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Automorphisms of p-Groups with Cyclic Commutator Subgroup.

FEDERICO MENEGAZZO (*)

ABSTRACT - We study the automorphism groups of finite, non abelian, 2-generated p-groups with cyclic commutator subgroup, for odd primes p. We exhibit presentations of the relevant groups, and compute the orders of $\mathrm{Aut}\, G$, $O_p(\mathrm{Aut}\, G)$, and of the linear group induced on the factor group $G/\Phi(G)$.

In this paper we give a systematic account of the automorphism groups of finite, non abelian, 2-generated p-groups with cyclic commutator subgroup, for odd primes p.

Special cases of this problem have of course been studied in connection with many questions, with the aim of providing examples and counterexamples; still, the general information available is remarkably scarce.

It is a remark by Ying Cheng [2] that in such groups G the central factor group $G/2(G)$ is metacyclic, hence modular; it follows that $|G|$ divides the order of $\mathrm{Aut}\, G$ [3]. Another known fact is that in any metabelian 2-generated p-group $G = \langle a, b \rangle$, for all choices of $x, y \in G'$, there is an automorphism α mapping a to ax and b to by [1]; moreover, if G' is cyclic and p is odd, such automorphisms are inner [2]. This implies that the order of $\mathrm{Inn}\, G$ is $|G'|^2$. $\mathrm{Aut}\, G$ naturally induces a group of linear transformations of the $\mathbb{Z}/p\mathbb{Z}$-vector space $G/\Phi(G)$, where $\Phi(G)$ is the Frattini subgroup; we denote this group—which in our case is a subgroup of $GL(2, \mathbb{Z}/p\mathbb{Z})$—by $\mathrm{Aut}_l G$, the l beeing a reminder of «linear». The kernel of this action, i.e.

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$\{\alpha \in \operatorname{Aut} G \mid g^{\alpha}\Phi(G) = \Phi(G), \forall g \in G\}$, is sometimes denoted by $\operatorname{Aut}^{\Phi}(G)$; for every p-group $\operatorname{Inn} G \leq \operatorname{Aut}^{\Phi}(G) \leq O_p(\operatorname{Aut} G)$.

We found it necessary to analise separately several cases; indeed, what we got is almost a classification (a classification of finite, non abelian, 2-generated p-groups with cyclic commutator subgroup, for odd primes p, has been given by Miech in [4]). For each case, we will exhibit presentations of the relevant groups and compute the orders of $\operatorname{Aut} G$, $O_p(\operatorname{Aut} G)$, $\operatorname{Aut}_l G$. In many instances, we have been able to display the effect of $\operatorname{Aut} G$ on two chosen generators of G; we hope that also in the remaining cases the information we provide is helpful.

1. In this section we will deal with metacyclic groups. Accordingly, we suppose that a group G has a cyclic normal subgroup $N = \langle b \rangle$ of order p^m, say, with cyclic factor group $G/N = \langle aN \rangle$ of order p^l. We may choose a such that $b^a = b^{1+p^s}$ for some s, $1 \leq s < m$. Since the order of $1 + p^s \bmod p^m$ is p^{m-s} we also have $m - s \leq l$. The centre is $Z(G) = \langle a^{p^{m-s}}, b^{p^{m-s}} \rangle$.

Suppose that G splits over N. Then $G = \langle a, b \mid a^{p^l} = b^{p^m} = 1, b^a = b^{1+p^s} \rangle$ is a presentation of G. If $\bar{b} = a^z b^w$ is a candidate for an $\operatorname{Aut} G$-image of b, we must have $\langle \bar{b}^{p^s} \rangle = G' = \langle b^{p^s} \rangle$ and $a^z = \bar{b} b^{-w} \in C_G(G')$. It follows that $(a^z b^w)^{p^s} = a^{zp^s} b^{wp^s} [b^w, a^z]^{\binom{p^s}{2}} \in \langle b^{p^s} \rangle$, and then $a^{zp^s} = 1$, $[b^w, a^z]^{\binom{p^s}{2}} = 1$, $(a^z b^w)^{p^s} = b^{wp^s}$, and $p \nmid w$. We also have $(a^z b^w)^a = a^z b^w b^{wp^s} = (a^z b^w)^{1+p^s}$: such a $\bar{b}$ is in fact in the $\operatorname{Aut} G$-orbit of b, and an automorphism mapping b to $\bar{b}$ and a to $\bar{a}$ exists if and only if $\bar{a}$ has order p^l and $\bar{b}^{\bar{a}} = \bar{b}^{1+p^s} = \bar{b}^a$, i.e. $\bar{a} \in a\Omega_l(C_G(\bar{b}))$, where $C_G(\bar{b}) = \langle \bar{b} \rangle \times \langle a^{p^{m-s}} \rangle$. We summarize our results:

(A.1) $$G = \langle a, b \mid a^{p^l} = b^{p^m} = 1, b^a = b^{1+p^s} \rangle$$

where $1 \leq s < m$ *and* $m - s \leq l$.

