functionals $\varphi(u) = (1/2)\|u\|^2 - \int_{\mathbb{R}^m} F(x, u)\, dx$, $\varphi_\infty(u) = (1/2)\|u\|^2 - \int_{\mathbb{R}^m} F_\infty(x, u)\, dx$, where $\|u\|^2 = \int_{\mathbb{R}^m} |\nabla u|^2 + |u|^2 dx$, and we look for solutions of (P) as critical points of $\varphi$.

As we will see the assumptions (f1)-(f5) are sufficient to guarantee the existence of at least one non zero critical point of the «periodic» functional $\varphi_\infty$. By (f5), if $u$ is a critical point of $\varphi_\infty$, then also $p * u = u(\cdot - p)$ is a critical point of $\varphi_\infty$ for any $p \in \mathbb{Z}^m$. If we translate this $u$ in a region where $f$ and $f_\infty$ are very close one to the other, one could expect that nearby this translate of $u$ there is a critical point of $\varphi$. In general this is not true as the following example shows.

Let $f(x, z) = \alpha(x_1)|z|^2 z$ with $\alpha \in C^1(\mathbb{R}, \mathbb{R})$, $\alpha(t) \geq \alpha_0 > 0$, $\dot\alpha(t) > 0$ $\forall t \in \mathbb{R}$, and assume that $u$ is a solution of (P). By standard bootstrap argument we get that $u \in H^2(\mathbb{R}^m)$, therefore $\varphi'(u)\partial_1 u = 0$. But, if $e_1 = (1, 0, \ldots, 0)$, we have

$$\varphi'(u)\partial_1 u = \frac{d}{ds}\varphi(u(\cdot + se_1))|_{s=0} = \int \dot\alpha(x_1)\frac{|u|^4}{4}\, dx = 0$$

which implies $u = 0$ (see [12]).

To avoid this situations we make a discreteness assumption on the set of the critical points of the functional at infinity.

We note first of all that $\varphi_\infty$ satisfies the geometrical hypotheses of the mountain pass theorem. Letting $\Gamma = \{\gamma \in C([0, 1], X);\ \gamma(0) = 0,\ \varphi_\infty(\gamma(1)) < 0\}$, we put $c = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} \varphi_\infty(\gamma(t))$.

Setting $K_\infty = \{u \in X\backslash\{0\}/\varphi'_\infty(u) = 0\}$, we assume

$$(*) \qquad \exists c^* > c \quad \text{such that } K_\infty^{c^*} = K_\infty \cap \{\varphi_\infty < c^*\} \quad \text{is denumerable.}$$

We note that the hypothesis $(*)$ excludes the asymptotically autonomous cases, because if $f_\infty$ does not depends on $x$ and $u \in K_\infty^{c^*}$ then $p * u \in \in K_\infty^{c^*}$ for any $p \in \mathbb{R}^m$.

In this setting we are able to prove the following

THEOREM 1.1. *If* (f1)-(f6) *and* $(*)$ *hold then* (P) *admits infinitely many distinct solutions.*

*Precisely there exists* $u \in X$, *solution to the equation* $-\Delta u + u = = f_\infty(x, u)$ *for which we have that* $\forall r > 0$ *there exist* $M = M(r) \in \mathbb{N}$ *and* $R = R(r) > 0$ *such that for any finite sequence* $\{p_1, \ldots, p_k\} \subset \mathbb{Z}^m$ *that verifies*

i) $|p_1| \geq R$ and $|p_i| \geq |p_{i-1}| + 2M \quad i = 2, \ldots, k$,

ii) $B_M(p_i) \subset A\backslash B_R(0) \quad i = 1, \ldots, k$,

*there exists a solution* $v$ *to* (P) *such that if we put* $|p_{k+1}| = +\infty$ *then*

$$\|v - p_1 * u\|_{B_{(1/2)(|p_1| + |p_2|)}(0)} \leq r,$$

$$\|v - p_i * u\|_{B_{(1/2)(|p_i| + |p_{i+1}|)}(0)\backslash B_{(1/2)(|p_i| + |p_{i-1}|)}(0)} \leq r \qquad i = 2, \ldots, k,$$

*where if* $A \subset \mathbb{R}^m$ *is measurable then* $\|u\|_A^2 = \int_A |\nabla u|^2 + |u|^2 dx$.

In particular, for $k = 1$ we get that if $p \in \mathbb{Z}^m$ verifies $B_M(p) \subset \subset A\backslash B_R(0)$ then there is a solution $v$ to (P) which is near $u(\cdot - p)$. Moreover for $k > 1$ if we choose any set of $k$ disjoint annuli centered in zero, each of which intersects the set $A\backslash B_R(0)$ in a ball of radius $M$ centered in a point of $\mathbb{Z}^m$, then there is a solution to (P) which is near a translate of $u$ in each of this balls. We call this type of solution $k$-bump solution.

The first proof via variational methods of the existence of 2-bump solutions was given by E. Séré in [22] solving the homoclinic existence problem for first order periodic and convex Hamiltonian systems. This paper inspired the work of V. Coti Zelati and P. H. Rabinowitz [9] on second order periodic Hamiltonian systems where was proved the existence of infinitely many k-bump homoclinic solutions for any $k \in \mathbb{N}$. In [10] they adapted these techniques to find the existence of infinitely many $k$-bump solutions for any $k \in \mathbb{N}$ for the problem (P) in the case in which $f(x, z)$ is periodic in $x$ and superquadratic in $z$. S. Alama and Y.

Y. Lee in [3] studied the problem (P) assuming $f$ asymptotic as $|x| \to \infty$ to a function $f_\infty$ of the type considered in [10]. In that paper they were able to prove that the problem (P) admits infinitely many $k$-bump solutions. All this results are based on assuming that there exists $c^* > c$ such that $K_\infty^{c^*}/\mathbb{Z}^m$ is finite (clearly, in the periodic case, the functional $\varphi_\infty$ is $\varphi$ itself).

The existence of $k$-bump homoclinic solutions with minimum distance between the bumps independent from $k$ was first proved by E. Séré in [23] for first order convex and periodic Hamiltonian systems. As a consequence, considering the $C^1_{\text{loc}}$-closure of the set of the multibump homoclinic orbits, using the Ascoli Arzelà Theorem, he finds solutions with possibly infinitely many bumps.

Also in Theorem 1.1 the minimum distance between the bumps of any $k$-bump solution depends only on $r$ (being given by $M(r)$). Using the Ascoli Arzelà theorem it is therefore possible to prove as in [23] the existence of a class of bounded solutions of the equation $-\Delta u + u = f(x, u)$. Precisely we have:

THEOREM 1.2. *Under the same assumptions of Theorem* 1.1, *it holds that for any* $r > 0$ *there exist* $M = M(r) \in \mathbb{N}$ *and* $R = R(r) > 0$ *such that for any sequence* $\{p_j\}_{j \in \mathbb{N}} \subset \mathbb{Z}^m$ *that verifies*

$$\text{i)}\quad |p_1| \geq R \text{ and } |p_i| \geq |p_{i-1}| + 2M \qquad i \geq 2,$$

$$\text{ii)}\quad B_M(p_i) \subset A \setminus B_R(0) \qquad i \in \mathbb{N},$$

*and for every sequence* $\sigma = (\sigma_j)_{j \in \mathbb{N}} \in \{0, 1\}^{\mathbb{N}}$ *there exists* $v_\sigma \in H^1_{\text{loc}}(\mathbb{R}^m)$ *satisfying* $-\Delta v_\sigma + v_\sigma = f(x, v_\sigma)$ *such that*

$$\|v_\sigma - \sigma_1(p_1 * u)\|_{B_{(1/2)(|p_1| + |p_2|)}(0)} \leq r,$$

$$\|v_\sigma - \sigma_i(p_i * u)\|_{B_{(1/2(|p_i| + |p_{i+1}|)}(0) \setminus B_{(1/2)(|p_i| + |p_{i-1}|)}(0)} \leq r \qquad i \geq 2.$$

The tools used in the proof of Theorem 1.1 are related to the ones developed in [18] and then improved with P. Caldiroli in [7] and with S. Abenda and P. Caldiroli in [1] studying the homoclinic existence problem for second order Hamiltonian systems. These arguments, inspired by [23] and [9], permits us to strengthens the results contained in [3] in a more general setting. In fact the superquadratic assumption (f4) is verified also by functions $f_\infty$ which change sign. Moreover the assumption $(*)$, is satisfied if the functional $\varphi_\infty$ is for example a Morse functional. In the one dimensional case ($m = 1$), it is possible to verify this condition via the Melnikov theory when $f_\infty$ is a periodic perturbation of particular autonomous problems (see [1]).

Another differences with the work of S. Alama and Y. Y. Lee [3] is the fact that $f$ is not assumed to be asymptotic to $f_\infty$ as $|x| \to \infty$ but only on a set large at infinity. This permits us to consider the problem (P) when $f$ is assumed to be asymptotic in different sets large at infinity to different functions. Precisely we consider the hypothesis

(f7) $\exists A_1, \ldots, A_l \subset \mathbb{R}^m$, large at infinity, $f_1, \ldots, f_l$ satisfying (f1)-(f5) for which $\forall \varepsilon > 0 \ \exists R > 0$ such that $\sup_{x \in A_\iota \setminus B_R(0)} |f(x, z) - f_\iota(x, z)| \leq \varepsilon(|z| + |z|^s) \ \forall z \in \mathbb{R}, \forall \iota \in \{1, \ldots, l\}$.

If for any $\iota \in \{1, \ldots, l\}$, we define $\varphi_\iota(u) = (1/2)\|u\|^2 - \int_{\mathbb{R}^m} F_\iota(x, u(x))\, dx$, $\mathcal{K}_\iota = \{u \in X \setminus \{0\};\ \varphi_\iota'(u) = 0\}$, $c_\iota$ the mountain pass level of $\varphi_\iota$ and we assume

($*_\iota$) $\exists c_\iota^* > c_\iota$ such that $\mathcal{K}_\iota^{c_\iota^*} = \mathcal{K}_\iota \cap \{\varphi_\iota < c_\iota^*\}$ is denumerable.

then, by Theorem 1.2, we have $l$ different sets of multibump solutions, each constructed with a suitable critical point of the functional $\varphi_\iota$.

In fact, we prove that there are also multibump solutions of (P) of mixed type, as said in the following theorem.

THEOREM 1.3. *Assume that* (f1)-(f5), (f7) *and* ($*_\iota$) *hold. There exists* $u_1, \ldots, u_l \in X$, *satisfying* $-\Delta u_\iota + u_\iota = f_\iota(x, u_\iota)$ *for which we have that for any* $r > 0$ *there exist* $M = M(r) \in \mathbb{N}$ *and* $R = R(r) > 0$ *such that for any sequences* $\{p_i\}_{i \in \mathbb{N}} \subset \mathbb{Z}^m$, $\{j_i\}_{i \in \mathbb{N}} \subset \{1, \ldots, l\}^{\mathbb{N}}$, *that verify*

i) $|p_1| \geq R$ and $|p_i| \geq |p_{i-1}| + 2M \quad i \geq 2$,

ii) $B_M(p_i) \subset A_{j_i} \setminus B_R(0) \quad i \in \mathbb{N}$,

*and for every sequence* $\sigma = (\sigma_i)_{i \in \mathbb{N}} \in \{0, 1\}^{\mathbb{N}}$ *there exists* $v_\sigma \in H^1_{\mathrm{loc}}(\mathbb{R}^m)$ *satisfying* $-\Delta v_\sigma + v_\sigma = f(x, v_\sigma)$ *such that*

$$\|v_\sigma - \sigma_1(p_1 * u_{j_1})\|_{B_{(1/2(|p_1| + |p_2|)}(0)} \leq r,$$

$$\|v_\sigma - \sigma_i(p_i * u_{j_i})\|_{B_{(1/2)(|p_i| + |p_{i+1}|)}(0) \setminus B_{(1/2)(|p_i| + |p_{i-1}|)}(0)} \leq r \quad i \geq 2.$$

*If* $\sigma_i \neq 0$ *only for a finite number of indices then* $v_\sigma$ *is actually a solution to* (P).

As last remark we point out that an analogous result was proved by S. Angenent in [2] in a different setting ($z - f(x, z)$ is assumed to be periodic in $x$ and bounded together with its derivatives), using essentially fixed point arguments. He proved his result under the assumption that the solution $u$ was such that the operator $-\Delta + I - f_z(x, u(x))$ had a bounded inverse. He was able to verify this hypothesis for periodic perturbation of particular autonomous problem which admits a unique (up to translations) radial solution, using a bifurcation theorem due to A. Weinstein [24]. It is known that the problem (P) when $f(x, z) = z^p$ admits a unique positive solution (see [13]) and it should be interesting to check if the hypothesis $(*)$ holds for periodic perturbations of this $f$.

The work is divided into four parts. In the first section we give some preliminaries. The second is devoted to the study of some properties of the periodic problem which we use in section three to construct a particular pseudogradient field of $\varphi$. Finally in section four we prove the existence theorem.

## 2. – A local compactness property.

In this section we study some properties of the functional $\varphi$ which are independent on the assumptions on the asymptotic behaviour of $f$. All the results contained here are true under the hypotheses (f1)-(f3). In the proofs that follow we shall always consider the case $m \geqslant 3$, the proofs for $m = 1$ or 2 being not more difficult.

We have to note first of all that (f1)-(f3) imply

$$\forall \varepsilon > 0 \ \exists A_\varepsilon > 0 / |f(x, z) h| \leqslant \varepsilon |z| |h| + A_\varepsilon |z|^s |h| \quad \text{for all } (x, z) \in \mathbb{R}^m \times \mathbb{R} \tag{2.1}$$

and obviously an analogous estimate holds also for $F(x, z)$. This permits us to say that $\varphi$ is well defined on $X$ because of the Sobolev Immersion Theorem. Actually the following holds:

PROPOSITION 2.2. $\varphi \in C^1(X, \mathbb{R})$.

