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A Property of the Variety of 2-Engel Groups.

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Introduction.

Suppose that \mathfrak{V} is a variety of groups defined by the law $w(x_1, \dots, x_n) = 1$, and assume that n is the least number of variables required to determine \mathfrak{V} . Following [KRS] we denote by \mathfrak{V}^* the class of groups G satisfying the following property:

«For every n infinite subsets X_1, \dots, X_n of G , there exist elements x_i in X_i , $i = 1, \dots, n$, such that the subgroup generated by $\{x_1, \dots, x_n\}$ is a \mathfrak{V} -group».

Clearly all finite groups satisfy the property for any \mathfrak{V} . The question we are interested in is:

«For which varieties \mathfrak{V} is every infinite \mathfrak{V}^* -group a \mathfrak{V} -group?»

For example, if \mathfrak{V} is the variety \mathfrak{A} of the abelian groups, then the law defining \mathfrak{A} is $w(x, y) = [x, y] = 1$, and, by definition of \mathfrak{A}^* , for every pairs X, Y of infinite subsets of $G \in \mathfrak{A}^*$, there exist $x \in X$, $y \in Y$ such that $xy = yx$. It follows, from a theorem proved by B. H. Neumann in [N], that G is centre-by-finite, so that $Z(G)$ is infinite. For any $x, y \in G$, we consider the infinite subsets $Z(G)x, Z(G)y$. By hypothesis we can find $z_1, z_2 \in Z(G)$ such that $1 = [z_1x, z_2y] = [x, y]$, so that $G \in \mathfrak{A}$.

Problems of similar nature are discussed in [KRS], where the variety \mathfrak{V} considered is the class \mathfrak{A}^2 of metabelian groups, and in [RS], where the authors studied the classes of locally nilpotent, locally soluble and locally finite groups. Furthermore in [LMR], the authors answer the question affirmatively, using a considerably weaker hypothe-

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sis, when \mathfrak{V} is the variety of nilpotent groups of nilpotency class $n - 1$. In fact they assume only that $[x_1, \dots, x_n] = 1$ instead of supposing that $\langle x_1, \dots, x_n \rangle$ is nilpotent of class $n - 1$. In the present paper we establish a positive result for the class \mathfrak{E}_2 of 2-Engel groups, by proving the following:

THEOREM. *Let G be an infinite group. If for every pair X, Y of infinite subsets of G there exist some x in X and y in Y such that $[x, y, y] = 1$, then G is a 2-Engel group.*

Our notation and terminology are standard (see for instance [Ro]). We shall write \mathfrak{E}_k^* to denote the class of groups G for which, whatever X, Y are infinite subsets of G , there exist x in X and y in Y such that $[x, y, \dots_k \dots, y] = 1$. Thus our Theorem states that $\mathfrak{E}_2^* = \mathfrak{E}_2 \cup \mathcal{F}$, where \mathcal{F} is the class of all finite groups. The proof we give relies upon a lemma proved in [S] which we restate here below for the reader's convenience:

LEMMA. *Let G be an infinite group in \mathfrak{E}_k^* . Then $C_G(x)$ is infinite for every x in G .*

Proofs.

We will need some preliminary results before proving our statement. The first of these is actually a straightforward consequence of the above Lemma.

LEMMA 1. *If G is an infinite group in the class \mathfrak{E}_k^* , then for any $x \in G$ there exists an infinite abelian subgroup A of G containing x .*

Furthermore we point out that:

REMARK. *If G is in \mathfrak{E}_k^* and its centre $Z(G)$ is infinite, then for any $x, y \in G$ the subsets $xZ(G), yZ(G)$ are infinite, hence there are $z_1, z_2 \in Z(G)$ such that:*

$$1 = [xz_1, yz_2, \dots_k \dots, yz_2] = [x, y, \dots_k \dots, y] \quad \forall x, y \in G.$$

Therefore G is a k -Engel group.

