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On a Question of Deaconescu About Automorphisms. - II.

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This note is concerned with groups G satisfying the property that $\mathrm{Aut}\, H \cong N_G(H)/C_G(H)$ for all subgroups H of G. Such groups were referred to in [3] as MD-groups. It was shown there that the only nontrivial finite MD-groups are $\mathbb{Z}_2$ and S_3, and Theorem 2.1 of the same paper gave a complete classification of infinite, metabelian MD-groups. Our first result here shows that all (infinite) soluble MD-groups are in fact metabelian. Indeed, rather more than this is proved. Recall that a group G is a *radical* group if the iterated series of Hirsch-Plotkin radicals reaches G. (Thus, for instance, locally nilpotent groups and hyperabelian groups are radical.) The class of radical groups is certainly closed under forming sections—in particular, every finite section of such a group is soluble.

We shall prove the following:

THEOREM 1. *Let G be an infinite, radical MD-group. Then G is metabelian and is thus an extension of a torsion-free, locally cyclic subgroup A having finite type at every prime p by a cyclic group $\langle x \rangle$ of order 2, such that $a^x = a^{-1}$ for all $a \in A$.*

As a consequence, we have:

COROLLARY. Let G be an infinite locally soluble MD-group. Then G is metabelian.

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Our other result concerns periodic MD-groups, and shows that $\mathbb{Z}_2$ and S_3 are the only such groups which are nontrivial.

THEOREM. 2. *Let G be a periodic MD-group. Then G is finite.*

PROOF OF THEOREM 1. In view of Theorem 2.1 of [3] (which also shows that a group G having the structure described is an MD-group) all we need do is show that an infinite radical MD-group is metabelian. Let G be a such a group. The automorphism group of each of the following has a finite insoluble section: $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z}_p \times \mathbb{Z}_p$ for $p > 3$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. It follows that every abelian subgroup of G has finite rank and hence, by a theorem of Baer and Heineken (see p. 89 of [5]), that G has finite rank. Since $\operatorname{Aut} C_{p^\infty}$ is uncountable, we see that every periodic abelian subgroup of G is reduced.

Now let H be the Hirsch-Plotkin radical of G and let P be a Sylow p-subgroup of H, for some prime p. Then H is hypercentral and P is reduced and hence finite. (For the relevant facts about locally nilpotent groups of finite rank, see Section 6.3 of [5].) If $p > 3$, then, by what was said in the previous paragraph, the centre of P is cyclic and, if nontrivial, contains a G-invariant cycle $\langle z \rangle$ of order p. Since $\operatorname{Aut} \mathbb{Z}_p$ is cyclic and of order $p-1$, while G/G' has exponent (at most) 2, by 1.3 of [3], we have a contradiction. Thus $P = 1$, and the torsion subgroup of H is a finite $(2, 3)$-group. It follows easily (see [5]) that H is nilpotent. If H is finite then so is G, contrary to hypothesis. Thus the centre Z of H contains an element a of infinite order. Since the torsion subgroup T of Z is finite, we may assume that $\langle a \rangle \lhd G$.

Suppose next that b is some nontrivial element of T. Then the map $a \to ab$, $t \to t$, for all $t \in T$, determines an automorphism α of $\langle a \rangle \times T = B$, say. Now $\operatorname{Aut} B$ is finite and so, by the MD-property, the natural embedding of $G/C_G(B)$ in $\operatorname{Aut} B$ is an isomorphism. So α is induced by conjugation by some element g of G, contradicting the fact that $\langle a \rangle \lhd G$. Thus Z and hence H is torsion-free. Further, because $\mathbb{Z} \times \mathbb{Z}$ does not embed in G we see that H is locally cyclic. It follows that $[H, G'] = 1$ since $\operatorname{Aut} H$ is abelian. But, in a radical group, the centraliser of the Hirsch-Plotkin radical H is contained in H (see [4, Lemma 2.32]). The result follows.

PROOF OF THE COROLLARY. Suppose that the MD-group G is locally soluble. We notice from the proof of Theorem 1 that every abelian subgroup of G has *bounded* rank (at most 3) and hence, by a result of Merzlyakov (see p. 89 of [5]), G has finite rank. But a locally soluble group with finite rank is hyperabelian ([5, Lemma 10.39]) and the result follows from Theorem 1.

PROOF OF THEOREM 2. Let G be a periodic MD-group and let p be any prime such that G contains an element of order p. Every infinite elementary abelian p-group has an automorphism of infinite order. The same is true of C_{p^∞} and so every abelian p-subgroup of G is finite. It follows that every locally finite p-subgroup is finite, and thus we may choose a maximal finite p-subgroup P of G. Let N, C respectively denote the normaliser and centraliser of P in G and assume that P does not have prime order. Then $\operatorname{Out} P$ contains an element of order p [1] and thus, if X/C is any Sylow p-subgroup of N/C, then $|X/C| > |P/Z(P)|$. Choosing X so that $P \leqslant X$, we see that there exists $x \in X$ such that $x^p \in PC$ but $x \notin PC$. Since G is periodic, we may assume that x is a p-element. But then $\langle x, P \rangle$ is a p-subgroup properly containing P, a contradiction. Thus all (locally) finite p-subgroups have prime order.

Next, suppose that G has a section K/L of order 4. Certainly K/L is not cyclic, since G has no elements of order 4, and so $K = \langle L, a, b \rangle$, where each of a and b has order 2. Now $\langle a, b \rangle$ is a periodic image of the infinite dihedral group and therefore it is finite and contains a 2-subgroup of order at least 4, a contradiction. We deduce that G has no subgroups H such that $\operatorname{Aut} H$ contains a subgroup of order 4. In particular G contains no subgroups of type $\mathbb{Z}_p \times \mathbb{Z}_q$ where p and q are odd primes. Since G itself contains no subgroups of order 4, it also cannot contain a subgroup of type $\mathbb{Z}_p \times \mathbb{Z}_2$. It follows that every abelian subgroup of G is of prime order. We now claim that G contains no elements of order p for any prime $p > 3$. Suppose (for a contradiction) that G has a subgroup $\langle a \rangle$ of order p, where p is the least such prime greater than 3. Then $C_G\langle a \rangle = \langle a \rangle$ and so the normaliser N of $\langle a \rangle$ has order $p(p-1)$. By the minimality of p, we see that $p - 1 = 2^k 3^l$ for some k, l and, since the Sylow subgroups of N are of prime order, we have $k = l = 1$ and $p = 7$. But $\operatorname{Aut} \mathbb{Z}_7 = \mathbb{Z}_6$. and G has no elements of order 6. This contradiction establishes our claim. We now see that every finite subgroup of G has order at most 6 and (of course) that G has exponent dividing 6. By M. Hall's solution of the Burnside problem for exponent six [2], G is locally finite and therefore finite. The result follows.

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