PROOF. We prove first that $\varphi$ is Gateaux differentiable. Given $h \in X$, by (2.1), we get that

$$\frac{1}{t} |F(x, u + th) - F(x, u)| = \frac{1}{t} \left| \int_0^t f(x, u + sh) h \, ds \right| \leqslant$$

$$\leqslant |h(x)|(|u(x)| + |h(x)|) + c_1 A_1 |h(x)|(|u(x)|^s + |h(x)|^s).$$

Being this last function in $L^1(\mathbb{R}^m)$ we can use the dominated convergence theorem to get that

$$\lim_{t \to 0} \frac{1}{t} |(u + th) - \varphi(u)| = \langle u, h \rangle - \int_{\mathbb{R}^m} f(x, u) h \, dx = \varphi'_G(u) h .$$

We prove now that $\varphi'_G$ is continuous. Let $u_n \to u$ and $\{u_{n_k}\} \subset \{u_n\}$. By the Sobolev Immersion theorem there exists $\{u_{n_{k_j}}\} \subset \{u_{n_k}\}$ and a function $v \in L^2(\mathbb{R}^m) \cap L^{s+1}(\mathbb{R}^m)$ such that $u_{n_{k_j}}(x) \to u(x)$ a.e. in $\mathbb{R}^m$ and $|u_{n_{k_j}}(x)| \leq v(x)$ a.e. in $\mathbb{R}^m$. Using again the dominated convergence theorem we get $\varphi'_G(u_{n_{k_j}}) \to \varphi'_G(u)$ in $X^*$. Since this can be done for any subsequence of $\{u_n\}$ the proposition is proved. ■

LEMMA 2.3. $\varphi(u) = (1/2)\|u\|^2 + o(\|u\|^2)$ *and* $\varphi'(u) u = \|u\|^2 + o(\|u\|^2)$ *as* $u \to 0$.

PROOF. If $\varepsilon > 0$ then, by (2.1), $\left| \int_{\mathbb{R}^m} f(x, u) u \, dx \right| \leq (\varepsilon + c_2 A_\varepsilon \cdot \|u\|^{s+1-2})\|u\|^2$ from which $\int_{\mathbb{R}^m} f(x, u) u \, dx = o(\|u\|^2)$. Analogously $\int_{\mathbb{R}^m} F(x, u) \, dx = o(\|u\|^2)$. ■

As a consequence we get a first compactness property of $\varphi$:

(2.4) $\exists \varrho > 0$ such that if $\|u_n\| \leq 2\varrho$ and $\varphi'(u_n) \to 0$ then $u_n \to 0$.

The hypotheses (f1)-(f3) are not sufficient to guarantee that the Palais Smale sequences are bounded in $X$. In the following we study the behavior of the bounded Palais-Smale sequences of $\varphi$. If $M \subset \mathbb{R}^m$ is measurable then we put $\|u\|_M^2 = \int_M |\nabla u|^2 + |u|^2 dx$.

LEMMA 2.5. *If* $u_n \to u$ *weakly in* $X$ *is such that* $\varphi(u_n) \to b$, $\varphi'(u_n) \to 0$ *then* $\varphi'(u) = 0$, $\varphi(u_n - u) \to b - \varphi(u)$ *and* $\varphi'(u_n - u) \to 0$.

PROOF. We prove first that $\varphi'(u) = 0$. Let $h \in X$ with $\|h\| = 1$. Fixing $\varepsilon > 0$ there exists $R > 0$ such that $\|h\|_{|x| > R} \leq \varepsilon$. Let $\{u_{n_k}\} \subset \{u_n\}$ and $v \in L^2(B_R(0)) \cap L^{s+1}(B_R(0))$ be such that $u_{n_k}(x) \to u(x)$ a.e. on $B_R(0)$, $|u_{n_k}(x)| \leq v(x)$ a.e. on $B_R(0)$. By (2.1) and the dominated convergence theorem we get

$$|\varphi'(u) h| = |\varphi'(u_{n_k}) h + \langle u - u_{n_k}, h \rangle - \int_{\mathbb{R}^m} (f(x, u) - f(x, u_{n_k})) h \, dx| \leq$$

$$\leqslant o(1) + \left| \int_{|x| > R} (f(x, u) - f(x, u_{n_k}))\, h\, dx \right| \leqslant$$

$$\leqslant o(1) + c_3 \int_{|x| > R} (|u| + |u_{n_k}|)|h| + (|u|^s + |u_{n_k}|^s)|\,|h|\, dx \leqslant$$

$$\leqslant o(1) + c_4(\|f\|_{|x| > R} + \|h\|^{s+1}_{|x| > R}) \leqslant o(1) + c_4(\varepsilon + \varepsilon^{s+1}).$$

Since $\varepsilon$ is arbitrary the claim follows.

Let's now prove that $\varphi'(u_n - u) \to 0$. Since

$$\varphi'(u_n - u)\, h = \varphi'(u_n)\, h - \int_{\mathbb{R}^m} (f(x, u_n - u) - f(x, u_n) + f(x, u))\, h\, dx,$$

it is sufficient to show that

$$\sup_{\|h\| = 1} \left| \int_{\mathbb{R}^m} (f(x, u_n - u) - f(x, u_n) + f(x, u))\, h\, dx \right| \to 0.$$

Given $\varepsilon > 0$ fix $R > 0$ such that $\|u\|_{|x| > R} < \varepsilon$. Then

$$\sup_{\|h\| = 1} \left| \int_{\mathbb{R}^m} (f(x, u_n - u) - f(x, u_n) + f(x, u))\, h\, dx \right| \leqslant$$

$$\leqslant \sup_{\|h\| = 1} \left| \int_{|x| \leqslant R} (\ldots)\, h\, dx \right| + \sup_{\|h\| = 1} \left| \int_{|x| > R} (\ldots)\, h\, dx \right|.$$

Consider the first addendum. Given $\{u_{n_k}\} \subset \{u_n\}$ there exists $\{u_{n_{k_j}}\} \subset \{u_{n_k}\}$ and a function $v \in L^2(B_R(0)) \cap L^{s+1}(B_R(0))$ such that $u_{n_{k_j}}(x) \to u(x)$ a.e. on $B_R(0)$ and $|u_{n_{k_j}}(x)| \leqslant v(x)$ a.e. on $B_R(0)$. We get

$$\left| \int_{B_R(0)} (f(x, u_{n_{k_j}} - u) - f(x, u_{n_{k_j}}) + f(x, u))\, h\, dx \right| \leqslant$$

$$\leqslant c_5 \left( \int_{B_R(0)} |f(x, u_{n_{k_j}}) - f(x, u_{n_{k_j}}) + f(x, u)|^{(s+1)/s} dx \right)^{s/(s+1)} \|h\|.$$

Since

$$|f(x, u_{n_{k_j}} - u) - f(x, u_{n_{k_j}}) + f(x, u)|^{(s+1)/s} \leqslant$$

$$\leqslant c_6(|v|^{(s+1)/s} + |v|^{s+1} + |u|^{(s+1)/s} + |u|^{s+1}) \in L^1(B_R(0))$$

we can use the dominated convergence theorem to get that

$$\int_{B_R(0)} |f(x, u_{n_{k_j}} - u) - f(x, u_{n_{k_j}}) + f(x, u)|^{(s+1)/s} dx = o(1).$$

Considering that this can be done for any subsequence of $\{u_n\}$ we actually get that

$$\sup_{\|h\|=1} \int_{B_R(0)} |f(x, u_n - u) - f(x, u_n) + f(x, u)| |h| dx = o(1).$$

For the second addendum we note first, by the choice of $R$,

$$\int_{|x|>R} f(x, u) h dx \leqslant c_6 \int_{|x|>R} |u| |h| + |u|^s |h| dx \leqslant c_7(\varepsilon + \varepsilon^s)\|h\|.$$

Since $f_x(x, 0) = 0$ we also infer that

$$\int_{|x|>R} |f(x, u_n - u) - f(x, u_n)| |h| dx \leqslant$$

$$\leqslant c_8 \int_{|x|>R} (1 + |u_n - u|^{s-1} + |u_n|^{s-1})|u| |h| dx \leqslant$$

$$\leqslant c_9\left(\|u\|_{|x|>R} + \left(\int_{|x|>R} (|u_n - u|^{s-1}|u|)^{(s+1)/s} dx\right)^{s/(s+1)} + \right.$$

$$\left. + \left(\int_{|x|>R} (|u_n|^{s-1}|u|)^{(s+1)/s} dx\right)^{s/(s+1)}\right)\|h\| \leqslant$$

$$\leqslant c_9\left(\|u\|_{|x|>R} + \left(\int_{|x|>R} |u_n - u|^{s+1} dx\right)^{(s-1)/(s+1)} \|u\|_{|x|>R} + \right.$$

$$\left. + \left(\int_{|x|>R} |u_n|^{s+1} dx\right)^{(s-1)/(s+1)} \|u\|_{|x|>R}\right)\|h\| \leqslant c_{10}\varepsilon\|h\|.$$

The proof that $\varphi(u_n - u) \to b - \varphi(u)$ is analogous. ■

LEMMA 2.6. *Let $\{u_n\}$ be a bounded sequence in $X$ and suppose that there exists $\varrho > 0$ such that* $\sup_{y \in \mathbb{R}^m} \int_{B_\varrho(y)} |u_n|^2 dx \to 0$ *as* $n \to \infty$. *Then* $u_n \to 0$ *in* $L^q(\mathbb{R}^n)$ *for any* $q \in (2, 2^*)$ *as* $n \to \infty$.

PROOF. We give for completeness the proof given in [10] (see Lemma 2.18).

Given $q \in (2, 2^*)$ let $\theta \in (0, 1)$ be such that $q = 2 + \theta(2^* - 2)$. By the Holder inequality

$$\int_{B_\varrho(y)} |u|^q dx = \int_{B_\varrho(y)} |u|^{2(1-\theta)} |u|^{2^*\theta} dx \leq$$

$$\leq \left( \int_{B_\varrho(y)} |u|^2 dx \right)^{1-\theta} \left( \int_{B_\varrho(y)} |u|^{2^*} dx \right)^{\theta},$$

$\forall u \in X$. So

$$\|u\|_{L^q(B_\varrho(y))} \leq \|u\|_{L^2(B_\varrho(y))}^{(2/q)(1-\theta)} \|u\|_{L^{2^*}(B_\varrho(y))}^{(2^*/q)\theta} = \|u\|_{L^2(B_\varrho(y))}^{1-\frac{a}{2}} \|u\|_{L^{2^*}(B_\varrho(y))}^{a}$$

where

$$a = \frac{2^*}{q}\theta = \frac{2^*}{q}\frac{(q-2)(m-2)}{4} = \frac{q-2}{q}m.$$

By the Sobolev Immersion theorem there exists $A_1 = A_1(q, m, \varrho)$ such that

$$\|u\|_{L^q(B_\varrho(y))} \leq A_1 \|u\|_{L^2(B_\varrho(y))}^{1-\frac{a}{2}} \|u\|_{B_\varrho(y)}^{a} \qquad \forall u \in X, \ \forall y \in \mathbb{R}^m.$$

Assume now that $aq \geq 2$ that is $q \geq 4/m + 2$. We have

$$\int_{B_\varrho(y)} |u|^q dx \leq A_1^q \|u\|_{L^2((B_\varrho(y))}^{q(1-a)} \|u\|^{aq-2} \int_{B_\varrho(y)} |\nabla u|^2 + |u|^2 dx.$$

Choosing a family of balls $\{B_\varrho(y_i)\}_{i \in \mathbb{N}}$ such that each point of $\mathbb{R}^m$ is contained in at least one and at most $k$ of such balls, summing over this family, we obtain

$$\|u\|_{L^q(\mathbb{R}^m)} \leq$$

$$\leq \sum_i \|u\|^q_{L^q(B_\varrho(y_i))} \leq A_1^q \|u\|^{aq-2} \sup_{y \in \mathbb{R}^m} \left( \int_{B_\varrho(y)} |u|^2 dx \right)^{q(1-a)} \sum_i \|u\|^2_{B_\varrho(y_i)} \leq$$

$$\leq kA_1^q \|u\|^{aq} \sup_{y \in \mathbb{R}^m} \left( \int_{B_\varrho(y)} |u|^2 dx \right)^{q(1-a)} \qquad \forall u \in X.$$

Setting in the above formula $u = u_n$ we get $u_n \to 0$ in $L^q(\mathbb{R}^m)$ for all $q \in [4/m + 2, 2^*)$. The proof now go on with another interpolation inequality.

If $q \in (2, \bar{q})$ $(\bar{q} = 4/m + 2)$, we have $q = 2\theta + \bar{q}(1-\theta)$ for some $\theta \in (0,1)$. By the Holder inequality we have $\|u_n\|^q_{L^q(\mathbb{R}^m)} \leq \leq \|u_n\|^{2\theta}_{L^2(\mathbb{R}^m)} \|u_n\|^{\bar{q}(1-\theta)}_{L^{\bar{q}}(\mathbb{R}^m)}$, for any natural $n$. The lemma follows from the fact that $u_n \to 0$ in $L^{\bar{q}}(\mathbb{R}^m)$. ■

LEMMA 2.7. *Let $u_n \to 0$ weakly in $X$ such that $\varphi'(u_n) \to 0$. Then, for any $R > 0$ we have $\|u_n\|_{|x|<R} \to 0$ and the sequence $\{u_n\}$ verifies either*

*a)* $u_n \to 0$ *or* *b)* $\exists \varrho, \eta > 0, \{y_n\} \subset \mathbb{R}^m / \limsup_{n \to \infty} \|u_n\|^2_{L^2(B_\varrho(y_n))} \geq \eta$.

PROOF. Let $R > 0$ and let $g_R \in C^\infty(\mathbb{R}^m, \mathbb{R})$ be such that $g_R(x) \geq 0$ for any $x \in \mathbb{R}^m$, $g_R(x) = 1$ if $x \in B_R(0)$, $\operatorname{supp} g_R \subset B_{2R}(0)$. Clearly $\|u_n\|^2_{B_R(0)} = \langle u_n, g_R u_n \rangle - \langle u_n, g_R u_n \rangle_{|x| \geq R}$. We prove first that $\langle u_n, g_R u_n \rangle \to 0$.

Since $\langle u_n, g_R u_n \rangle = \varphi'(u_n) g_R u_n + \int_{\mathbb{R}^m} f(x, u_n) g_R u_n \, dx$ and since $\|g_R u_n\| \leq c_{10}$, it's enough to show that $\int_{\mathbb{R}^m} f(x, u_n) g_R u_n \, dx \to 0$. But this is a consequence of the Lebesgue dominated convergence theorem since $\operatorname{supp} g_R \subset B_{2R}(0)$.

Therefore we have

$$\|u_n\|^2_{B_R(0)} = o(1) - \int_{|x| \geq R} \nabla g_R \nabla u_n u_n \, dx - \int_{|x| \geq R} g_R (|\nabla u_n|^2 + u_n^2) \, dx.$$

Since $\int_{|x| \geq R} g_R(|\nabla u_n|^2 + u_n^2) \, dx \geq 0$, to prove that $\|u_n\|_{B_R(0)} \to 0$ it is sufficient to show that $\int\limits_{|x|\geq R} \nabla g_R \nabla u_n u_n \, dx \to 0$. But this is true because

$$\left| \int\limits_{|x|\geq R} \nabla g_R \nabla u_n u_n \, dx \right| \leq$$

$$\leq \left( \int\limits_{R\leq |x| \leq 2R} |\nabla g_R \nabla u_n|^2 dx \right)^{1/2} \left( \int\limits_{R\leq |x| \leq 2R} |u_n|^2 dx \right)^{1/2}$$

and because $u_n \to 0$ in $L^2(B_{2R}(0) \backslash B_R(0))$.