The effect of $\operatorname{Aut} G$ *on the generators* a, b *is*

$$\begin{cases} b \mapsto a^z b^w, \\ a \mapsto a a^{\lambda p^{m-s}} (a^z b^w)^{\mu}, \end{cases}$$

where $zp^s \equiv 0 \ (p^l)$, $w \not\equiv 0 \ (p)$, $\mu p^l \equiv 0 \ (p^m)$.

— *If* $l \geq m$:

$$|\operatorname{Aut} G| = (p-1)p^{l+m+2s-1}, \qquad |\operatorname{Inn} G| = p^{2(m-s)},$$

$$|O_p(\operatorname{Aut} G)| = p^{l+m+2s-1}, \qquad |\operatorname{Aut}_l G| = p(p-1).$$

— *If* $m > l > s$:

$$|\mathrm{Aut}\, G| = (p-1)p^{2l+2s-1}, \qquad |\mathrm{Inn}\, G| = p^{2(m-s)},$$

$$|O_p(\mathrm{Aut}\, G)| = p^{2l+2s-1}, \qquad |\mathrm{Aut}_l\, G| = p-1.$$

— *If* $s \geqslant l$:

$$|\mathrm{Aut}\, G| = (p-1)p^{3l+s-1}, \qquad |\mathrm{Inn}\, G| = p^{2(m-s)},$$

$$|O_p(\mathrm{Aut}\, G)| = p^{3l+s-1}, \qquad |\mathrm{Aut}_l\, G| = p(p-1).$$

Suppose now that in our metacyclic p-group G there is no cyclic normal subgroup N having a cyclic complement. G has a presentation $G = \langle a, b \,|\, b^{p^m} = 1,\ b^a = b^{1+p^s},\ a^{p^l} = b^{p^h}\rangle$. $1 \neq b^{p^h} \in Z(G)$ yields $m > h \geqslant m - s$; b cannot have maximum order, so $l > h$; for the same reason, $G' \nleq \langle a \rangle$, which means $s < h$. As a first approximation to the $\mathrm{Aut}\, G$-orbit of b, we look for elements $g = a^z b^w$ generating normal subgroups N of order p^m containing $G' = \langle b^{p^s}\rangle$. This happens if and only if $\langle g^{p^s}\rangle = \langle b^{p^s}\rangle$, *i.e.* $a^{zp^s} \in \langle a \rangle \wedge \langle b \rangle = \langle a^{p^l}\rangle$, or $a^z \in \langle a^{p^{l-s}}\rangle$, and $w \not\equiv 0\ (p)$. We have $[\langle a^{p^{l-s}}\rangle : \langle a^{p^l}\rangle]\,\phi(p^m) = p^s \phi(p^m)$ such elements g, where ϕ is the Euler function, which generate p^s subgroups N as above. For every choice of N, $G = \langle a, N\rangle$ and the automorphism group induced on N by conjugation in G is the group generated by the power $1 + p^s$; it is therefore possible to choose $\bar{a} \in G$ so that $G/N = \langle \bar{a}N\rangle$ and $g^{\bar{a}} = g^{1+p^s}$ for every generator g of N. The choice of $\bar{a}$ is not unique: all the elements in the coset $\bar{a}C_G(N)$ (and they only) share the same properties. The order of $C_G(N) = NC_{\langle a\rangle}(N) = N\langle a^{p^{m-s}}\rangle$ is p^{l+s}: once N is given, we have p^{l+s} possible choices for $\bar{a}$. Comparing the orders, for every such $\bar{a}$ we find $\langle \bar{a}\rangle \wedge N = \langle \bar{a}^{p^l}\rangle = N^{p^h}$: it is then possible to choose a generator $\bar{b}$ of N such that $\bar{a}^{p^l} = \bar{b}^{p^h}$. Again, the choice of $\bar{b}$ is not unique: the possible choices are the elements of the coset $\bar{b}\Omega_h(N)$. All told, the number of pairs $(\bar{a}, \bar{b})$ satisfying the given presentation of G is (p^s choices for N) times (p^{l+s} choices of $\bar{a}$) times (p^h choices of $\bar{b}$). In summary

