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A Collar Neighborhood Theorem for a Complex Manifold.

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SUMMARY - For a real paracompact smooth manifold D with smooth boundary M the collar neighborhood theorem is well known. But for an intrinsically defined complex manifold D with a smooth boundary M, there is no such analogous theorem (see [7], [8]). This is closely related to the failure, in general, of the Newlander-Nirenberg theorem up-to-the boundary (see [9], [10]); which can occur in the presence of some pseudoconcavity of M. However the up-to-the boundary version of the Newlander-Nirenberg theorem is valid if the boundary M is strictly pseudoconvex (see [5]), or even when M is weakly pseudoconvex (see [3]). This is of course a local result near a boundary point $p \in M$. Thus the question arises as to when these local extensions, of the complex structure of D across M, can be pieced together to give a global collar neighborhood whose complex structure is an extension of the complex structure from D. We show here that it can be done when the boundary M is strictly pseudoconvex. When $\dim_C D = 1$, there is no condition at all required on M. Of course when D is a real analytic manifold with real analytic boundary M, and the integrable almost complex structure on D is also real analytic up-tp-the boundary M, then the collar neighborhood exists without any assumption about the Levi convexity of M. This follows by the identity theorem from complex analysis.

1. Existence of the collar neighborhood.

Let Ω be a paracompact (*i.e.* countable at infinity) smooth manifold of dimension $2n$, $n \geqslant 2$, and let D be an open domain in Ω, with a smooth boundary $M = bD$.

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We assume tht M is a closed connected differentiable real submanifold of Ω, of dimension $2n-1$ and countable at infinity.

Let $J_0: T\Omega \to T\Omega$ be a smooth almost complex structure on Ω, formally integrable on $\overline{D}$. Then we have the following.

THEOREM 1. *Assume that $M = bD$ is strictly pseudoconvex for the structure J_0. Then we can find an open submanifold ω of Ω, containing $\overline{D}$, and a complex structure $J: T\omega \to T\omega$, such that $J|D = J_0|D$.*

PROOF. The statement follows by an argument which uses Zorn's lemma and the local Newlander-Nirenberg theorem up-to-the boundary (see [5], [3]).

We introduce the family $\mathfrak{X}$ of pairs (X, J), where X is an open submanifold of Ω containing D, and $J: TX \to TX$ an integrable almost complex structure on X such that $J|D = J_0|D$. As $(D, J_0|D) \in \mathfrak{X}$, the family $\mathfrak{X}$ is non-empty.

On $\mathfrak{X}$ we define an equivalence relation by setting

$$(X_1, J_1) \sim (X_2, J_2)$$

iff

(i) $X_1 \cap M = X_2 \cap M$;

(ii) there is an open neighborhood G_{X_1, X_2} of $X_1 \cap M = X_2 \cap M$ in Ω such that $J_1 | G_{X_1, X_2} = J_2 | G_{X_1, X_2}$.

We denote by $\tilde{\mathfrak{X}}$ the quotient $\mathfrak{X}/\sim$ and by $[X, J]$ the equivalence class of $(X, J) \in \mathfrak{X}$ in $\tilde{\mathfrak{X}}$.

In $\tilde{\mathfrak{X}}$ we define an order relation $<$ by setting

$$[X_1, J_1] < [X_2, J_2]$$

iff:

(*a*) $X_1 \cap M \subsetneqq X_2 \cap M$;

(*b*) J_1 and J_2 agree on an open neighborhood G_{X_1, X_2} of $X_1 \cap M$ in $X_1 \cap X_2$.

We want to show that $\tilde{\mathfrak{X}}$ is inductive; *i.e.* that every chain in $\tilde{\mathfrak{X}}$ has an upper bound in $\tilde{\mathfrak{X}}$. Le $\mathcal{C}$ be a chain in $\tilde{\mathfrak{X}}$ for the ordering $<$. If $\mathcal{C}$ is finite, it has a maximum, which is therefore a majorant of $\mathcal{C}$. Assume now that $\mathcal{C}$ is infinite. Let

$$M_0 = \cup \{X \cap M \mid [X, J] \in \mathcal{C}\}.$$

This is an open subset of M. Let $\mathcal{W} = \{W_\nu \mid \nu \in N\}$ be a countable open covering in Ω of M_0 which is locally finite, and with $\overline{W}_\nu$ compact for every ν, and $\overline{W}_\nu \cap M \subset M_0$. We define by recurrence

$$\mathcal{W}_0 = \{W_0\}$$

$$\mathcal{W}_{k+1} = \left\{W_\nu \in \mathcal{W} \mid W_\nu \cap \cup \mathcal{W}_k \neq \emptyset,\ W_\nu \notin \bigcup_{j \leq k} \mathcal{W}_j\right\}.$$