The first part of the lemma is so proved. It's easy to prove the alternative.

Assume that (*b*) does not hold. In that case $\lim\limits_{n\to\infty} \sup\limits_{y\in \mathbb{R}^m} \|u\|_{B_\varrho(y)} = 0$ for any $\varrho > 0$. Since $\{u_n\}$ is bounded, by Lemma 2.6 we get that $u_n \to 0$ in $L^{s+1}(\mathbb{R}^m)$. From this, since $\int f(x, u_n) u_n \, dx \leq \varepsilon \|u_n\|^2_{L^2(\mathbb{R}^m)} + A_\varepsilon \|u_n\|^{s+1}_{L^{s+1}(\mathbb{R}^m)}$, we get $\int\limits_{\mathbb{R}^m} f(x, u_n) u_n \, dx \to 0$. Therefore $\|u_n\|^2 = \varphi'(u_n) u_n + \int\limits_{\mathbb{R}^m} f(x, u_n) u_n \, dx \to 0$ as we claimed. ∎

Therefore if $u_n$ is a Palais-Smale sequence which converges weakly to a certain point $u$, then $u_n$ converges to $u$ in $H^1_{\text{loc}}(\mathbb{R}^m)$. Moreover if $u_n$ does not converge to $u$ in $X$, then fixed any $R > 0$ we have $\limsup\limits_{n\to\infty} \|u_n\|_{|x|>R} \geq r > 0$. This mass $r$ cannot be smaller than a certain positive fixed value as the next lemma says.

LEMMA 2.8. *Let $u_n \to u$ weakly in $X$, $\varphi'(u_n) \to 0$. If there exists $R > 0$ such that $\limsup\limits_{n\to\infty} \|u_n\|_{|x|>R} \leq \varrho$ then $u_n \to u$ in $X$.*

PROOF. Fix $T > 0$ such that $\|u\|_{|x|\geq T} \leq \varrho/2$. Putting $M = \max\{R, T\}$ we have by Lemmas 2.5, 2.7, that $\|u_n - u\|_{|x|\leq M} \to 0$. Therefore

$$\|u_n - u\|^2 = o(1) + \|u_n - u\|^2_{|x|>M} =$$

$$= o(1) + \frac{\varrho^2}{4} + \varrho \|u_n\|_{|x|>M} + \|u_n\|^2_{|x|>M}$$

from which we get $\limsup \|u_n - u\| < 2\varrho$. Since $\varphi'(u_n - u) \to 0$ we derive from (2.4) that $u_n \to u$. ∎

This is a first local compactness property of the functional which will be useful in the following. From it we derive easily

LEMMA 2.9. *If* $\operatorname{diam}\{u_n\} < \varrho$ *and* $\varphi'(u_n) \to 0$ *then* $\{u_n\}$ *has an accumulation point.*

PROOF. Let $\operatorname{diam}\{u_n\} = \varrho_0$ and $T > 0$ such that $\|u_1\|_{|x|>T} \leqslant \varrho - \varrho_0$. In that case $\|u_n\|_{|x|>T} \leqslant \|u_n - u_1\|_{|x|>T} + \varrho - \varrho_0 \leqslant \varrho$. Since $\{u_n\}$ is bounded it has a subsequence $\{u_{n_k}\}$ which converges weakly in $X$ to a certain point $u$. Then, by Lemma 2.8, $u_{n_k} \to u$. ■

## 3. – The periodic case.

Here we will study some properties of the functional $\varphi_\infty$. Obviously all the results given in the previous section remain valid for $\varphi_\infty$. First of all we see how the further hypothesis (f4) implies that the functional $\varphi_\infty$ satisfies the geometrical hypotheses of the mountain pass theorem.

By Lemma 2.3 we just know that there exists $r > 0$ such that $\varphi_\infty(u) \geqslant (1/4)r^2$ for any $u \in \partial\mathcal{B}_r(0)$. Then we note that the assumption (f4) gives information about the behavior of $F_\infty$ at infinity with respect to $z$. In fact, one can infer that given $z_1 \neq 0$ then if $z/z_1 \geqslant 1$ we have

$$(3.1) \qquad F_\infty(x, z) \geqslant \left[F_\infty(x, z_1) - \frac{a}{\beta - 2} z_1^2\right] \left|\frac{z}{z_1}\right|^\beta + \frac{a}{\beta - 2} z^2$$

$$\forall x \in \mathbb{R}^m, \quad \frac{z}{z_1} \geqslant 1.$$

LEMMA 3.2. *There exists* $u_1 \in E$ *such that* $\varphi_\infty(u_1) < 0$.

PROOF. Let $(x_0, z_0) \in \mathbb{R}^m \times \mathbb{R}$ be given by (f4), then $\delta_0 = F_\infty(x_0, z_0) - (a/(\beta - 2)) z_0^2 > 0$. By continuity there exists $\varepsilon > 0$ such that $F_\infty(x, z_0) - (a/(\beta - 2)) z_0^2 \geqslant (1/2)\delta_0$ for any $x \in B_\varepsilon(z_0)$. Chosen $\varrho \in C^\infty(\mathbb{R}^m, \mathbb{R}^+)$ with $\operatorname{supp} \varrho \subset B_\varepsilon(x_0)$, we define $u_0(t) = z_0 \varrho(t)$. Then $\varphi_\infty(\lambda u_0) = (\lambda^2/2)\|u_0\|^2 - \int_{A_\lambda} F_\infty(x, \lambda u_0)\, dx - \int_{B_\lambda} F_\infty(x, \lambda u_0)\, dx$ where $A_\lambda = \{x\colon \lambda\varrho(x) < 1\}$ and $B_\lambda = \mathbb{R}^m \backslash A_\lambda$. Then $\int_{A_\lambda} |F_\infty(t, \lambda u_0)| \leqslant$

$\leqslant |B_\varepsilon(x_0)| \max\{|F_\infty(x, z)|: x \in \mathbb{R}^m, |z| \leqslant |z_0|\}$, whereas, by (3.1),

$$\int_{B_\lambda} F_\infty(x, \lambda u_0)\,dx \geqslant \lambda^\beta \int_{B_\lambda} \left[F_\infty(x, z_0) - \frac{\alpha}{\beta - 2} z_0^2\right] |\varrho|^\beta\,dx +$$

$$+\lambda^2 \frac{\alpha}{\beta-2} \int_{B_\lambda} u_0^2\,dx \geqslant \frac{1}{2}\delta_0 \|\varrho\|^\beta_{L^\beta(B_\lambda)} \lambda^\beta .$$

Therefore $\varphi_\infty(\lambda u_0) \to -\infty$ as $\lambda \to +\infty$ and the thesis follows. ∎

This shows that the functional at infinity verifies the geometrical hypotheses of the mountain pass theorem. Then, if we define $\Gamma = \{\gamma \in C([0, 1], E): \gamma(0) = 0,\ \varphi_\infty(\gamma(1)) < 0\}$ and $c = \inf_{\gamma \in \Gamma} \max_{s \in [0, 1]} \varphi_\infty(\gamma(s))$, we infer that $c$ is a positive, asymptoticall critical value for $\varphi_\infty$.

Now we use again (f4) to show that the Palais-Smale sequences of $\varphi_\infty$ are in fact bounded sequences in $X$.

LEMMA 3.3. *If* $\{u_n\} \subset X$ *is such that* $\varphi'_\infty(u_n) \to 0$ *and* $\limsup \varphi_\infty(u_n) < +\infty$, *then* $\{u_n\}$ *is bounded in* $E$ *and* $\liminf \varphi(u_n) \geqslant 0$. *In particular any Palais-Smale sequence for* $\varphi_\infty$ *is bounded in* $E$.

PROOF. From (f4), we easily get that

$$\left(\frac{1}{2} - \frac{1}{\beta} - \frac{\alpha}{\beta}\right)\|u\|^2 - \frac{1}{\beta}\|\varphi'_\infty(u)\|\|u\| \leqslant \varphi_\infty(u) \qquad \forall u \in X. \tag{3.4}$$

Now, given a sequence $\{u_n\} \subset X$ such that $\varphi'_\infty(u_n) \to 0$ and $\limsup \varphi_\infty(u_n) < +\infty$, from (3.4) we obtain $\|u_n\| \leqslant C$ for all $n \in \mathbb{N}$, $C$ being a positive constant. Consequently we have that $\varphi_\infty(u_n) \geqslant -C\|\varphi'_\infty(u_n)\|$ and this implies that $\liminf \varphi_\infty(u_n) \geqslant 0$. ∎

Using the periodicity we can now prove that the problem at infinity always admits a non zero solution which is obtained as weak limit of a suitable translated of the Palais-Smale sequence given by the mountain pass theorem.

THEOREM 3.5. *The problem*:

$$(\mathrm{P}_\infty) \qquad -\Delta u + u = f_\infty(x, u), \qquad u \in H^1(\mathbb{R}^m),$$

*admits a non zero solution.*

PROOF. Let $\{u_n\}$ be the Palais-Smale sequence given by the mountain pass theorem. By Lemma 3.3 we can assume that $u_n \to u$ weakly in

$X$. If $u \neq 0$ then by Lemma 2.5 the theorem is proved. Assume $u_n \to 0$ weakly in $X$. Since $\varphi_\infty(u_n) \to c > 0$ it cannot be $u_n \to 0$ so the alternative (b) of Lemma 2.7 holds and (up to a subsequence) $\exists \alpha_1 \alpha_2 > 0$, $\{y_n\} \subset \subset \mathbb{R}^m$ for which $\lim_{n \to \infty} \|u_n\|^2_{L^2(B_{\alpha_1}(y_n))} \geq \alpha_2$. Define $v_n = u_n(\cdot - [y_n])$ where $[y_n] = ([y_{n,1}], \ldots, [y_{n,m}])$. Then $\varphi_\infty(v_n) \to c$, $\varphi'_\infty(v_n) \to 0$ and $\|v_n\| \leq C$. Let $v_n \to v$ (up to a subsequence) weakly in $X$. By Lemma 2.7 we have $\|v_n - v\|_{L^2(B_{\alpha_1+1}(0))} \to 0$ therefore $\|v\|^2_{L^2(B_{\alpha_1+1}(0))} \geq \alpha_2 > 0$. So $v$ is a non zero critical point of $\varphi_\infty$. ∎

Therefore $K_\infty = \{v \in X \backslash \{0\} / \varphi'_\infty(v) = 0\} \neq \emptyset$ and using 2.4 we have also that

$$\inf_{v \in K_\infty} \|v\| = \lambda > 0 . \tag{3.6}$$

As we have point out in the introduction, the fact that the set of critical points of the functional at infinity is not empty does not guarantee that the problem (P) has non trivial solutions. We will prove that if the set of the critical points of the functional at infinity is numerable then this forces the functional $\varphi$ itself to have infinitely many critical points.

To study better the Palais-Smale sequences, following [23], [7], we introduce two sets of real numbers. Letting

$$S^b_{\mathrm{PS}} = \{(u_n) \subset E \colon \lim \varphi'_\infty(u_n) = 0,\ \limsup \varphi_\infty(u_n) \leq b\}$$

we define

$$\Phi^b = \{l \in \mathbb{R} \colon \exists (u_n) \in S^b_{\mathrm{PS}} \text{ s.t. } \varphi_\infty(u_n) \to l\}$$

the set of the asymptotic critical values lower than $b$ and

$$D^b = \{r \in \mathbb{R} \colon \exists (u_n), (\overline{u}_n) \in S^b_{\mathrm{PS}} \text{ s.t. } \|u_n - \overline{u}_n\| \to r\}$$

the set of the asymptotic distances between two Palais-Smale sequences under $b$.

The sets $\Phi^b$ and $D^b$ are actually closed subsets of $\mathbb{R}$ (see [7]) and we have:

(3.7) given $b > 0$, for any $r \in \mathbb{R}^+ \backslash D^b$ there exists $d_r > 0$ such that $[r - 3d_r, r + 3d_r] \subset \mathbb{R}^+ \backslash D^b$ and there exists $\mu_r > 0$ such that $|\varphi'_\infty(u)| \geq \mu_r$ for any $u \in \mathcal{A}_{r-3d_r, r+3d_r}(K^b_\infty) \cap \{\varphi_\infty \leq b\}$,

(3.8) given $b > 0$, for any $l \in (0, b) \backslash \Phi^b$ there exists $\delta > 0$ such that $[l - \delta, l + \delta] \subset (0, b) \backslash \Phi^b$ and there exists $\nu > 0$ such that $|\varphi'_\infty(u)| \geq \nu$ for any $u \in \{l - \delta \leq \varphi_\infty \leq l + \delta\}$,

where, if $S \subset X$ and $0 \leq r_1 \leq r_2$, $\mathcal{A}_{r_1, r_2}(S) = \bigcup_{x \in S} B_{r_2}(x) \backslash B_{r_1}(x)$.

Using Lemmas 2.5, 2.7, 3.3 and (3.6) together with the periodicity assumption it is possible to characterize the Palais-Smale sequences of $\varphi_\infty$. As in [10] we prove the following.