(A.2) $$G = \langle a, b \,|\, b^{p^m} = 1,\ b^a = b^{1+p^s},\ a^{p^l} = b^{p^h}\rangle$$

where $1 \leqslant s < h < m$ *and* $m - s \leqslant h < l$.

$$|\mathrm{Aut}\, G| = p^{l+h+2s}, \ |\mathrm{Inn}\, G| = p^{2(m-s)}, \ |\mathrm{Aut}_l\, G| = p.$$

REMARK. For any $n \geqslant 6$, fix m such that $3 \leqslant m \leqslant n/2$, $l = n - m$, $s = 1$, $h = m - 1$: we get a group G of order p^n with $|\mathrm{Aut}\, G| = p^{n+1}$.

2. In this section we will deal with groups G which are not metacyclic, and have nilpotency class 2. We first fix the notation: a, b are generators for G such that $G/G' = \langle aG' \rangle \times \langle bG' \rangle$, aG' has order p^l and bG' has order p^m. Then $u = [b, a]$ generates G', u has order p^n with $1 \leqslant n \leqslant l$, $n \leqslant m$, and of course $u \in Z(G)$, $Z(G) = \langle a^{p^n}, b^{p^n}, u \rangle$.

Suppose first that $\langle b \rangle \wedge G' = \langle a \rangle \wedge G' = 1$. G has a presentation $G = \langle a, b, u \,|\, a^{p^l} = b^{p^m} = u^{p^n} = 1,\ b^a = bu,\ u^a = u^b = u \rangle$, and the elements of G can be uniquely written in the form $a^x b^y u^z$, $x \in \mathbb{Z}/p^l\mathbb{Z}$, $y \in \mathbb{Z}/p^m\mathbb{Z}$, $z \in \mathbb{Z}/p^n\mathbb{Z}$. We can also assume $l \geqslant m$. If θ is any automorphism of G/G' and we fix elements $\bar{a} \in (aG')^\theta$, $\bar{b} \in (bG')^\theta$ and set $\bar{u} = [\bar{b}, \bar{a}]$, it is immediately seen that $\langle \bar{a}, \bar{b} \rangle = G$ and $\bar{a}, \bar{b}, \bar{u}$ satisfy the relations, so that the assignment $a \mapsto \bar{a}$, $b \mapsto \bar{b}$, $u \mapsto \bar{u}$ extends to an automorphism of G. This means that the obvious homomorphism $\mathrm{Aut}\, G \to \mathrm{Aut}\,(G/G')$ is onto; its kernel is $\mathrm{Inn}\, G$, of order p^{2n}. The structure of $\mathrm{Aut}\,(G/G')$ is well known: if $l = m$ it is isomorphic with $GL(2, \mathbb{Z}/p^l\mathbb{Z})$, while if $l > m$ it is isomorphic to the group of all matrices $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ where $x \in \mathbb{Z}/p^l\mathbb{Z}$, $y, w \in \mathbb{Z}/p^m\mathbb{Z}$, $z \in p^{l-m}\mathbb{Z}/p^n\mathbb{Z}$, x and $w \not\equiv 0 \ (p)$.

We obtained:

(B.1) $G = \langle a, b, u \,|\, a^{p^l} = b^{p^m} = u^{p^n} = 1, b^a = bu, u^a = u^b = u \rangle$

with $l \geqslant m \geqslant n \geqslant 1$.