For

$$\widetilde{W}_k = \cup \mathcal{W}_k$$

we have

(1) $\overline{\widetilde{W}_k}$ is compact;

(2) $\widetilde{W}_k \cap \widetilde{W}_{k+2} = \emptyset,\ \forall k$;

(3) $\cup \widetilde{W}_k \cap M = M_0$.

To construct a majorant for the chain $\mathcal{C}$ we proceed in the following way. We set

$$U_k = \widetilde{W}_k \cap M .$$

We note that $\overline{U}_k$ is a compact subset of M_0 for every k and then we can find $(X_1, J_1) \in \mathfrak{X}$ with $[X_1, J_1] \in \mathcal{C}$ such that

$$\overline{U}_0 \cup \overline{U}_1 \subset X_1 \cap M .$$

Let $V_0 \subset \widetilde{W}_0 \cap X_1$ be an open neighborhood of U_0 in Ω and let us set $\omega_1 = D \cup V_0$. We define an integrable almost complex structure on ω_1 by $\widetilde{J}_1 \colon T\omega_1 \to T\omega_1$ being the restriction of J_1 to $D \cup V_0 \subset X_1$.

Next we choose $(X_2, X_2) \in \mathfrak{X}$ with $[X_2, J_2] \in \mathcal{C}$ and $[X_1, J_1] \prec [X_2, J_2]$ such that

$$\overline{U}_0 \cup \overline{U}_1 \cup \overline{U}_2 \subset X_2 \cap M .$$

By point (*b*) in the definition of the order relation, we can find an open neighborhood G_{X_1, X_2} of $X_1 \cap M$ in Ω such that $G_{X_1, X_2} \subset X_1 \cap X_2$ and $J_1 \mid G_{X_1, X_2} = J_2 \mid G_{X_1, X_2}$.

Then we can find an open neighborhood V_1 of U_1 in $\widetilde{W}_1 \cap G_{X_1, X_2}$. We set $\omega_2 = \omega_1 \cup V_1$ and we can define on ω_2 an integrable almost complex structure $\widetilde{J}_2 \colon T\omega_2 \to T\omega_2$ by $\widetilde{J}_2 \mid \omega_1 = \widetilde{J}_1$ and $\widetilde{J}_2 \mid V_1 = J_2 \mid V_1 = J_1 \mid V_1$.

By recurrence we prove the following: for every ν we can find

(α) an open neighborhood V_ν of U_ν in $\tilde{W}_\nu$;

(β) an element $(X_\nu, J_\nu) \in \mathfrak{X}$ with $[X_\nu, J_\nu] \in \mathcal{C}$, such that $[X_{\nu-1}, J_{\nu-1}] < [X_\nu, J_\nu]$;

(γ) an open neighborhood $G_{X_{\nu-1}, X_\nu}$ of $M \cap X_{\nu-1}$ in $X_{\nu-1} \cap X_\nu$ on which $J_{\nu-1} = J_\nu$ such that

$$V_\nu \subset G_{X_{\nu-1}, X_\nu},$$

$$\overline{U}_0 \cup \ldots \cup \overline{U}_{\nu+1} \subset X_\nu .$$

Because $V_\nu \cap V_{\nu+j} = \emptyset$ for $j \geqslant 2$, if we set

$$X = D \cup V_0 \cup V_1 \cup \ldots$$

and we define $J\colon TX \to TX$ by

$$J | D_0 = J_0 ,$$

$$J | V_\nu = J_\nu | V_\nu ,$$

we obtain an integrable almost complex structure on X.

We have $(X, J) \in \mathfrak{X}$, $X \cap M = M_0$ and for each $[Y, J_Y] \in \mathcal{C}$, the structures J and J_Y agree by construction on a neighborhood of $Y \cap M$.

Hence $[X, J]$ is a majorant of $\mathcal{C}$.