LEMMA 3.9. *Let $\{u_n\} \subset X$ be such that $\varphi_\infty(u_n) \to b$ and $\varphi'_\infty(u_n) \to \to 0$. Then there are $v_0 \in K_\infty \cup \{0\}, v_1, \ldots, v_k \in K_\infty$, a subsequence of $\{u_n\}$, denoted again $\{u_n\}$, and corresponding sequences $\{y_n^1\}, \ldots, \{y_n^k\} \in \mathbb{Z}^m$ such that, as $n \to \infty$:*

$$\|u_n - [v_0 + v_1(\cdot - y_n^1) + \ldots + v_k(\cdot - y_n^k)]\| \to 0,$$

$$\varphi_\infty(v_0) + \ldots + \varphi_\infty(v_k) = b,$$

$$|y_n^j| \to +\infty \qquad (j = 1, \ldots, k),$$

$$|y_n^i - y_n^j| \to +\infty \qquad (i \neq j).$$

PROOF. By Lemma 3.3 $\{u_n\}$ is a bounded sequence in $X$ and we can assume that $\exists \lim_{n \to \infty} \|u_n\| \in \mathbb{R}$ and that $u_n$ converges weakly to some $v_0$ in $X$. If $u_n \to v_0$ the lemma follows. Otherwise the case (*b*) of the alternative in Lemma 2.7 holds for the sequence $u_n - v_0$: $\exists \varrho, \eta > 0, \{y_n\} \subset \mathbb{Z}^m$, such that, up to a subsequence, $\lim_{n \to \infty} \|u_n - v_0\|^2_{L^2(B_\varrho(y_n))} \geq \eta$. Putting $u_n^1 = (u_n - v_0)(\cdot + y_n)$ we observe that there exists $v_1 \in K_\infty$ such that $u_n^1 \to v_1$ weakly in $X$, $\lim_{n \to \infty} \|u_n - v_0 - v_1(\cdot - y_n^1)\|^2 = \lim_{n \to \infty} \|u_n\|^2 - \|v_0\|^2 - \|v_1\|^2$. If $\lim_{n \to \infty} \|u_n\|^2 = \|v_0\|^2 + \|v_1\|^2$ the lemma follows because in that case $0 = \lim_{n \to \infty} \varphi_\infty(u_n - v_0 - v_1(\cdot - y_n^1)) = \lim_{n \to \infty} \varphi_\infty(u_n - v_0) - \varphi_\infty(v_1) = b - \varphi_\infty(v_0) - \varphi_\infty(v_1)$. If $\lim_{n \in \mathbb{N}} \|u_n\|^2 > \|v_0\|^2 + \|v_1\|^2$ we have that the sequence $\{u_n - v_0 - v_1(\cdot - y_n^1)\}$ verifies the case (*b*) of the alternative in Lemma 2.7 and we can continue as above. After a number of steps not greater than $\lim_{n \to \infty} \|u_n\| / \inf \sqrt{K_\infty} \|u\|$, the lemma follows. ■

This characterization reflects on the structure of the sets $\Phi^b$ and $D^b$. In fact, as in [7] (see Lemma 3.10), one can prove now that

$$\Phi^b = \{\textstyle\sum \varphi_\infty(v_i) \colon v_i \in K_\infty\} \cap [0, b],$$

$$D^b = \left\{ \left( \sum_{j=1}^{k} \|v_j - \bar{v}_j\|^2 \right)^{1/2} : \right.$$
$$\left. k \in \mathbb{N},\ v_i, \bar{v}_i \in K_\infty \cup \{0\},\ \sum_1^k \varphi_\infty(v_i) \leq b, \sum_1^k \varphi_\infty(\bar{v}_i) \leq b \right\}.$$

Using the hypothesis $(*)$ given in the introduction this permits us to bound from below the norm of the gradient $\varphi'_\infty$ in large regions of $X$. In fact if we assume

$(*)$ *there exists* $c^* > c$ *such that* $K_\infty^{c^*}$ *is countable*

then both the sets $D^{c^*}$ and $\Phi^{c^*}$ are countable too. Being $D^{c^*}$ and $\Phi^{c^*}$ also closed we then have that

(3.10) $D^{c^*}$ does not contain any neighborhood of 0 in $\mathbb{R}^+$,

(3.11) $]0, c^*[\backslash\Phi^{c^*}$ is open and dense in $[0, c^*]$.

By (3.7) and (3.10) we have that around $K_\infty^{c^*}$ there is a sequence of annuli of radii smaller and smaller on which there are not Palais-Smale sequences at a level less then or equal to $c^*$. Analogously, by (3.8) and (3.11), fixed any $\varepsilon \in (0, c^* - c)$ there exist two closed intervals $[a_1, b_1] \subset (c - \varepsilon, c)$ and $[a_2, b_2] \subset (c, c + \varepsilon)$ such that the sets $\{a_1 \leqslant \leqslant \varphi_\infty \leqslant b_1\}$ and $\{a_2 \leqslant \varphi_\infty \leqslant b_2 b\}$ do not contain Palais-Smale sequences.

Using (3.7), (3.10) and the local compactness property given by Lemma 2.9 it is possible to show as in [7] that the functional $\varphi_\infty$ admits a local mountain pass type critical point (see [19]).

DEFINITION 3.12. Let $f$ be a functional of class $C^1$ on a Banach space $X$ and let $\Omega$ be a nonempty open subset of $X$.

We say that two points $x_0, x_1 \in \Omega$ are connectible in $\Omega \cap \{f < c\}$ if there is a path $p \in C([0, 1], X)$ joining $x_0$ and $x_1$, with range $p \subset \Omega$ and such that $\max_p f < c$.

A critical point $\bar{x} \in X$ for $f$ is called of *local mountain pass-type* for $f$ on $\Omega$ if $\bar{x} \in \Omega$ and for any neighborhood $\mathcal{N}$ of $\bar{x}$ subset of $\Omega$ the set $\{f < f(\bar{x})\} \cap \mathcal{N}$ contains two points not connectible in $\Omega \cap \{f < f(\bar{x})\}$.

We refer to Section 4 of [7] for the proof of the following

LEMMA 3.13. *If* $\varphi_\infty$ *verifies* $(*)$ *then it admits a non zero critical point of mountain pass type. In particular there exist* $\bar{c} \in [c, c^*)$ *and* $\bar{r} \in (0, \varrho)$ *such that for any sequence* $(r_n) \subset \mathbb{R}_+ \backslash D^*$, $r_n \to 0$ *there is a sequence* $(v_n) \subset K_\infty(\bar{c})$, $v_n \to \bar{v} \in K_\infty(\bar{c})$ *having this property: for any* $n \in \mathbb{N}$

*and for any* $h > 0$ *there is a path* $\gamma_n \in C([0, 1], X)$ *satisfying*:

(i) $\gamma_n(0),\ \gamma_n(1) \in \partial B_{r_n}(v_n)$;

(ii) $\gamma_n(0)$ *and* $\gamma_n(1)$ *are not connectible in* $B_{\bar{r}}(\bar{v}) \cap \{\varphi_\infty < \bar{c}\}$;

(iii) *range* $\gamma_n \subseteq \overline{B}_{r_n}(v_n) \cap \{\varphi_\infty \leq \bar{c} + h\}$;

(iv) *range* $\gamma_n \cap A_{r_n - (1/2) d_{r_n}, r_n}(v_n) \subseteq \{\varphi_\infty \leq \bar{c} - h_n\}$;

(v) $\operatorname{supp} \gamma_n(s) \subset [-R_n, R_n]$ *for any* $s \in [0, 1]$,

*where* $R_n > 0$ *is independent of* $s$, $h_n = (1/8)\, d_{r_n} \mu_{r_n}$ *and* $d_{r_n}$ *and* $\mu_{r_n}$ *are defined by* (3.7).

REMARK 3.14. Clearly the property given in this sections are true for all the functionals $\varphi_\iota$ ($\iota = 1, \ldots, l$) if the assumptions ($*_\iota$) are verified. In the following we write $c_\iota$ as the mountain pass level of $\varphi_\iota$, $\bar{c}_\iota$ as the local mountain pass level near $\bar{v}_\iota$ etc.

## 4. – The construction of a pseudogradient vector field.

In this section we will study some consequences of the assumption (f7) with which we ask that there exist $A_1, \ldots, A_l \subset \mathbb{R}^m$, large at infinity on where $f$ is asymptotic to $l$ different periodic and superquadratic functions $f_1, \ldots, f_l$ as $|x| \to \infty$.

First of all we show that by (f7) if a function $u$ is translated in a region where $f$ and $f_\iota$ are close one to the other then $\varphi'(u)$ is near $\varphi'_\iota(u)$ (here $\varphi_\iota(u)(1/2)\|u\|^2 - \int_{\mathbb{R}^m} F_\iota(x, u)\, dx$ is the functional associated to the function $f_\iota$ ($\iota = 1, \ldots, l$)).

LEMMA 4.1. *For any* $\delta > 0$ *and* $C > 0$ *there exists* $R > 0$ *such that if* $\|u\| \leq C$ *and* $\operatorname{supp} u \subset A_\iota \backslash B_R(0)$ *then* $\|\varphi'(u) - \varphi'_\iota(u)\| \leq \delta$.

PROOF. For any $\varepsilon > 0$ we choose $R > 0$ such that, if $\operatorname{supp} u \subset A_\iota \backslash B_R(0)$, we have $|f(x, u(x)) - f_\iota(x, u(x))| \leq \varepsilon(|u(x)| + |u(x)|^s)$ for almost every $x \in \mathbb{R}^m$. Then $|\varphi'(u)h - \varphi'_\iota(u)h| \leq \varepsilon \int_{A_\iota} |u|\,|h| + |u|^s |h|\, dx \leq \varepsilon c_{10} \|u\| \|h\|$ and the lemma follows. ■

Given $k, N \in N$ we say that $p = (p_1, \ldots, p_k) \in P(k, N)$ if $p_j \in \mathbb{Z}^k$ for any $j$ and $|p_j| \geq |p_{j-1}| + 4N^2 + 6N$ for $j \geq 2$.

If $p \in P(k, N)$ we define the annuli

$$\mathcal{U}_1 = \left\{ |x| \leq \frac{1}{2}(|p_1| + |p_2|) \right\},$$

$$\mathcal{U}_j = \left\{ \frac{1}{2}(|p_j| + |p_{j-1}|) \leq |x| \leq \frac{1}{2}(|p_j| + |p_{j+1}|) \right\} \quad (j = 2, \ldots, k)$$

and

$$M_i = \{ |p_i| + 2N(N+1) < |x| < |p_{i+1}| - 2N(N+1) \} \quad (i = 1, \ldots, k),$$

where $|p_{k+1}| = +\infty$. Since $p \in P(k, N)$ the thickness of the annulus $M_i$ is alway greater than or equal to $2N$.

Given $r > 0$, $p \in P(k, N)$, $V = (V_1, \ldots, V_l) \in X^l$, $J = (j_1, \ldots, j_k) \in \{1, \ldots, l\}^k$ we define the set

$$\mathcal{B}_r(V; p; J) = \left\{ u \in X / \max_{i=1, \ldots, k} \|u - V_{j_i}(\cdot - p_i)\|_{\mathcal{U}_i} < r \right\}.$$

It is easy to see that if $N$ is sufficiently large (depending on $V$ and $r$) then $\mathcal{B}_r(V; p; J)$ is a nonempty open subset of $X$. Moreover the elements of $\mathcal{B}_r(V; p; J)$ are multibump functions of the mixed type. In fact if $u \in \mathcal{B}_r(V; p; J)$ then $u$ is near the function $V_{j_i}(\cdot - p_i)$ on the annulus $\mathcal{U}_i$.

Defining

$$\varphi_{\iota, i}(u) = (1/2)\|u\|^2_{\mathcal{U}_i} - \int_{\mathcal{U}_i} F_\iota(x, u(x))\, dx,$$

$$u \in X, \quad (i = 1, \ldots, k), \qquad (\iota = 1, \ldots, l),$$

we investigate some properties of the functionals $\varphi$ and $\varphi_{\iota, i}$ on the set $\mathcal{B}_r(V; p; J)$.

We note that for any given $V \in E^l$, $r > 0$, if $\tilde{N} = \tilde{N}(V, r) \in \mathbb{N}$ is such that $\|V_\iota\|_{|x| > \tilde{N}} \leq r$ for any $\iota \in \{1, \ldots, l\}$ then given $k \in \mathbb{N}$, $N > \tilde{N}$, $p \in P(k, N)$ and $J \in \{1, \ldots, l\}^k$ then $\forall u \in \mathcal{B}_r(V; p; J)$ and $\forall i \in \{1, \ldots, k\}$ there exists $j \in \{N+2, \ldots, 2N+1\}$ such that

$$\|u\|^2_{jN \leq ||x| - |p_i|| \leq (j+1)N} \leq \frac{4r^2}{N}. \tag{4.2}$$

Therefore if $u \in \mathcal{B}_r(V; p; J)$ then for any $i \in \{1, \ldots, k\}$, the annulus $\mathcal{U}_i$ contains two annular regions of thickness $N$, symmetric with respect to $p_i$, over which the norm of $u$ is small as we want if $N$ is sufficiently large. Moreover, by construction, $M_i$ never intersects any one of these annuli.

We call $j_{u,i}$ the smallest index in $\{N+2, \dots, 2N+1\}$ which verifies (4.2).

For any $\varepsilon \in (0, r)$ there exists $N_\varepsilon \in \mathbb{N}$, $N_\varepsilon \geq \max\{\tilde{N}(V, r), 2\}$ such that

$$\max_{\iota = 1, \dots, l} \left\{ \|V_\iota\|^2_{|x| > N_\varepsilon}, \frac{4r^2}{N_\varepsilon} \right\} < \frac{\varepsilon}{2}.$$

So if $k \in \mathbb{N}$, $N > N_\varepsilon$ and $p \in P(k, N)$, then $\forall u \in \mathcal{B}_r(V; p; J)$ and $\forall i \in \in \{1, \dots, k\}$ we get that

$$\|u\|^2_{j_{u,i}N \leq ||x| - |p_i|| \leq (j_{u,i}+1)N} < \frac{\varepsilon}{2}. \tag{4.3}$$

Then, fixed $u \in \mathcal{B}_r(V; p; J)$, we define the following subsets of $\mathbb{R}^m$:

$$E_{u,i} = \{|p_i| + (j_{u,i}+1)N \leq |x| \leq |p_{i+1}| - (j_{u,i+1}+1)N\} \qquad (i = 1, \dots, k),$$

$$E_u = \bigcup_{i=1}^{k} E_{u,i},$$

$$\tilde{E}_{u,i} = \{x \in \mathbb{R}^m / \operatorname{dist}(x, E_{u,i}) \leq N\} \qquad (i = 1, \dots, k),$$

$$\tilde{E}_u = \bigcup_{i=1}^{k} \tilde{E}_{u,i},$$

$$\mathcal{F}_{u,i} = \mathcal{U}_i \cap (\tilde{E}_u \setminus E_u) \qquad (i = 1, \dots, k).$$

With this notation (4.3) can be rewritten in the form

$$\|u\|^2_{\mathcal{F}_{u,i}} \leq \frac{\varepsilon}{2} \qquad \forall u \in \mathcal{B}_r(V; p; J), \ \forall i \in \{1, \dots, k\}. \tag{4.4}$$

We plainly recognize also that

$$\|u\|^2_{\tilde{E}_{u,i} \setminus E_{u,i}} \leq \varepsilon \qquad \forall u \in \mathcal{B}_r(V; p; J), \ \forall i \in \{1, \dots, k\}. \tag{45}$$

By construction $M_i \subset E_{u,i}$, therefore the thickness of $E_{u,i}$ is greater than or equal to $N$ $\forall i \in \{1, \dots, k\}$, $\forall u \in \mathcal{B}_r(V; p; J)$. This is true also for the connected parts of the sets $\mathcal{F}_{u,i}$ and $\tilde{E}_{u,i} \setminus E_{u,i}$.