— *If* $l = m$:

$$|\mathrm{Aut}\, G| = p^{2n+4(l-1)+1}(p^2 - 1)(p - 1), \qquad |\mathrm{Inn}\, G| = p^{2n},$$

$$|O_p(\mathrm{Aut}\, G)| = p^{2n+4(l-1)}, \qquad |\mathrm{Aut}_l\, G| = p(p^2 - 1)(p - 1).$$

— *If* $l > m$:

$$|\mathrm{Aut}\, G| = p^{2n+3m+l-2}(p - 1)^2, \qquad |\mathrm{Inn}\, G| = p^{2n},$$

$$|O_p(\mathrm{Aut}\, G)| = p^{2n+3m+l-2}, \qquad |\mathrm{Aut}_l\, G| = p(p - 1)^2.$$

[Notice that, if $G/G' = \langle \bar{a}G' \rangle \times \langle \bar{b}G' \rangle$, then $\{\bar{a}, \bar{b}\} = \{a^\alpha, b^\alpha\}$ for some $\alpha \in \mathrm{Aut}\, G$, so in particular $\langle \bar{a} \rangle \wedge G' = \langle \bar{b} \rangle \wedge G' = 1$. This fact will avoid any overlapping with the discussion that follows.]

Suppose now that for no direct decomposition $G/G' = \langle aG' \rangle \times \langle bG' \rangle$ it is possible to choose a and b such that $\langle a \rangle \wedge G' = \langle b \rangle \wedge G' = 1$, but there is one such decomposition with $\langle a \rangle \wedge G' = 1$ (and $\langle b \rangle \wedge G' \neq 1$). G has a presentation $G = \langle a, b, u \mid a^{p^l} = u^{p^n} = 1, b^{p^m} = u^{p^k}, b^a = bu, u^a = u^b = u \rangle$, where $1 \leqslant k < n$ (we are no longer assuming $l \geqslant m$). If $\bar{b} = a^r b^s u^t$ is in the Aut G-orbit of b, then $\langle \bar{b}^{p^m} \rangle = (G')^{p^k} = \langle u^{p^k} \rangle$; this happens if and only if $a^{rp^m} = 1$ and $s \not\equiv 0\ (p)$. For such a $\bar{b}$ we have $[\bar{b}, a] = u^s$, $\bar{b}^{p^m} = b^{sp^m} = u^{sp^k} = [\bar{b}, a]^{p^k}$ and $G = \langle a, \bar{b} \rangle$, so that there is in fact an automorphism of G mapping b to $\bar{b}$ and fixing a. If θ is any automorphism of G mapping b to $\bar{b}$, then $[\bar{b}, a^\theta]^{p^k} = \bar{b}^{p^m} = [\bar{b}, a]^{p^k}$ and $(a^\theta)^{p^l} = 1$; so the condition on a^θ is $a^\theta \in a\Omega_l(C_G(\bar{b}^{p^k}))$, where $C_G(\bar{b}^{p^k}) = \langle a^{p^{n-k}}, \bar{b}, G' \rangle$, hence $\Omega_l(C_G(\bar{b}^{p^k})) = \langle a^{p^{n-k}} \rangle \Omega_l(\langle \bar{b} \rangle) G'$. We may now state

(B.2) $\quad G = \langle a, b, u \mid a^{p^l} = u^{p^n} = 1, b^{p^m} = u^{p^k}, b^a = bu, u^a = u^b = u \rangle$

where $l \geqslant n$, $m \geqslant n$, $1 \leqslant k < n$.

The effect of Aut G *on the generators* a, b *is*

$$\begin{cases} b \mapsto a^r b^s u^t, \\ a \mapsto a a^{\lambda p^{n-k}} c, \end{cases}$$

where $a^r \in \Omega_m(\langle a \rangle)$, $s \not\equiv 0(p)$, $c \in \Omega_l(\langle a^r b^s u^t \rangle) G'$.

— *If* $l \leqslant m$:

$$|\operatorname{Aut} G| = (p-1)p^{3l+m+2k-1}, \qquad |\operatorname{Inn} G| = p^{2n},$$

$$|O_p(\operatorname{Aut} G)| = p^{3l+m+2k-1}, \qquad |\operatorname{Aut}_l G| = p(p-1).$$

— *If* $m < l < m + n - k$:

$$|\operatorname{Aut} G| = (p-1)p^{2l+2m+2k-1}, \qquad |\operatorname{Inn} G| = p^{2n},$$

$$|O_p(\operatorname{Aut} G)| = p^{2l+2m+2k-1}, \qquad |\operatorname{Aut}_l G| = p - 1.$$

— *If* $m + n - k \leqslant l$:

$$|\operatorname{Aut} G| = (p-1)p^{l+3m+n+k-1}, \qquad |\operatorname{Inn} G| = p^{2n},$$

$$|O_p(\operatorname{Aut} G)| = p^{l+3m+n+k-1}, \qquad |\operatorname{Aut}_l G| = p(p-1).$$