LEMMA 2. *Let $G = \langle y, A \rangle$ be an infinite group in \mathfrak{E}_2^* , where A is an infinite abelian subgroup of G . Then there exists an infinite subset T of the set $B = \{a \in A \mid [a, y, y] = 1\}$ such that $t_1 t_2^{-1} \in B$ for any t_1, t_2 in T .*

PROOF. Consider the set $Y = \{y^a \mid a \in A\}$. If Y is finite, then the index $|A : C_A(y)|$ is finite too, hence $C_A(y)$ is infinite and contained in the centre of G , $Z(G)$. This means that $Z(G)$ is infinite, and, by the previous remark, G is a 2-Engel group. In this case we choose $T = B = A$.

So we may assume, without loss of generality, that Y is infinite. Suppose now that the set $A \setminus B$ is infinite and consider the two infinite sets Y and $A \setminus B$. By hypothesis there are elements $a \in A \setminus B$ and $b \in A$ such that $1 = [a, y^b, y^b] = [a, y, y]$. But this is a contradiction since a is not in B . Thus $A \setminus B$ has to be finite and B is an infinite subset of A .

If A has a torsion-free element a , then it is possible to construct an infinite strictly decreasing chain of infinite subgroups of A

$$A \geq \langle a \rangle > \langle a^2 \rangle > \dots > \langle a^{2^n} \rangle > \dots$$

Since $A \setminus B$ is finite, there exists $n \in \mathbb{N}$ such that $\langle a^{2^n} \rangle$ is completely contained in B . Then we set $T = \langle a^{2^n} \rangle$. We have now to examine what happens when A is a torsion group. In this case the subgroup H generated by $A \setminus B$ is finite, and A/H is infinite. Choose any transversal T for H in A containing 1. This is an infinite subset of A contained in B and, for any pair of distinct elements of T , t_1, t_2 , we have $t_1 t_2^{-1} \notin H$. Since $1 \in T$, we have $t_1 t_2^{-1} \in B$, for every t_1, t_2 in T . This proves our claim.

We are now in a position to prove the theorem stated in the introduction.

THEOREM. *If G is an infinite group in the class \mathcal{E}_2^* , then G is a 2-Engel group.*

PROOF. Our purpose is to show that $[x, y, y] = 1$, for every x, y in G . By Lemma 1 we may assume, without loss of generality, that G is the group generated by y and A , where A is an infinite abelian subgroup of G containing x , i.e. $G = \langle y, A \rangle$.

If we consider the subset $B = \{a \in A \mid [a, y, y] = 1\}$ of A , Lemma 2 guarantees the existence of an infinite subset T of B such that for any $t_1, t_2 \in T$, $t_1 t_2^{-1} \in B$. Set $\bar{T} = \{y^t \mid t \in T\}$ and consider the following two cases:

Case 1. \bar{T} finite. Since T is contained in the union of finitely many cosets of $C_A(y)$, it follows that $C_A(y)$ is infinite. Thus the centre of G , containing $C_A(y)$, is infinite too and the claim follows from the Remark.

Case 2. \bar{T} infinite. We will show that $[a, y, y] = 1$ for every a in A . The subsets $a\bar{T}$, \bar{T} of G are infinite for every $a \in A$ and, therefore, we can find $t_1, t_2 \in T$ such that $1 = [ay^{t_1}, y^{t_2}, y^{t_2}]$. But

$$\begin{aligned} [ay^{t_1}, y^{t_2}, y^{t_2}] &= [(ay)^{t_1}, y^{t_2}, y^{t_2}] = [(ay)^{t_1 t_2^{-1}}, y, y]^{t_2} = \\ &= [ay^{t_1 t_2^{-1}}, y, y] = [[a, y]^{y^{t_1 t_2^{-1}}}, y][y^{t_1 t_2^{-1}}, y, y]. \end{aligned}$$

Now we notice that, since $t_1 t_2^{-1}$ is in B , $y^{t_1 t_2^{-1}} \in C_G(y)$ for every t_1, t_2 in T . Hence $[y^{t_1 t_2^{-1}}, y, y] = 1$, and $y^{y^{(t_1 t_2^{-1})^{-1}}}$, so that we have

$$1 = [[a, y]^{y^{t_1 t_2^{-1}}}, y] = [a, y, y] \quad \forall a \in A,$$

and the theorem is proved.

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