For $i \in \{1, \ldots, k\}$, we define the cut-off functions:

$$\beta_{u,i}(x) = \begin{cases} 1 & x \in E_{u,i}, \\ 0 & x \notin \tilde{E}_{u,i}, \end{cases}$$

with $\beta_{u,i}$ continuous on $\mathbb{R}^m$ and linear if restricted on the connected parts of $\tilde{E}_u \backslash E_u$ intersected with any straight line passing through the origin. We put also $\beta_{u,0} \equiv 0$.

Then, for $i \in \{1, \ldots, k\}$, we set:

$$\bar{\beta}_{u,i}(x) = \begin{cases} 0 & x \notin \mathcal{U}_i, \\ 1 - \beta_{u,i-1} - \beta_{u,i} & x \in \mathcal{U}_i. \end{cases}$$

If $\beta$ is any one of the above cut-off functions then $|\nabla\beta(x)| \leqslant 1/N$, for a.e. $x \in \mathbb{R}^m$, therefore since $N \geqslant 2$, it is easy to see that, if $A$ is measurable $\subset \mathbb{R}^m$ then $\|\beta u\|_A^2 \leqslant 2\|u\|_A^2$, $\forall u \in X$. Moreover if $u \in \mathcal{B}_r(V; p; J)$ and $i \in \{1, \ldots, k\}$, then by (4.5), we get

$$(4.6)\quad \langle u, \beta_{u,i} u\rangle = \|u\|^2_{E_{u,i}} + \int_{\tilde{E}_{u,i} \backslash E_{u,i}} [\nabla\beta_{u,i} \nabla u u + \beta_{u,i}(|\nabla u|^2 + |u|^2)]\, dx \geqslant$$

$$\geqslant \|u\|^2_{E_{u,i}} - \frac{1}{4}\|u\|^2_{\tilde{E}_{u,i}\backslash E_{u,i}} \geqslant \|u\|^2_{E_{u,i}} - \frac{1}{4}\varepsilon.$$

Now we define, for $i \in \{1, \ldots, k\}$, the functions

$$\chi_i(u) = \begin{cases} 1 & \|u\|^2_{E_{u,i}} \geqslant \varepsilon, \\ \dfrac{1}{k} & \text{otherwise}, \end{cases}$$

and we set finally

$$W_u = \sum_{i=1}^{k} \chi_i(u)\beta_{u,i} u.$$

If we define the finite cone $C = \{y \in \mathbb{R}^m;\ |y| < 1/2,\ 1/4 < y_1 < 1/2\}$ then the embedding constant relative to the immersion $H^1(\Omega) \to L^{s+1}(\Omega)$ can be chosen to be independent of $\Omega$ if $\Omega$ is an open set of $\mathbb{R}^m$ which verifies the cone property with respect to $C$.

This implies, by (f2) (f3), that we can fix $r_0 \in (0, \min\{\bar{r}, \sqrt{2} - 1\})$

such that if $u, w \in X$ then

$$\text{(4.7)}\qquad \|u\|_A \leqslant r_0 \Rightarrow \int_A F(x,u)\,dx \leqslant \frac{1}{8}\|u\|_A^2$$

$$\text{and}\quad \int_A f(x,u)\,w\,dx \leqslant \frac{1}{8}\|u\|_A\|w\|_A$$

for any open set $A \subset \mathbb{R}^m$ which satisfies the cone property with respect to $C$. We can assume that $r_0$ is such that (4.7) holds also if we consider $f_\iota$ ($\iota = 1, \ldots, l$) instead of $f$.

Using (4.6), (4.7), we can prove now that:

LEMMA 4.8. *Let* $r \in (0, (1/4) r_0)$ *and* $0 < \varepsilon < r^2$. *Then* $\forall u \in \mathcal{B}_r(V; p; J)$ *we have*

$$\varphi'(u) W_u \geqslant \frac{1}{2} \sum_{j=1}^{k} \chi_j(u)(\|u\|_{E_{u,j}}^2 - \varepsilon),$$

$$\varphi'_{\iota,i}(u) W_u \geqslant \frac{1}{2} \sum_{j=1}^{k} \chi_j(u)(\|u\|_{\mathcal{U}_i \cap E_{u,j}}^2 - \varepsilon).$$

PROOF. We have that $N > 2$, and the thickness of the annuli $E_{u,j}$ and of the ones whose union is $\tilde{E}_{u,j} \setminus E_{u,j}$ is greater than or equal to $N$. Therefore these sets satisfy the cone property with respect to $C$. Moreover $\|u\|_{E_{u,j}} \leqslant 4r \leqslant r_0$ (in fact $\|u\|_{E_{u,j} \cap \mathcal{U}_i} \leqslant 2r$ $\forall i \in \{1, \ldots, k\}$) and $\|u\|_{\tilde{E}_{u,j} \setminus E_{u,j}} \leqslant \varepsilon^{1/2} < r_0$. Therefore, by (4.6), and (4.7), we get

$$\varphi'(u) W_u \geqslant \sum_{j=1}^{k} \chi_j(u) \Bigg( \|u\|_{E_{u,j}}^2 - \frac{\varepsilon}{4} - \int_{E_{u,j}} f(x,u)\,u\,dx - \int_{\tilde{E}_{u,j} \setminus E_{u,j}} f(x, \beta_{u,i} u)\,\beta_{u,i} u\,dx \Bigg) \geqslant$$

$$\geqslant \sum_{j=1}^{k} \chi_j(u) \left( \frac{7}{8} \|u\|_{E_{u,i}}^2 - \frac{1}{4}\varepsilon - \frac{1}{4}\varepsilon \right) \geqslant \frac{1}{2} \sum_{j=1}^{k} \chi_j(u)(\|u\|_{E_{u,j}}^2 - \varepsilon).$$

The computation is perfectly analogous for $\varphi_{\iota,i}$. ∎

By Lemma 4.8 we always have that

$$\varphi'(u)W_u \geq \frac{1}{2}\sum_{j=1}^{k}\chi_j(u)(\|u\|^2_{E_{u,j}} - \varepsilon) \geq$$

$$\geq \frac{1}{2}\sum_{\{j/\|u\|^2_{E_{u,j}}<\varepsilon\}}\chi_j(u)(\|u\|^2_{E_{u,j}} - \varepsilon) \geq -\frac{\varepsilon}{2}$$

and analogously

$$\varphi'_{\iota,i}(u)W_u \geq -\frac{\varepsilon}{2} \qquad \forall i \in \{1, \ldots, k\}$$

for all $u \in \mathcal{B}_r(V; p; J)$.

Moreover if $\|u\|_{\mathcal{U}_i \cap E_{u,j}}$ is greater then $2\varepsilon^{1/2}$, for a certain couple of index $(i, j)$, then $W_u$ indicates an increasing direction both for $\varphi$ and $\varphi_{\iota,i}$. We note also that $W_u$ has support in a region where each $V_{j_i}(\cdot - p_i)$ is small and it holds that $\langle W_u, u\rangle_{M_i} \geq (1/k)\|u\|^2_{M_i}$ for any $u \in B_r(V; p; J)$ and for any $i \in \{1, \ldots, k\}$.

Given $J = (j_1, \ldots, j_k) \in \{1, \ldots, l\}^k$, $R > 0$ we say that $p \in P_R(k, N, J)$ if $p \in P(k, N)$ and $B_{N(N+1)}(p_i) \subset A_{j_i} \backslash B_R(0)$ $(i = 1, \ldots, k)$.

Let $b_\iota$ be any nonzero critical level of $\varphi_\iota$, and $r \in (0, (1/8)r_0) \backslash \bigcup_{\iota=1}^{l} D_\iota^{c_\iota^*}$, $r_1$, $r_2$, $r_3$ be such that $r - 3d_r < r_1 < r_2 < r_3 < r + 3d_r < (1/4)r_0$ where $d_r = \min\{d_r^\iota;\ \iota = 1, \ldots, l\}$ $(\iota = 1, \ldots, l)$.

Let also $b_{-,\iota}$, $b_{+,\iota}$ and $\delta$ be such that $[b_{-,\iota} - \delta, b_{-,\iota} + 2\delta] \subset\, ]0, b_\iota[\backslash \Phi_\iota^{c_\iota^*}$ and $[b_{+,\iota} - \delta, b_{+,\iota} + 2\delta] \subset\, ]b_\iota, c_\iota^*[\backslash \Phi_\iota^{c_\iota^*}$.

PROPOSITION 4.9. *There exists $\mu = \mu(r) > 0$ and $\varepsilon_1 = \varepsilon_1(r, b_{+,\iota}, b_{-,\iota}, \delta) > 0$, $R > 0$ such that: $\forall v_\iota \in \mathcal{K}_\iota(b_\iota)$ $(\iota = 1, \ldots, l)$, $\forall \varepsilon \in ]0, \varepsilon_1[$ there exists $N \in \mathbb{N}$, such that, for any $k \in \mathbb{N}$, $J = (j_1, \ldots, j_k) \in \{1, \ldots, l\}^k$ and $p \in P_R(k, N, J)$, there exists a locally lipschitz continuous function $W: X \to X$ which verifies*

($W_0$) $\|W(u)\|_{\mathcal{U}_j} \leq 2$ $\forall u \in X$, $j = 1, \ldots, k$, $\varphi'(u)W(u) \geq 0$ $\forall u \in X$, $W(u) = 0$ $\forall u \in X \backslash \mathcal{B}_{r_3}(V; p; J)$ $(V = (v_1, \ldots, v_l))$,

($W_1$) $\varphi'_{j_i,i}(u)W(u) \geq \mu$ *if* $r_1 \leq \|u - v_{j_i}(\cdot - p_i)\|_{\mathcal{U}_i} \leq r_2$, $u \in \mathcal{B}_{r_2}(V; p; J) \cap \bigcap_{i=1}^{k} \varphi_{j_i,i}^{b_{+,j_i}+\delta}$,

($W_2$) $\varphi'_{j_i,i}(u)W(u) \geq 0$ $\forall u \in (\varphi_{j_i,i}^{b_{+,j_i}+\delta} \backslash \varphi_{j_i,i}^{b_{+,j_i}}) \cup (\varphi_{j_i,i}^{b_{-,j_i}+\delta} \backslash \varphi_{j_i,i}^{b_{-,j_i}})$,

($W_3$) $\langle u, W(u)\rangle_{M_j} \geq 0$ $\forall j \in \{1, \ldots, k\}$ *if* $\max\|u\|^2_{M_i} \geq 4\varepsilon$.

*Moreover if* $\mathcal{K} \cap \mathcal{B}_{r_2}(V; p; J) = \emptyset$ *then there exists* $\mu_k > 0$ *such that*

$$(\mathrm{W}_4) \qquad \varphi'(u)\,W(u) \geq \mu_k \ \forall\, u \in \mathcal{B}_{r_2}(V; p; J).$$

PROOF. Let $\tilde{r}_1 = r_1 - (1/2)(r_1 - r + 3d_r)$, $\tilde{r}_3 = r_3 + (1/2)(r + 3d_r - r_3)$ and let $\mu_r$ be given by 3.8.

Let also

$$\nu = \inf\{\|\varphi_\iota'(u)\|/u \in (\varphi_\iota^{b_{+,\iota}+2\delta} \setminus \varphi_\iota^{b_{+,\iota}-\delta}) \cup (\varphi_\iota^{b_{-,\iota}+2\delta} \setminus \varphi_\iota^{b_{-,\iota}-\delta});\ \iota = 1, \ldots, l\};$$

by Remark 3.9 we have that $\nu > 0$.

Let $C = 2\sup\{\|u\|;\ u \in \mathcal{K}_\iota(b_\iota),\ \iota = 1, \ldots, l\} + r_0$; by Lemma 4.1 there exists $R > 0$ such that if $\|u\| \leq C$ and $\operatorname{supp} u \subset A_\iota \setminus B_R(0)$ then $\|\varphi_\iota'(u) - \varphi'(u)\| \leq (1/4)\min\{\nu, \mu_r\}$.

Let

$$\varepsilon_1^{1/2} = \min\left\{\frac{(r_1 - r + 3d_r)}{12}, \frac{(r + 3d_r - r_3)}{12}, \frac{\mu_r}{16}, \frac{\nu}{16}, \frac{\delta^{1/2}}{6}\right\}.$$

Let's fix $v_\iota \in \mathcal{K}_\iota(b_\iota)$ $(\iota = 1, \ldots, l)$, $\varepsilon \in (0, \varepsilon_1)$, $k \in \mathbb{N}$, $N > N_\varepsilon$, $J = (j_1, \ldots, j_k) \subset \{1, \ldots, l\}^k$ and $(p_1, \ldots, p_k) \in P_R(k, N, J)$.

We construct the vector field $W_u$ on $\mathcal{B}_{r_3}(v; p; J)$, using Lemma 4.8 with $r = r_3$. We will now define another vector field analyzing the different cases.

*Case* 1) $u \in (\mathcal{B}_{r_3}(V; p; J) \setminus \mathcal{B}_{r_1}(V; p; J)) \cap \bigcap_{i=1}^{k} \varphi_{j_i, i}^{b_{+,j_i}+3\delta/2}$.

We set $\mathfrak{I}_1(u) = \{i \in \{1, \ldots, k\}/\|u - v_{j_i}(\cdot - p_i)\|_{\mathcal{U}_i} \geq r_1\}$. Obviously $\mathfrak{I}_1(u) \neq \emptyset$.

Let $i \in \mathfrak{I}_1(u)$ and $\xi_1 = (1/2)\min\{r_1 - \tilde{r}_1, \tilde{r}_3 - r_3\}$.