To finish with the class 2 case, we have to deal with the following situation: for any choice of generators a, b of G such that $G/G' =$

$=\langle aG'\rangle\times\langle bG'\rangle$ we have $\langle a\rangle\wedge G'\neq 1$ and $\langle b\rangle\wedge G'\neq 1$. G will have a presentation $G=\langle a, b, u\,|\,u^{p^n}=1,\ a^{p^l}=u^{p^h},\ b^{p^m}=u^{p^k},\ b^a=bu,\ u^a=$ $=u^b=u\rangle$, where $1\leqslant h<n$, $1\leqslant k<n$, and we may assume $l\geqslant m$. There are further restrictions, namely: $h>k$ and $l>m+h-k$. In fact, from $h\leqslant k$ it would follow $b^{p^m}=u^{p^k}=(u^{p^h})^{p^{k-h}}=a^{p^{l+k-h}}$, $(b(a^{-1})^{p^{l+k-h-m}})^{p^m}=1$ and the generators a, $b_0=b(a^{-1})^{p^{l+k-h-m}}$ would satisfy $G/G'=\langle aG'\rangle\times\langle b_0G'\rangle$ and $\langle b_0\rangle\wedge G'=1$. Similarly, if $h>k$ but $l\leqslant m+h-k$, we could set $a_0=a(b^{-1})^{p^{m+h-k-l}}$ and obtain $G/G'=$ $=\langle a_0G'\rangle\times\langle bG'\rangle$ and $\langle a_0\rangle\wedge G'=1$. We then assume $h>k$ and $l>m+h-k$ and look for subgroups $C=\langle c\rangle$ where $c=a^xb^yu^z$ and $D=\langle d\rangle$ where $d=a^rb^su^t$, such that $C^{p^l}=(G')^{p^h}$ and $D^{p^m}=(G')^{p^k}$. Now $c^{p^l}=a^{xp^h}b^{yp^l}=u^{xp^h}(u^{yp^k})^{p^{l-m}}=u^{p^h(x+yp^{l-m+k-h})}$ generates $\langle u^{p^h}\rangle$ if and only if $x\not\equiv 0$ (p): there are $\phi(p^l)p^{m+n}$ such elements, which generate $\phi(p^l)p^{m+n}/\phi(p^{l+n-h})=p^{m+h}$ subgroups C. And $d^{p^m}=a^{rp^m}b^{sp^m}=$ $=a^{rp^m}u^{sp^k}$ generates $\langle u^{p^k}\rangle$ if and only if $a^{rp^m}\in\langle a\rangle\wedge G'=\langle a^{p^l}\rangle=\langle u^{p^h}\rangle$ and $s\not\equiv 0$ (p): there are $\phi(p^m)p^{m+n}$ such elements, which generate $\phi(p^m)p^{m+n}/\phi(p^{m+n-k})=p^{m+k}$ subgroups D. It is also clear that, for any choice of C and D as above, $G=\langle C, D\rangle$ and arbitrary generators c of C (in place of a) and d of D (in place of b) satisfy, together with $v=[d, c]$ in place of u, relations very similar to the original ones, the difference being that some coefficients λ,μ might appear in $c^{p^l}=v^{\lambda p^h}$, $d^{p^m}=v^{\mu p^k}$ $(\lambda,\mu\not\equiv 0$ $(p))$. But there certainly are particular generators $\bar a$, $\bar b$ of C, D respectively (a convenient choice is c^μ, d^λ), which satisfy, with $\bar u=[\bar b,\bar a]$, all the relations (*e.g.*, ab, b is not a «good pair», but ab, $b^{1+p^{l-m+k-h}}$ is one). For such a choice of $\bar a$, $\bar b$, we find that $\bar a^i$, $\bar b^j$ $(i\not\equiv 0$, $j\not\equiv 0$ $(p))$ will again satisfy the relations if and only if $[\bar b^j,\bar a^i]=\bar u^{ij}$ is such that $\bar u^{ip^h}=\bar a^{ip^l}=\bar u^{ijp^h}$ and $\bar u^{jp^k}=\bar b^{jp^m}=\bar u^{ijp^k}$, *i.e.* if and only if $j\equiv$ $\equiv 1$ (p^{n-h}), $i\equiv 1$ (p^{n-k}): and we have p^{l+m} such pairs. So the number of automorphisms of G is: (number of C's) times (number of D's) times p^{l+m}. We can now state:

(B.3) $G=\langle a, b, u\,|\,u^{p^n}=1,\ a^{p^l}=u^{p^h},\ b^{p^m}=u^{p^k},\ b^a=bu,\ u^a=u^b=u\rangle$

where $m\geqslant n>h>k\geqslant 1, l>m+h-k$.

$$|\mathrm{Aut}\,G|=p^{l+3m+h+k},\qquad |\mathrm{Inn}\,G|=p^{2n},\qquad |\mathrm{Aut}_l G|=p\,.$$

3. From now on, we suppose G is not metacyclic, and its nilpotency class is greater than two. Let us fix some notation. G' is cyclic of order p^n, and $G/C_G(G')$ is cyclic and non-trivial, so that G is generated by

two elements a, b such that $b \in C_G(G')$, a acts on G' as the power $1 + p^s$ $(0 < s < n)$ and $G' = \langle [b, a] \rangle$; we denote by p^m the order of bG' and by p^l the order of $a\langle b, G' \rangle$. The Aut G-orbit of b is contained in $C_G(G') = \langle a^{p^{n-s}}, b, G' \rangle$ and the Aut G-orbit of a is contained in the coset $aC_G(G')$. We set $u = [b, a]$; since $\langle b, G' \rangle$ is abelian and normal, $[b^y u^z, a] = u^{y+zp^s}$. We will show that if $i \geqslant 0$ then $[b, a^{p^i}] \equiv u^{p^i}$ (mod $\langle u^{p^{s+i}} \rangle$). This is true for $i = 0$; so, suppose $i > 0$ and, by induction, $[b, a^{p^{i-1}}] = u^{p^{i-1} + \lambda p^{s+i-1}} = u^{p^{i-1}(1+\lambda p^s)}$. $a^{p^{i-1}}$ acts on G' as some power $1 + \mu p^{s+i-1}$, and $1 + (1 + \mu p^{s+i-1}) + \ldots + (1 + \mu p^{s+i-1})^{p-1} = p(1 + \nu p^{s+i-1})$ for some ν, so that (on G') the endomorphism $1 + a^{p^{i-1}} + \ldots + (a^{p^{i-1}})^{p-1}$ is the power $p(1 + \nu p^{s+i-1})$. And then $[b, a^{p^i}] = [b, (a^{p^{i-1}})^p] = u^{p^{i-1}(1+\lambda p^s)p(1+\nu p^{s+i-1})} \in u^{p^i} \langle u^{p^{s+i}} \rangle$, which establishes our claim. So, in particular, $[b, a^{p^{n-s}}] = u^{p^{n-s}}$. And if $a^{xp^{n-s}} b^y u^z$, $a^{1+\lambda p^{n-s}} b^\mu u^\nu$ are arbitrary elements of $C_G(G')$ and $aC_G(G')$, respectively, we can compute the useful formula

$$[a^{xp^{n-s}} b^y u^z, a^{1+\lambda p^{n-s}} b^\mu u^\nu] = u^{y + p^{n-s}(y\lambda - x\mu) + zp^s}.$$

We can now easily compute $\langle a \rangle \wedge C_G(b) = \langle a \rangle \wedge Z(G) = \langle a^{p^n} \rangle \geqslant \langle a \rangle \wedge \langle b, G' \rangle = \langle a^{p^l} \rangle$, which gives $l \geqslant n$; $\langle b \rangle \wedge C_G(a) = \langle b \rangle \wedge Z(G) = \langle b^{p^n} \rangle$; $C_G(b) = \langle a^{p^n}, b, u \rangle$ and $Z(G) = \langle a^{p^n} \rangle C_{\langle b, u \rangle}(a) = \langle a^{p^n}, b^{p^n}, b^{-p^s} u \rangle$. Since $\langle aZ(G) \rangle \wedge \langle bZ(G) \rangle = Z(G)$, we see that $G/Z(G)$ is metacyclic of order p^{2n}, and $bZ(G)^{aZ(G)} = bZ(G)^{1+p^s}$.

In this section we deal with the following special case: $C_G(G')/G'$ contains a direct factor $\langle bG' \rangle$ of G/G'; and $\langle bG' \rangle$ has a complement $\langle aG' \rangle$ in G/G' such that $\langle a \rangle \wedge G' = 1$.