By Zorn's lemma, $\tilde{\mathfrak{X}}$ contains a maximal element $[\omega, J]$. We need to prove that $\omega \supset M$.

Let $M_0 = \omega \cap M$ and suppose to the contrary, that $M_0 \neq M$. Let $\rho\colon \Omega \to \boldsymbol{R}$ be a defining function for D in Ω, *i.e.* we assume that $\rho < 0$ on D, $\rho = 0$ on M, $d\rho \neq 0$ on M and $\rho > 0$ on $\Omega - \overline{D}$.

Let $\{\varphi_\nu\}$ be a partition of unity on a neighborhood of M_0 in ω, with $\varphi_\nu \geqslant 0$, and $\operatorname{supp} \varphi_\nu$ compact for every ν. Then, for a suitable choice of a sequence $\{\varepsilon_\nu\}$ of positive real numbers,

$$\tilde{D} = \{\rho < \textstyle\sum \varepsilon_\nu \varphi_\nu\}$$

is an open neighborhood of D in Ω, with $M_0 \subset \tilde{D}$ and $b\tilde{D}$ smooth and strictly pseudoconvex for the extension of the integrable almost complex structure J to $\overline{\tilde{D}}$.

It $p \in M - M_0$, then p is a boundary point of $\tilde{D}$ and then, by the Newlander-Nirenberg theorem up to the boundary, we can find an open submanifold B of Ω, containing $\tilde{D} \cup \{p\}$, on which a complex structure J' is defined, extending the complex structure J on $\tilde{D}$. But then $[\omega, J] = [\tilde{D}, J|\tilde{D}] < [B, J']$ and this gives a contradiction to the fact that $[\omega, J]$ was maximal in $\tilde{\mathfrak{X}}$. Therefore $\omega \supset M$ and the proof is complete.

2. Remarks.

1) When $\dim_C D = 1$, so $\dim_R M = 1$, one can take any smooth extension J of the almost complex structure $J_0 \,|\overline{D}$ some open neighborhood $\omega \supset \overline{D}$. This J is then a complex structure on ω since there is no formal integrability requirement in complex dimenson one. Thus Theorem 1 holds without any condition on M.

2) Suppose $\dim_C D \geqslant 2$ and, instead of assuming that M is strictly pseudoconvex, we assume that at every point of M the Levi form has at least one negative eigenvalue. Then we cannot appeal to the up-to-the boundary version of the Newlander-Nirenberg theorem, because there are known counterexamples (see [8]). So let us assume instead that M is locally embeddable at each point. The existence of a collar neighborhood (ω, J) of $(D, J_0 \,|D)$, as in Theorem 1, then follows by a result of Dwilewicz [4].

3) In fact, we can do away with the global hypothesis that M be the abstract boundary of a complex manifold D, as in Theorem 1, and replace it by a microlocal hypothesis: let M be a smooth paracompact (*i.e.* countable at infinity) abstract strictly pseudoconvex CR manifold (of hypersurface type). Consider the following condition:

(A) For every p on M, the given CR structure on M has a local extension to the germ of a complex structure on the pseudoconvex side of M. Here the extension is intended in the sense of an abstract boundary; *i.e.*, there is a local smooth integrable almost complex structure which extends the CR structure to the pseudoconvex side near each point.

We ask the question: does M have a global embedding as a closed CR hypersurface in some open complex manifold X? Assume M has a real dimension $2n - 1$ with $n > 1$.

THEOREM 2. *M has such a global embedding if and only if the microlocal condition (A) is satisfied.*

PROOF. The condition is obviously necessary. To show it is also sufficient, first note that by the Newlander-Nirenberg result up to the boundary (see Hanges-Jacobowitz [5]), we have that M is locally embeddable at each point. Hence we have the Hans Lewy local extension of CR functions to the pseudoconvex side of M. If follows that the local extensions of the CR structure piece together, in effect, producing our D from Theorem 1. Rather than go into details, we refer the reader to Dwilewicz [4], where this type of argument is treated very explicitly. Then we apply our Theorem 1.