We consider the two possible subcases:

$$\|u\|_{\mathcal{U}_i \cap E_u} \geq \xi_1 \qquad or \qquad \|u\|_{\mathcal{U}_i \cap E_u} < \xi_1 .$$

In the first one, using Lemma 4.8 and the fact that $\varepsilon_1^{1/2} \leq \xi_1/3$, we get (putting $E_0 = \emptyset$)

$$(4.10) \qquad \varphi'(u)\,W_u \geq \frac{1}{2}(\|u\|_{E_{u,i-1}}^2 + \|u\|_{E_{u,i}}^2 - 2\varepsilon) - \sum_{\{l/\|u\|_{E_{u,l}}^2 < \varepsilon\}} \chi_l(u)\frac{\varepsilon}{2} \geq$$

$$\geq \frac{1}{2}(\|u\|_{\mathcal{U}_i \cap E_u}^2 - 2\varepsilon) - \sum_{\{l/\|u\|_{E_{u,l}}^2 < \varepsilon\}} \chi_l(u)\frac{\varepsilon}{2} \geq \frac{\xi_1^2}{2} - 2\varepsilon \geq \frac{\xi_1^2}{4},$$

and analogously

$$\text{(4.11)} \qquad \varphi'_{j_i,i}(u)\,W_u \geqslant \frac{\xi_1^2}{2} - 2\varepsilon \underset{k}{\geqslant} \frac{\xi_1^2}{4}.$$

For all $u \in (\mathcal{B}_{r_3}(V;p;J)\backslash\mathcal{B}_{r_1}(V;p;J)) \cap \bigcap_{i=1} \varphi_{j_i,i}^{b_+,j_i+\delta}$ if $\|u\|_{\mathcal{U}_i \cap E_u} \geqslant \xi_1$ and $i \in \mathfrak{I}_1(u)$ we put $\mathcal{W}_{u,i} = 0$.

In the second subcase we firstly note that, arguing as in (4.3), there exists $j_u \in \{1, \ldots, N\}$ such that $\|u\|^2_{j_uN \leqslant |x-p_i| \leqslant (j_u+1)N} < \varepsilon/2$. We put $\tilde{B}_u = B_{(j_u+1)N}(p_i)$ and $B_u = B_{j_uN}(p_i)$ noting that $\operatorname{dist}(\tilde{B}_u, \tilde{E}_u) \geqslant N$ and therefore that the set $\mathcal{U}_i\backslash(\tilde{B}_u \cup \tilde{E}_u)$ has the cone property with respect to $C$.

We define also the cut-off function $\eta_u \in C(\mathrm{R}^m, \mathrm{R})$ such that $\eta_u(x) = 1$ if $x \in B_u$, $\eta_u(x) = 0$ if $x \notin \tilde{B}_u$, and in such a way $\eta_u$ is linear if restricted on the connected parts of $\tilde{B}_u\backslash B_u$ intersected with any line passing through $p_i$.

Consider the following alternative:

i) $\|u\|_{\mathcal{U}_i\backslash(\tilde{B}_u \cup \tilde{E}_u)} \geqslant \xi_1/2$, or

ii) $\|u\|_{\mathcal{U}_i\backslash(\tilde{B}_u \cup \tilde{E}_u)} < \xi_1/2$.

If i) holds we put $\mathcal{W}_{u,i} = (1-\eta_u)\,\bar{\beta}_{u,i}u$ and by (4.7) we get

$$\varphi'(u)\,\mathcal{W}_{u,i} \geqslant \|u\|^2_{\mathcal{U}_i\backslash(\tilde{B}_u \cup \tilde{E}_u)} - 3\varepsilon - \frac{1}{8}\|u\|^2_{\mathcal{U}_i\backslash(\tilde{B}_u \cup \tilde{E}_u)} \geqslant \frac{\xi_1^2}{8} - 3\varepsilon \geqslant \frac{\xi_1^2}{16}.$$

Analogously we get

$$\varphi'_{j_i,i}(u)\,\mathcal{W}_{u,i} \geqslant \frac{\xi_1^2}{16}.$$

We finally observe that

$$\text{(4.12)} \quad \min\{\varphi'_{j_i,i}(u)(\mathcal{W}_{u,i}+W_u),\, \varphi'(u)(\mathcal{W}_{u,i}+W_u)\} \geqslant \frac{\xi_1^2}{16} - \frac{\varepsilon}{2} \geqslant \frac{\xi_1^2}{32}.$$

If ii) holds we claim that $\eta_u u \in \mathcal{A}_{r-3d_r,\,r+3d_r}(v_{j_i}(\cdot - p_i)) \cap \varphi_{j_i}^{c^*_{j_i}}$.

In fact we firstly note that since $(5/4)\xi_1^2 - 3\varepsilon^{1/2}\xi_1 - 7\varepsilon \leqslant r_1^2 - (r-3d_r)^2$ we have

$$\|\eta_u u - v_{j_i}(\cdot - p_i)\|^2 \geqslant \|u - v_{j_i}(\cdot - p_i)\|^2_{B_u} =$$

$$= \|\cdots\|^2_{\mathcal{U}_i} - \|\cdots\|^2_{E_u \cap \mathcal{U}_i} - \|\cdots\|^2_{\mathfrak{F}_{u,i}} + - \|\cdots\|^2_{\mathcal{U}_i\backslash(\tilde{E}_u \cup \tilde{B}_u)} - \|\cdots\|^2_{\tilde{B}_u\backslash B_u} \geqslant$$

$$\geqslant r_1^2 - (\xi_1^2 + 2\varepsilon^{1/2}\xi_1 + \varepsilon) - 3\varepsilon - \left(\frac{\xi_1^2}{4} + \varepsilon^{1/2}\xi_1 + \varepsilon\right) - 3\varepsilon =$$

$$= r_1^2 - \frac{5}{4}\xi_1^2 - 3\varepsilon^{1/2}\xi_1 - 7\varepsilon \geqslant (r-3d_r)^2.$$

On the other hand since $(3/2)\xi_1 + r_3 + 3\varepsilon^{1/2} \leq r + 3d_r$ we get

$$\|\eta_u u - v_{j_i}(\cdot - p_i)\|^2 \leq \|\eta_u u - v_{j_i}(\cdot - p_i)\|^2_{\mathcal{U}_i} + \varepsilon \leq$$

$$\leq (\|(1-\eta_u)u\|_{\mathcal{U}_i} + \|u - v_{j_i}(\cdot - p_i)\|_{\mathcal{U}_i})^2 + \varepsilon \leq$$

$$\leq (\|u\|_{\mathcal{U}_i \cap E_u} + \|u\|_{\mathcal{F}_{u,i}} + \|u\|_{\mathcal{U}_i \setminus (\tilde{E}_u \cup \tilde{B}_u)} + 2^{1/2}\|u\|_{\tilde{B}_u \setminus B_u} + r_3)^2 + \varepsilon \leq$$

$$\leq \left(\frac{3}{2}\xi_1 + r_3 + 3\varepsilon^{1/2}\right)^2 \leq (r + 3d_r)^2 .$$

To end the proof of the claim we note that, since $\|u\|_{\mathcal{U}_i \setminus B_u} \leq (1/2) r_0$, by (4.7) we have that $(1/2)\|u\|^2_{\mathcal{U}_i \setminus B_u} - \int_{\mathcal{U}_i \setminus B_u} F_{j_i}(x, u)\, dx \geq 0$. Therefore $\varphi_{j_i}(\eta_u u) = \varphi_{j_i, i}(\eta_u u) \leq v_{j_i, i}(u) + \|u\|^2_{\tilde{B}_u \setminus B_u} \leq \varphi_{j_i, i}(u) + \varepsilon \leq b_{+, j_i} + 2\delta < c^*_{j_i}$ as we claimed.

Therefore there exists $Z_{u,i} \in X$, $\|Z_{u,i}\| \leq 1$ such that

$$\varphi'_{j_i, i}(\eta_u u) Z_{u,i} = \varphi'_{j_i}(\eta_u u) Z_{u,i} \geq \frac{\mu_r}{2} .$$

Since $p \in P_R(k, N, J)$ we have $\operatorname{supp} \eta_u u \subset A_{j_i} \setminus B_R(0)$. Moreover $\|\eta_u u\| \leq 2^{1/2}\|u\|_{\mathcal{U}_i} \leq 2^{1/2}(\|v_{j_i}\| + r_0) \leq C$, which implies by the choice of $R$ that

$$\varphi'(\eta_u u) Z_{u,i} = \varphi'_{j_i}(\eta_u u) Z_{u,i} + (\varphi'(\eta_u u) - \varphi'_{j_i}(\eta_u u)) Z_{u,i} \geq \frac{\mu_r}{4} .$$

As last step we note that

$$|\varphi'_{j_i, i}(\eta_u u) Z_{u,i} - \varphi'_{j_i, i}(u)\eta_u Z_{u,i}| =$$

$$= \Big| \langle \eta_u u, Z_{u,i} \rangle_{\tilde{B}_u \setminus B_u} - \langle u, \eta_u Z_{u,i} \rangle_{\tilde{B}_u \setminus B_u} -$$

$$- \int_{\tilde{B}_u \setminus B_u} (f_{j_i}(x, \eta_u u) - f_{j_i}(x, u)\eta_u) Z_{u,i}\, dx \Big| \leq$$

$$\leq \frac{2}{N}\|u\|_{\tilde{B}_u \setminus B_u} + \frac{1}{4}\|u\|_{\tilde{B}_u \setminus B_u} \leq \varepsilon^{1/2} \leq \frac{\mu_r}{8}$$

and the same argument gives also

$$|\varphi'(\eta_u u) Z_{u,i} - \varphi'(u)\eta_u Z_{u,i}| \leqslant \frac{\mu_r}{8}.$$

From the two above inequality it follows that

$$\min\{\varphi'(u)\eta_u Z_{u,i},\ \varphi'_{j_i,i}(u)\eta_u Z_{u,i}\} \geqslant \frac{\mu_r}{8}.$$

In this case we put $\mathcal{W}_{u,i} = (1/2)\eta_u Z_{u,i}$ observing that

$$\text{(4.13)}\quad \min\{\varphi'(u)(\mathcal{W}_{u,i} + W_u),\ \varphi'_{j_i,i}(u)(\mathcal{W}_{u,i} + W_u)\} \geqslant \frac{\mu_r}{16} - \frac{\varepsilon}{2} \geqslant \frac{\mu_r}{32}.$$

We now set $2\mu = \min\{(\mu_r/32,\ \xi_1^2/32\}$ and

$$\mathcal{V}_{u,1} = \begin{cases} W_u + \sum_{i \in \mathfrak{I}_1(u)} \mathcal{W}_{u,i} & u \in (\mathcal{B}_{r_3}(V;p;J) \setminus \mathcal{B}_{r_1}(V;p;J)) \cap \bigcap_{i=1}^{k} \varphi_{j_i,i}^{b_+ + 3\delta/2}, \\ 0 & \text{otherwise}, \end{cases}$$

obtaining by (4.10)-(4.13) that $\forall u \in (\mathcal{B}_{r_3}(V;p;J) \setminus \mathcal{B}_{r_1}(V;p;J)) \cap \cap \bigcap_{i=1}^{k} \varphi_{j_i,i}^{b_+,j_i + 3\delta/2}$

$$\text{(4.14)}\quad \begin{cases} \varphi'(u)\mathcal{V}_{u,1} \geqslant 2\mu, \\ \varphi'_{j_i,i}(u)\mathcal{V}_{u,1} \geqslant 2\mu \qquad \forall i \in \mathfrak{I}_1(u), \\ \langle u, \mathcal{V}_{u,1}\rangle_{M_l} = \langle u, W_u\rangle_{M_l} \geqslant \frac{1}{k}\|u\|^2_{M_l} \qquad l = 0, \ldots, k. \end{cases}$$

We also note that $\|\mathcal{V}_{u,1}\|_{\mathcal{U}_i} \leqslant \|\mathcal{W}_{u,i}\|_{\mathcal{U}_i} + \|W_u\|_{\mathcal{U}_i} \leqslant 1/\sqrt{2}(1 + r_0) < 1$ for any $i \in \{1, \ldots, k\}$.

*Case* 2) $u \in \mathcal{B}_{r_3}(V;p;J) \cap \left(\bigcup_{i=1}^{k} (\varphi_{j_i,i})_{b_+,j_i}^{b_+,j_i + \delta}\right)$.

We put $\mathfrak{I}_2^+(u) = \{i \in \{1, \ldots, k\} / u \in (\varphi_{j_i,i})_{b_+,j_i}^{b_+,j_i + \delta}\}$ and fix $i \in \in \mathfrak{I}_2^+(u)$.

Fixing also $\xi_2^2 = \delta/4$ it can be either

$$\|u\|_{\mathcal{U}_i \cap E_u} \geqslant \xi_2 \qquad or \qquad \|u\|_{\mathcal{U}_i \cap E_u} < \xi_2.$$

In the first subcase, considering that $\varepsilon_1 \leqslant \xi_2^2/9$, we get as above that

$$\varphi'(u)W_u \geqslant \frac{1}{2}\xi_2^2 - 2\varepsilon \geqslant \frac{1}{4}\xi_2^2 \quad \text{and} \quad \varphi'_{j_i,i}(u)W_u \geqslant \frac{1}{2}\xi_2^2 - 2\varepsilon \geqslant \frac{1}{4}\xi_2^2.$$

For all

$$u \in \mathcal{B}_{r_3}(V; p; J) \cap \left( \bigcup_{i=1}^{k} (\varphi_{j_i,i})_{b_{+,j_i}}^{b_{+,j_i}+\delta} \right)$$

and $i \in \mathfrak{J}_2(u)$, if $\|u\|_{\mathcal{U}_i \cap E_u} \geqslant \xi_2$ we put $\tilde{\mathcal{W}}_{u,i} = 0$.

In the second subcase we proceed as in the case 1 considering the alternative

i) $\|u\|_{\mathcal{U}_i \setminus (\tilde{B}_u \cup \tilde{E}_u)} \geqslant \xi_2/2$, or

ii) $\|u\|_{\mathcal{U}_i \setminus (\tilde{B}_u \cup \tilde{E}_u)} < \xi_2/2$.

If i) holds then putting $\mathcal{W}_{u,i} = (1 - \eta_u)\,\bar{\beta}_{u,i}u$ arguing as in case 1 we get

$$\min\{\varphi'(u)(\mathcal{W}_{u,i} + W_u),\ \varphi'_{j_i}(u)(\mathcal{W}_{u,i} + W_u)\} \geqslant \frac{\xi_2^2}{32}. \tag{4.15}$$

If ii) holds then we claim that $\eta_u u \in (\varphi_{j_i})_{b_{+,j_i}-\delta}^{b_{+,j_i}+2\delta}$.