To begin, we suppose, in addition, that b can be chosen to satisfy $\langle b \rangle \wedge G' = 1$. Then G has a presentation $G = \langle a, b, u \mid a^{p^l} = b^{p^m} = u^{p^n} = 1, b^a = bu, u^a = u^{1+p^s}, u^b = u \rangle$, where $l \geqslant n$, $m \geqslant n$. If $\bar{b} = a^x b^y u^z$ is to be in the Aut G-orbit of b, then $\bar{b} \in C_G(G')$, $\bar{b} \notin \Phi(G)$, $\bar{b}^{p^m} = 1$, so that $a^x \in \Omega_m(\langle a^{p^{n-s}} \rangle)$, $y \not\equiv 0$ (p). And if $\bar{a}$ is in the Aut G-orbit of a, then $\bar{a} = ac$ for some $c \in \Omega_l(C_G(G')) = \langle a^{p^{n-s}} \rangle \Omega_l(\langle b \rangle) \langle u \rangle$. Conversely, for any choice of $\bar{a}$, $\bar{b}$ in agreement with these requirements, if we set $\bar{u} = [\bar{b}, \bar{a}]$, we get a generating triple for G which satisfies the defining relations. Hence we can state:

(C.1) $\quad G = \langle a, b, u \mid a^{p^l} = b^{p^m} = u^{p^n} = 1, b^a = bu, u^a = u^{1+p^s}, u^b = u \rangle$

where $l \geqslant n$, $m \geqslant n$, $0 < s < n$.

The effect of Aut G *on the generators* a, b *is*

$$\begin{cases} a \mapsto a^{1+\lambda p^{n-s}} b^{\mu} u^{\nu}, \\ b \mapsto a^{x} b^{y} u^{z}, \end{cases}$$

where $b^{\mu} \in \Omega_l(\langle b \rangle)$, $a^{x} \in \Omega_m(\langle a^{p^{n-s}} \rangle)$, $y \not\equiv 0 \ (p)$.

— *If* $m \leqslant l - n + s$:

$$|\operatorname{Aut} G| = (p-1) p^{l+3m+n+s-1}, \qquad |\operatorname{Inn} G| = p^{2n},$$

$$|O_p(\operatorname{Aut} G)| = p^{l+3m+n+s-1}, \qquad |\operatorname{Aut}_l G| = p(p-1).$$

— *If* $l - n + s < m \leqslant l$:

$$|\operatorname{Aut} G| = (p-1) p^{2l+2m+2s-1}, \qquad |\operatorname{Inn} G| = p^{2n},$$

$$|O_p(\operatorname{Aut} G)| = p^{2l+2m+2s-1}, \qquad |\operatorname{Aut}_l G| = p(p-1).$$

— *If* $l < m$:

$$|\operatorname{Aut} G| = (p-1) p^{3l+m+2s-1}, \qquad |\operatorname{Inn} G| = p^{2n},$$

$$|O_p(\operatorname{Aut} G)| = p^{3l+m+2s-1}, \qquad |\operatorname{Aut}_l G| = p - 1.$$

In this second part of the section we still assume that G has generators a, b such that $G/G' = \langle aG' \rangle \times \langle bG' \rangle$ and $\langle a \rangle \wedge G' = 1$ as in the first part, but now $\langle b \rangle \wedge G' \neq 1$. G can be presented in the form: $G = \langle a, b, u \mid a^{p^l} = u^{p^n} = 1,\ b^a = bu,\ u^a = u^{1+p^s},\ u^b = u,\ b^{p^m} = u^{rp^k} \rangle$ with $l \geqslant n$, $m > k$, $r \not\equiv 0 \ (p)$, $0 < k < n$, $p^m \equiv rp^{k+s}(p^n)$; conversely, from the presentation above one can easily read that no element $g \in C_G(G')$ exists such that $\langle gG' \rangle$ is a non-trivial direct factor of G/G' and $\langle g \rangle \wedge G' = 1$.