4) Going back to the situation of section 1, suppose that our strictly pseudoconvex M is compact, and forms the abstract boundary of an open Stein manifold D. Then it follows from the work of Andreotti and Grauert [1] that $\bar{D}$ has a Stein neighborhood in the collar. A related result, for the case where the boundary M of D is assumed in the concrete sense, was found by Heunemann [6]. Let us for convenience now take $\dim M = 2n + 1$ with $n > 0$. We may then apply the well-known results (see Narasimhan [11] and Bishop [2]) and conclude that M has a global closed CR embedding in $\mathbf{C}^{2n+3}$ and a global closed CR immersion in $\mathbf{C}^{2n+2}$. But this does not give the best result. Indeed we have

THEOREM 3. *Let M be a smooth compact stricly pseudoconvex* CR *manifold (of hypersurface type) with* $\dim M = 2n + 1$, $n \geqslant 1$. *Then*

($n \geqslant 2$): *M has a global closed* CR *embedding in $\mathbf{C}^{2n+2}$ and a global closed* CR *immersion in $\mathbf{C}^{2n+1}$.*

($n = 1$): *M^3 has a global closed* CR *embedding in $\mathbf{C}^4$ and a global closed* CR *immersion in $\mathbf{C}^3$ provided that M^3 forms the abstract boundary of an open Stein surface D^2.*

PROOF. When $\dim M \geqslant 5$, it follows by the theorem of Boutet de Monvel (*) that M has a global closed CR embedding in $\mathbf{C}^n$, for some N. When $\dim M \geqslant 3$, we get such an embedding into $\mathbf{C}^N$ by using the remark just before Theorem 3. So in any case we get an embedding in $\mathbf{C}^N$, for some N. It then suffices to take a generically chosen holomorphic projection into $\mathbf{C}^{2n+2}$ or $\mathbf{C}^{2n+1}$. For further details, see the more general Proposition at the end of Hill-Nacinovich (**).

(*) L. Boutet de Monvel, *Intégration des équations de Cauchy-Riemann induites formelles,* Sem. Goulaouic-Lions-Schwartz, 9 (1975).

(**) Hill-Nacinovich, *The Topology of Stein* CR *Manifolds and the Lefschetz Theorem,* Ann. Inst. Fourier, Grenoble, 43, 2 (1993), pp. 459-468.

3. On the non-uniqueness of the collar neighborhood.

Let us first consider the case $n = 1$. On $C - [1, \infty]$ we denote by $\alpha(z)$ the branch of $\sqrt{1 - z}$ with positive real part.

Then we consider on the closed unit disc $\overline{D} = \{ |z| \leqslant 1 \rangle$ the function

$$\varphi(z) = \begin{cases} Az + \exp\left(- \dfrac{1}{\alpha(z)} \right) & \text{for } z \neq 1\,, \\ A & \text{for } z = 1\,. \end{cases}$$

For every $A \in C$ this defines a Whitney function on the closed disc. For $|A|$ large it is a biholomorphism of the open disc D onto an open domain G of C. By Whitney's theorem, for large A, φ extends to a diffeomorphism $\tilde{\varphi}$ of a neighborhood U of $\overline{D}$ in C onto a neighborhood V of $\overline{G}$ in C.

Then we consider the two complex structures on U defined by the single coordinate patch (U, z) and $(U, \tilde{\varphi}(z))$ respectively. We claim that the two structures do not agree on any neighborhook of 1, while they obviously agree on the open disc D and hence on $\overline{D}$. Indeed, $\varphi | D$ is holomorphic on D for both structures, but has no analytic extension beyond 1 for the first one, as $Az - \varphi$ would then extend to a non-zero analytic function flat at 1. It obviously extends for the second structure, being the restriction to D of the coordinate function.

We can now easily construct examples of non-uniqueness in several variables. If $n > 1$, denoting by e_1 the vector $(1, 0, \ldots, 0)$ in C^n, we consider $\Omega = \{ z \in C^n \mid |z - e_1/2| < 1/2 \}$. With $D = \{ t \in C \mid |t| < 1 \}$ and $U, \tilde{\varphi}$ as above we realize that the two structures defined on the neighborhood $\tilde{\Omega} = U \times D^{n-1}$ of $\overline{\Omega}$ by the single coordinate patch $(\tilde{\Omega}, z^1, \ldots, z^n\}$ and $(\tilde{\Omega}, \tilde{\varphi}(z^1), z^2, \ldots, z^n)$ respectively cannot possibly agree on an open neighborhood of e_1.

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