In fact

$$\|u\|_{\mathcal{U}_i}^2 - \|\eta_u u\|_{\mathcal{U}_i}^2 \leqslant \|u\|_{\mathcal{U}_i \cap E_u}^2 + \|u\|_{\mathcal{F}_{u,i}}^2 + \|u\|_{\mathcal{U}_i \setminus (\tilde{E}_u \cup \tilde{B}_u)}^2 +$$

$$+ \|u\|_{\tilde{B}_u \setminus B_u}^2 + \|\eta_u u\|_{\tilde{B}_u \setminus B_u}^2 \leqslant \frac{5}{4}\xi_2^2 + 2\varepsilon$$

and

$$\int_{\mathcal{U}_i} F_{j_i}(x, u) - F_{j_i}(x, \eta_u u)\,dx =$$

$$= \int_{\mathcal{U}_i \setminus \tilde{B}_u} F_{j_i}(x, u)\,dx + \int_{\tilde{B}_u \setminus B_u} F_{j_i}(x, u) - F_{j_i}(x, \eta_u u)\,dx \leqslant$$

$$\leqslant \frac{1}{8}\|u\|_{\mathcal{U}_i \setminus \tilde{B}_u}^2 + \frac{1}{2}\|u\|_{\tilde{B}_u \setminus B_u}^2 \leqslant \frac{1}{2}(\xi_2^2 + \varepsilon).$$

We finally derive that

$$|\varphi_{j_i,i}(u) - \varphi_{j_i,i}(\eta_u u)| \leqslant \xi_2^2 + \varepsilon < \delta$$

which implies $\eta_u u \in (\varphi_{j_i,i})_{b_+,j_i-\delta}^{b_+,j_i+2\delta}$ as we claimed. So there exists $Z_{u,i} \in X$, $\|Z_{u,i}\| \leq 1$ such that $\varphi'_{j_i}(\eta_u u) Z_{u,i} = \varphi'_{j_i,i}(\eta_u u) Z_{u,i} \geq \nu/2$.

As in case 1), since $\operatorname{supp} \eta_u u \subset A_{j_i} \backslash B_R(0)$, we get $\varphi'(\eta_u u) Z_{u,i} \geq \nu/4$.

Moreover, again as in the case 1), since $\varepsilon^{1/2} < \nu/16$

$$\min\{\varphi'(u)\eta_u Z_{u,i}, \varphi'_{j_i,i}(u)\eta_u Z_{u,i}\} \geq \frac{\nu}{8}$$

and we put in this case $\tilde{\mathcal{W}}_{u,i} = (1/2)\eta_u Z_{u,i}$ noting that

$$\min\{\varphi'(u)(W_u + \tilde{\mathcal{W}}_{u,i}), \varphi'_{j_i,i}(u)(W_u + \tilde{\mathcal{W}}_{u,i})\} \geq \geq \frac{\nu}{16} - \frac{\varepsilon}{2} \geq \frac{\nu}{32}. \tag{4.16}$$

Let now $\nu^+ = \min\{\nu/32, \xi_2^2/32\}$ and

$$\mathcal{V}_{u,2} = \begin{cases} W_u + \sum_{i \in \mathfrak{I}_2^+(u)} \tilde{\mathcal{W}}_{u,i} & u \in \bigcup_{i=1}^k (\varphi_{j_i,i})_{b_+,j_i}^{b_+,j_i+\delta} \cap \mathcal{B}_{r_3}(V; p; J), \\ 0 & \text{otherwise}, \end{cases}$$

obtaining, as in the case 1), that

$$\forall u \in \mathcal{B}_{r_3}(V; p; J) \cap \left(\bigcup_{i=1}^k (\varphi_{j_i,i})_{b_+,j_i}^{b_+,j_i+\delta}\right),$$

$$\begin{cases} \varphi'(u)\mathcal{V}_{u,2} \geq \nu^+, \\ \varphi'_{j_i,i}(u)\mathcal{V}_{u,2} \geq \nu^+ \qquad \forall \in \mathfrak{I}_2^+(u), \\ \langle u, \mathcal{V}_{u,2}\rangle_{M_l} = \langle u, W_u\rangle_{M_l} \geq \dfrac{1}{k}\|u\|_{M_l}^2 \qquad l \in \{0, \dots, k\}. \end{cases} \tag{4.17}$$

As in the case 1) it is easy to prove that $\max_i \|\mathcal{V}_{u,2}\|_{\mathcal{U}_i} \leq 1$.

*Case* 3) $u \in \mathcal{B}_{r_3}(V; p; J) \cap \left(\bigcup_{i=1}^k (\varphi_{j_i,i})_{b_-,j_i}^{b_-,j_i+\delta}\right)$.

As in case 2) we put $\xi_3 = \delta/4$, $\nu^- = \min\{\nu/32, \xi_3^2/32\}$, and $\mathfrak{I}_2^-(u) = \{i \in \{1, \dots, k\}/u \in (\varphi_{j_i,i})_{b_-,j_i}^{b_-,j_i+\delta}\}$, getting that $\forall u \in \mathcal{B}_{r_3}(V; p; J) \cap$

$\cap \left( \bigcup_{i=1}^{k} (\varphi_{j_i, i})_{b_{-, j_i}}^{b_{-, j_i} + \delta} \right)$ there exists $\mathcal{V}_{u,3} \in X$ such that $\max_i \|\mathcal{V}_{u,3}\|_{\mathcal{U}_i} \leq 1$ and

$$(4.18) \qquad \begin{cases} \varphi'(u)\,\mathcal{V}_{u,3} \geq \nu^-, \\ \varphi'_{j_i, i}(u)\,\mathcal{V}_{u,3} \geq \nu^- \qquad \forall i \in \mathfrak{J}_2^-(u) \\ \langle u, \mathcal{V}_{u,3}\rangle_{M_l} = \langle u, W_u\rangle_{M_l} \geq \frac{1}{k}\|u\|_{M_l}^2 \qquad l \in \{1, \ldots, k\}. \end{cases}$$

We put $\mathcal{V}_{u,3} = 0$ if $u \notin \mathcal{B}_{r_3}(V; p; J) \cap \left( \bigcup_{i=1}^{k} (\varphi_{j_i, i})_{b_{-, j_i}}^{b_{-, j_i} + \delta} \right)$.

*Case* 4) $u \in \mathcal{B}_{r_1}(V; p; J)$.

In this case we distinguish between the two subcases:

$$\max_{1 \leq l \leq k} \|u\|_{M_l}^2 \geq 4\varepsilon \qquad or \qquad \max_{0 \leq l \leq k} \|u\|_{M_l}^2 < 4\varepsilon .$$

In the first case, if we have $\|u\|_{M_{\bar{l}}}^2 = \max_{0 \leq l \leq k} \|u\|_{M_l}^2 \geq 4\varepsilon$, we get using Lemma 4.8 that

$$\varphi'(u)\,W_u \geq \frac{1}{2}(\|u\|_{E_{u,\bar{l}}}^2 - \varepsilon) - \frac{1}{2}\varepsilon \geq \frac{1}{2}(\|u\|_{M_{\bar{l}}}^2 - \varepsilon) - \frac{1}{2}\varepsilon \geq \varepsilon$$

and we set $\mathcal{V}_{u,4} = W_u$.

In the second case, by the local compactness property of $\varphi$ (Lemma 2.8), we obtain that if $K \cap \mathcal{B}_{r_1}(V; p; J) = \emptyset$ then there exists $V_u \in X$, $\|V_u\| \leq 1$ and there exists $\mu'_k > 0$, independent of $u$, such that $\varphi'(u)\,V_u \geq \mu'_k/2$. We set $\mathcal{V}_{u,4} = V_u$.

Let also $\mathcal{V}_{u,4} = 0$ if $u \notin \mathcal{B}_{r_1}(V; p; J)$.

We can conclude that if put $2\mu_k = \min\{\varepsilon, \mu'_k/2\}$ we have $\forall u \in \mathcal{B}_{r_1}(V; p; J)$ that $\varphi'(u)\,\mathcal{V}_{u,4} \geq 2\mu_k$ and if $\max_{j=1,\ldots,k} \|u\|_{M_j}^2 \geq 4\varepsilon$ then

$$(4.19) \qquad \langle u, \mathcal{V}_{u,4}\rangle_{M_l} = \langle u, W_u\rangle_{M_l} \geq \frac{1}{k}\|u\|_{M_l}^2 \qquad l \in \{0, \ldots, k\}.$$

For $u \in X$ we put $\mathcal{V}_u = \sum_{i=1}^{4} \mathcal{V}_{u,i}$ noting that $\max_i \|\mathcal{V}_u\|_{\mathcal{U}_i} \leq 2$. Then the proposition follows with a classical pseudogradient construction, by using a suitable partition of unity and suitable cutoff functions.

## 5. - Multiplicity result.

In this section we will state and prove the main Theorem.

THEOREM 5.1. *Assume that* (f1)-(f5), (f7) *and* $(*_\iota)$ $(\iota = 1, \ldots, l)$ *hold. Let* $v_\iota$ *be the critical point of* $\varphi_\iota$ $(\iota = 1, \ldots, l)$ *given by* 3.14. *Then for any* $r > 0$ *there is* $N \in \mathbb{N}$, $R > 0$, *such that for every* $k \in \mathbb{N}$ $J \in \{1, \ldots, k\}^l$ *and* $p \in P_R(k, N, J)$ *we have* $\mathcal{K} \cap \mathcal{B}_r(V; p; J) \neq \emptyset$.

PROOF. Suppose the contrary, then there exists $\tilde{r} \in (0, r_0/8)$ such that for any $\tilde{N} \in \mathbb{N}$, $\tilde{R} > 0$ there are $k \in \mathbb{N}$, $J \in \{j_1, \ldots, j_k\}^l$ and $p \in P_{\tilde{R}}(k, \tilde{N}, J)$ for which $\mathcal{K} \cap \mathcal{B}_{\tilde{r}}(V; p; J) = \emptyset$ $(V = (v_1, \ldots, v_l))$. Let $(v_n^\iota) \subset \mathcal{K}_\iota(\bar{c}_\iota)$ and $(r_n) \subset \mathbb{R}^+$, be the sequences given by 3.14. Since $v_n^\iota \to v_\iota$ and $r_n \to 0$ we can choose $n \in \mathbb{N}$ such that $\|v_n^\iota - v_\iota\| < \tilde{r}/2$, $r_n < \tilde{r}/2 - 3d_{r_n}$ and $\mathcal{B}_{2r_n}(v_n^\iota) \subset \mathcal{B}_{\tilde{r}}(\bar{v}_\iota)$. In particular we have that $\mathcal{B}_{r_n}(V_n; p; J) \subset \mathcal{B}_{\tilde{r}}(V; p; J)$ $(V_n = (v_n^1, \ldots, v_n^l))$.

Fixing this $n$, fix also any $r_-$, $r$, $r_+$ such that $r_n - 3d_{r_n} < r_- < r < r_+ < r_n + 3d_{r_n}$ and fix $c_{-,\iota}$, $c_{+,\iota}$, $\delta$ such that $]c_{-,\iota} - \delta, c_{-,\iota} + 2\delta[ \subset \bar{c}_\iota - 1/4 \min\{h_n, \mu(r - r_-)\}, \bar{c}[ \setminus \Phi_\iota^{c_\iota^*}$ and $]c_{+,\iota} - \delta, c_{+,\iota} + 2\delta[ \subset ]\bar{c}_\iota, \min\{c_\iota^*, \bar{c}_\iota + (1/4)\mu(r - r_-)\}[ \setminus \Phi_\iota^{c_\iota^*}$.

By 3.14 we can choose $\gamma_\iota \in C([0, 1], X)$ such that

(i) $\gamma_\iota(0)$, $\gamma_\iota(1) \in \partial\mathcal{B}_{r_n}(v_n^\iota) \cap \varphi_\iota^{\bar{c}_\iota - (1/2)h_n}$;

(ii) $\gamma_\iota(0)$ and $\gamma_\iota(1)$ are not connectible in $\{\varphi_\iota < \bar{c}_\iota\} \cap \mathcal{B}_{\tilde{r}}(v_\iota)$;

(iii) range $\gamma_\iota \subset \bar{\mathcal{B}}_{r_n}(v_n^\iota) \cap \varphi_\iota^{c_{+,\iota}}$;

(iv) range $\gamma_\iota \cap \mathcal{A}_{r_n - (1/2)d_{r_n}, r_n}(v_n^\iota) \subset \varphi_\iota^{\bar{c}_\iota - (1/2)h_n}$;

(v) $\operatorname{supp} \gamma_\iota(s) \subseteq [-T, T]$ for any $s \in [0, 1]$, being $T > 0$ independent on $s$.

Fix $0 < \varepsilon < \min\{\varepsilon_1, (1/9)(\bar{c} - c_{-,\iota}), (1/2)d_{r_n}^2\}$. We can also assume, enlarging $T$ if necessary, that $\|v_n^\iota\|_{|x| \geq T}^2 \leq \varepsilon/4$ and, we fix an integer $N_1 \geq \max\{T, N, 4\}$ where $N$ is given by Proposition 4.9 for these value of $r$, $b_{+,\iota} = c_{+,\iota}$, $b_{-,\iota} = c_{-,\iota}$, $\varepsilon/4$ instead of $\varepsilon$ and $V_n$ instead of $V$.

If $R > 0$ is given by the Proposition 4.9, since $\mathcal{K} \cap \mathcal{B}_{r_n}(V; p; J) = \emptyset$ for a $p \in P_R(k, N, J)$, there exists a locally Lipschitz continuous function $W$: $X \to X$ which satisfies the properties $(W_0)$-$(W_4)$. Let us consider the flow associated to the following Cauchy problem

$$\frac{d\eta}{ds}(s, u) = -W(\eta(s, u)), \qquad \eta(0, u) = u.$$

Plainly, by $(\mathcal{W}_0)$ for any $u \in X$ this Cauchy problem admits a unique solution $\eta(\cdot, u)$ defined on $\mathbb{R}^+$ and the function $\eta$ is continuous on $\mathbb{R}^+ \times E$. Moreover, the function $s \mapsto \varphi(\eta(s, u))$ is nonincreasing.

We define the function $G\colon Q = [0,1]^k \to X$ by setting $G(\theta) = \sum_{i=1}^{k} \gamma_{j_i}(\theta_i)(\cdot - p_i)$ for $\theta = (\theta_1, \dots, \theta_k) \in Q$. We put $F_i^0 = \{\theta \in Q\colon \theta_i = 0\}$ and $F_i^1 = \{\theta \in Q\colon \theta_i = 1\}$ and we note that $G(\theta)|_{\mathcal{U}_i} = \gamma_{j_i}(0)$ if $\theta \in F_i^0$, $G(\theta)|_{\mathcal{U}_i} = \gamma_{j_i}(1)$ if $\theta \in F_i^1$.

Moreover $G(\theta)|_{\mathcal{U}_i} = \gamma_{j_i}(\theta_i)(\cdot - p_i)$ and $\operatorname{supp} \gamma_{j_i}(\theta_i)(\cdot - p_i) \subseteq [-R + p_i, R + p_i] \subset \mathcal{U}_i \setminus (M_i \cup M_{i-1})$. Therefore $\varphi_{j_i, i}(G(\theta)) = \varphi_{j_i}(\gamma_{j_i}(\theta_i))$ for any $i \in \{1, \dots, k\}$ and for any $\theta = (\theta_1, \dots, \theta_k) \in Q$.

To prove the theorem, we make the following claim.

Claim. *There exists $\tau > 0$ such that the continuous function $\bar{G}\colon Q \to X$ given by $\bar{G}(\theta) = \eta(\tau, G(\theta))$ satisfies:*

(vi) $\bar{G} = G$ *on* $\partial Q$;

(vii) $\max_i \| \bar{G}(\theta)\|_{M_i}^2 \leq \varepsilon$ *for any* $\theta \in Q$;

(viii) *there is a path $\xi$ inside $Q$ joining two opposite faces $F_{\bar{i}}^0$ and $F_{\bar{i}}^1$ such that, along $\xi$, the function $\varphi_{j_{\bar{i}}} \circ \bar{G}$ takes values under $c_{-, j_{\bar{i}}} + \varepsilon$; namely: $\exists\, \bar{i} \in \{1, \dots, k\}$ and $\xi = (\xi_1, \dots, \xi_k) \in C([0,1], Q)$ such that $\xi_{\bar{i}}(0) = 0$, $\xi_{\bar{i}}(1) = 1$ and $\bar{G}(\xi(s)) \in \varphi_{j_{\bar{i}}}^{c_{-, j_{\bar{i}}} + \varepsilon}$ for any $s \in [0,1]$.*

Assume the claim holds and introduce a cut-off function $\chi \in C(\mathbb{R}^m, \mathbb{R})$, such that $\chi(x) = 0$ if $x \notin \mathcal{U}_{\bar{i}}$, $\chi(t) = 1$ if $t \in \mathcal{U}_{\bar{i}} \setminus (M_{\bar{i}} \cup M_{\bar{i}-1})$ and linear on the connected parts of the intersection of any line passing through the origin with the set $\mathcal{U}_{\bar{i}} \cap (M_{\bar{i}} \cup M_{\bar{i}-1})$.

Define $g \in C([0,1], X)$ by setting $g(s) = \chi \bar{G}(\xi(s))$ for $s \in [0,1]$. We observe that, because of (vi) and (viii) we have $g(0) = \chi \bar{G}(\xi(0)) = \chi G(\xi(0)) = \gamma_{j_{\bar{i}}}(0)(\cdot - p_{\bar{i}})$ and similarly $g(1) = \gamma_{j_{\bar{i}}}(1)(\cdot - p_{\bar{i}})$. We have also that the path $g$ is contained in the ball $B_{\bar{r}}(v_{j_{\bar{i}}}(\cdot - p_{\bar{i}}))$. Indeed, first of all $\|v_n^{j_{\bar{i}}} - v_{j_{\bar{i}}}\| \leq r_0/16 < \bar{r}/2$. Secondly, since $\operatorname{supp} g_{j_{\bar{i}}}(s) \subset \mathcal{U}_{\bar{i}}$, $\varepsilon < (1/2) d_{r_n}^2$ and since $B_{r_+}(V_n, p, J)$ is invariant under $\eta$, we get

$$\|g_{j_{\bar{i}}}(s) - v_n^{j_{\bar{i}}}(\cdot - p_{\bar{i}})\|^2 \leq \|g_{j_{\bar{i}}}(s) - v_n^{j_{\bar{i}}}(\cdot - p_{\bar{i}})\|_{\mathcal{U}_i}^2 + \varepsilon \leq$$

$$\leq \max_{\theta \in Q} (\|\chi(\bar{G}(\theta) - v^{j_{\bar{i}n}}(\cdot - p_{\bar{i}}))\|_{\mathcal{U}_{\bar{i}}} + 2\varepsilon)^2 \leq 2(r_+ + \varepsilon)^2 + \varepsilon \leq 4 r_n^2 \leq \frac{\bar{r}^2}{4}.$$

Translating by $-p_{\bar{i}}$ the path $g$, we get a curve joining $\gamma_{j_{\bar{i}}}(0)$ with $\gamma_{j_{\bar{i}}}(1)$ in $\mathcal{B}_{\bar{r}}(v_{j_{\bar{i}}})$. Showing that on $g$ the functional $\varphi_{j_{\bar{i}}}$ remains under the level $\bar{c}_{j_{\bar{i}}}$ we will get a contradiction with the property (ii) of $\gamma_{j_{\bar{i}}}$.

To prove this, we notice that

$$\varphi_{j_{\bar{i}}}(g(s)) = \varphi_{j_{\bar{i}},\bar{i}}(g(s)) \leq \varphi_{j_{\bar{i}},\bar{i}}(\bar{G}(\xi(s))) + \frac{1}{2}\|g(s)\|^2_{\mathcal{U}_{\bar{i}} \cap (M_{\bar{i}} \cup M_{\bar{i}-1})} + \tag{5.2}$$

$$+ \int_{\mathcal{U}_{\bar{i}} \cap (M_{\bar{i}} \cup M_{\bar{i}-1})} F_{j_{\bar{i}}}(x, \bar{G}(\xi(s))) - \int_{\mathcal{U}_{\bar{i}} \cap (M_{\bar{i}} \cup M_{\bar{i}-1})} F_{j_{\bar{i}}}(x, g(s)).$$

By (viii), $\varphi_{j_{\bar{i}},\bar{i}}(\bar{G}(\xi(s))) \leq c_{-,j_{\bar{i}}} + \varepsilon$. Moreover, by (vii), $(1/2)\|g(s)\|^2_{\mathcal{U}_{\bar{i}} \cap (M_{\bar{i}} \cup M_{\bar{i}-1})} \leq \|\bar{G}(\xi(s))\|^2_{M_{\bar{i}}} + \|\bar{G}(\xi(s))|^2_{M_{\bar{i}-1}} \leq 2\varepsilon$. Finally,

$$\int_{\mathcal{U}_{\bar{i}} \cap (M_{\bar{i}} \cup M_{\bar{i}-1})} |V(t, \bar{G}(\xi(s)))| \leq \|\bar{G}(\xi(s))\|^2_{\mathcal{U}_{\bar{i}} \cap (M_{\bar{i}} \cup M_{\bar{i}-1})} \leq 2\varepsilon$$

and

$$\int_{\mathcal{U}_{\bar{i}} \cap (M_{\bar{i}} \cup M_{\bar{i}-1})} |F_{j_{\bar{i}},\bar{i}}(x, g(s))| \leq 2\|\bar{G}(\xi(s))\|^2_{\mathcal{U}_{\bar{i}} \cap (M_{\bar{i}} \cup M_{\bar{i}-1})} \leq 4\varepsilon.$$

Putting together all these estimates in (5.2), and considering that $\varepsilon < (1/9)(\bar{c} - c_-)$, we finally get that $\varphi_{\bar{i}}(g(s)) \leq c_{-,j_{\bar{i}}} + 9\varepsilon < \bar{c}_{j_{\bar{i}}}$, which contradicts (ii).

To check the claim we firstly note that the properties (vi) and (vii) are true for any $\tau > 0$, and follows easily from $(W_2)$ and $(W_3)$.

We divide the proof of (viii) in some lemmas.

LEMMA 5.3. *There is $\tau > 0$ such that for any $u \in \mathcal{B}_{r_-}(V_n; p; J) \cap \bigcap_{i=1}^{k} \varphi_{j_i,i}^{c_+,j_i}$ there exists $\bar{i} \in \{1, \ldots, k\}$ for which $\eta(\tau, u) \in \varphi_{j_{\bar{i}},\bar{i}}^{c_-,j_{\bar{i}}}$.*

PROOF. Set $\sigma = 2 \operatorname{diam} \varphi(\mathcal{B}_{r_+}(V_n; p; J))$. Since $\varphi(\mathcal{B}_{r_+}(V_n; p; J))$ is a bounded set, $\sigma < +\infty$. Put $\tau = \sigma/\mu_k$ and let $u \in \mathcal{B}_{r_-}(V_n; p; J) \cap \bigcap_{i=1}^{k} \varphi_{j_i,i}^{c_+,j_i}$. By $(W_2)$ the curve $s \mapsto \eta(s, u)$ remains in $\bigcap_{i=1}^{k} \varphi_{j_i,i}^{c_+,j_i}$. Moreover it must goes out of $\mathcal{B}_r(V_n; p; J)$ at some $\bar{s} \in ]0, \tau[$, otherwise, by $(W_4)$,

$$\varphi(u) - \varphi(\eta(\tau, u)) = \int_0^\tau \varphi'(\eta(s, u)) \mathcal{W}(\eta(s, u))\, ds \geq \mu_k \tau = \sigma$$

in contrast with the definition of $\sigma$. Then, there are $\bar{i} \in \{1, \ldots, k\}$ and an interval $[s_1, s_2] \subset ]0, \tau[$ such that $\|\eta(s_1, u) - v_n^{j_{\bar{i}}}(\cdot - p_{\bar{i}})\|_{\mathcal{U}_{\bar{i}}} = r_-$, $\|\eta(s_2, u) - v_n^{j_{\bar{i}}}(\cdot - p_{\bar{i}})\|_{\mathcal{U}_{\bar{i}}} = r$ and $r_- < \|\eta(s, u) - v_n^{j_{\bar{i}}}(\cdot - p_{\bar{i}})\|_{\mathcal{U}_j} < r$ for any $s \in ]s_1, s_2[$. Then by $(W_1)$ and since by $(W_2)$ $\eta(s, u) \in \varphi_{j_{\bar{i}},\bar{i}}^{c_+,\bar{i}}$ for any

$s \geqslant 0$, we get

$$\varphi_{j_{\bar{i}},\bar{i}}(\eta(s_2, u)) \leqslant \varphi_{j_{\bar{i}},\bar{i}}(\eta(s_1, u)) - \\ - \int_{s_1}^{s_2} \varphi'_{j_{\bar{i}},\bar{i}}(\eta(s, u))\, \mathcal{W}(\eta(s, u))\, ds \leqslant c_{+, j_{\bar{i}}} - \mu(s_2 - s_1).$$

But since $\|\mathcal{W}(\eta(s, u))\|_{\mathcal{U}_{\bar{i}}} \leqslant 2$ for any $s \geqslant 0$ we get also

$$r - r_- \leqslant \|\eta(s_2, u) - \eta(s_1, u)\|_{\mathcal{U}_{\bar{i}}} \leqslant \int_{s_1}^{s_2} \|\mathcal{W}(\eta(s, u))\|_{\mathcal{U}_{\bar{i}}} \leqslant 2(s_2 - s_1)$$

from which $\varphi_{j_{\bar{i}},\bar{i}}(\eta(s_2, u)) \leqslant c_{+, j_{\bar{i}}} - (1/2)\mu(r - r_-) < c_{-, j_{\bar{i}}}$. By $(W_2)$ we get that $\eta(s, u) \in \varphi_{j_{\bar{i}},\bar{i}}^{c_-, j_{\bar{i}}}$ for any $s \geqslant s_2$ and in particular that $\eta(\tau, u) \in \varphi_{j_{\bar{i}},\bar{i}}^{c_-, j_{\bar{i}}}$. ■

LEMMA 5.4. *For any* $\theta \in Q$ *there is* $\bar{i} \in \{1, \ldots, k\}$ *such that* $\varphi_{j_{\bar{i}},\bar{i}}(\bar{G}(\theta)) \leqslant c_{-, j_{\bar{i}}}$.

PROOF. Assume first that $G(\theta) \in \mathcal{B}_{r_-}(V_n; p; J)$. Then, since by construction $G(\theta) \in \bigcap_{i=1}^{k} \varphi_{j_i, i}^{c_+, j_i}$, we obtain the result by Lemma 5.3.

In the other case there exists $\bar{i} \in \{1, \ldots, k\}$ such that

$$r_n - \frac{1}{2} d_{r_n} \leqslant r_- \leqslant \|G(\theta) - v_n^{j_{\bar{i}}}(\cdot - p_{\bar{i}})\|_{\mathcal{U}_{\bar{i}}} = \\ = \|\gamma_{j_{\bar{i}}}(\theta_{\bar{i}})(\cdot - p_{\bar{i}}) - v_n^{j_{\bar{i}}}(\cdot - p_{\bar{i}})\|_{\mathcal{U}_{\bar{i}}} \leqslant \|\gamma_{j_{\bar{i}}}(\theta_{\bar{i}}) - v_n^{j_{\bar{i}}}\|$$

so using the properties (iii) and (iv) of $\gamma_{j_{\bar{i}}}$ we get

$$\varphi_{j_{\bar{i}},\bar{i}}(G(\theta)) = \varphi_{j_{\bar{i}},\bar{i}}(\gamma(\theta_{\bar{i}})(\cdot - p_{\bar{i}})) = \varphi_{j_{\bar{i}}}(\gamma(\theta_{\bar{i}})) \leqslant \bar{c}_{j_{\bar{i}}} - \frac{1}{2} h_n \leqslant c_{-, j_{\bar{i}}}.$$

By Lemma 5.3 we then have that $\eta(s, G(\theta)) \in \varphi_{j_{\bar{i}},\bar{i}}^{c_-, j_{\bar{i}}}$ for any $s \geqslant 0$ and the lemma follows. ■

Now we can conclude the proof of property (viii). We proceed by contradiction assuming the contrary. That is, for any $i \in \{1, \ldots, k\}$ the set $D_i = (\varphi_{j_i, i} \circ \bar{G})^{-1}([c_{-, j_i} + \varepsilon, +\infty[)$ separates $F_i^0$ from $F_i^1$ in $Q$. Let $C_i$ be the component of $Q \backslash D_i$ containing $F_i^1$ and let $\sigma_i \colon Q \to \mathbb{R}$ be the function given by

$$\sigma_i(\theta) = \begin{cases} \operatorname{dist}(\theta, D_i) & \text{if } \theta \in Q \backslash C_i\,, \\ -\operatorname{dist}(\theta, D_i) & \text{if } \theta \in C_i\,. \end{cases}$$

Then $\sigma_i$ is a continuous function on $Q$ such that $\sigma_i|_{F_i^0} \geqslant 0$, $\sigma_i|_{F_i^1} \leqslant 0$ and $\sigma_i(\theta) = 0$ if and only if $\theta \in D_i$. Using a theorem by Carlo Miranda (see [17]) we get that there exists $\theta \in Q$ such that $\sigma_i(\theta) = 0$ for all $i \in \in \{1, \ldots, k\}$ which means that $\bigcap_{i=1}^{k} D_i \neq \emptyset$. But this is in contrast with Lemma 5.4. ■

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