If $\bar{b} = a^{xp^{n-s}} b^y u^z \in C_G(G')$ is in the Aut G-orbit of b, then $\langle \bar{b}^{p^m} \rangle = (G')^{p^k}$; the group $C_G(G')$ has class $\leqslant 2$ and b^{p^m} belongs to its centre, so that $\bar{b}^{p^m} = a^{xp^{n-s+m}} b^{yp^m} u^{zp^m}$, and $\langle \bar{b}^{p^m} \rangle = (G')^{p^k}$ if and only if $a^{xp^{n-s+m}} = 1$ and $y \not\equiv 0 \ (p)$. On the other hand, suppose $a^{xp^{n-s}} \in \Omega_m(\langle a^{p^{n-s}} \rangle)$ and $y \not\equiv 0 \ (p)$; set $\bar{b} = a^{xp^{n-s}} b^y u^z$ and $\bar{u} = [\bar{b}, a] = [b^y u^z, a] = u^{y+zp^s}$. Then $G = \langle a, \bar{b} \rangle$, all the relations except maybe the last one hold, and $\bar{b}^{p^m} = b^{yp^m} u^{zp^m} = u^{yrp^k + zp^m} = u^{rp^k(y+zp^s)} = \bar{u}^{rp^k}$: hence, there is an automorphism of G mapping b to $\bar{b}$ and fixing a. For a given $\bar{b}$ as above, the conditions on $\bar{a} = ac$, where $c \in C_G(G')$, in order that some $\theta \in \operatorname{Aut} G$ exists which satisfies $a^\theta = \bar{a}$, $b^\theta = \bar{b}$ are the following: $\bar{a}^{p^l} = 1$, and $[\bar{b}, \bar{a}]^{p^k} =$

$= [\bar{b}, a]^{p^k}$, *i.e.* $c \in \Omega_l(C_G(G')) = \langle a^{p^{n-s}} \rangle \Omega_l(\langle \bar{b} \rangle) G'$ and $c \in C_G(\bar{b}^{p^k}) = \langle a^{p^{n-k}} \rangle \langle \bar{b}, G' \rangle$, which means $\bar{a} = a a^{\lambda} \bar{b}^{\mu} u^{\nu}$ with $a^{\lambda} \in \langle a^{p^{n-s}} \rangle \wedge \langle a^{p^{n-k}} \rangle$, $\bar{b}^{\mu} \in \Omega_l(\langle \bar{b} \rangle)$. To compute the orders explicitly, we now only have to make the necessary distinctions, according to the relative sizes of k and s, $|\bar{b}| = p^{m+n-k}$ and p^l, $|a^{p^{n-s}}| = p^{l-n+s}$ and p^m. We get the following summary:

(C.2) $G = \langle a, b, u \,|\, a^{p^l} = u^{p^n} = 1, b^a = bu, u^a = u^{1+p^s}, u^b = u, b^{p^m} = u^{rp^k} \rangle$

where $l \geq n$, $m > k$, $r \not\equiv 0$ (p), $0 < k < n$, $0 < s < n$, $p^m \equiv rp^{k+s}$ (p^n).

— *If* $m \leq l - n + s$ *and* $s \leq k$:

$$|\mathrm{Aut}\, G| = (p-1)\, p^{l+3m+n+s-1}, \qquad |\mathrm{Inn}\, G| = p^{2n},$$
$$|O_p(\mathrm{Aut}\, G)| = p^{l+3m+n+s-1}, \qquad |\mathrm{Aut}_l\, G| = p(p-1).$$

— *If* $m \leq l - n + s$, $k < s$ *and* $m + n - k \leq l$:

$$|\mathrm{Aut}\, G| = (p-1)\, p^{l+3m+n+k-1}, \qquad |\mathrm{Inn}\, G| = p^{2n},$$
$$|O_p(\mathrm{Aut}\, G)| = p^{l+3m+n+k-1}, \qquad |\mathrm{Aut}_l\, G| = p(p-1).$$

— *If* $m \leq l - n + s$, $k < s$ *and* $m + n - k > l$:

$$|\mathrm{Aut}\, G| = (p-1)\, p^{2l+2m+2k-1}, \qquad |\mathrm{Inn}\, G| = p^{2n},$$
$$|O_p(\mathrm{Aut}\, G)| = p^{2l+2m+2k-1}, \qquad |\mathrm{Aut}_l\, G| = p-1.$$

— *If* $m > l - n + s$, $s < k$ *and* $m + n - k \leq l$:

$$|\mathrm{Aut}\, G| = (p-1)\, p^{2l+2m+s+k-1}, \qquad |\mathrm{Inn}\, G| = p^{2n},$$
$$|O_p(\mathrm{Aut}\, G)| = p^{2l+2m+s+k-1}, \qquad |\mathrm{Aut}_l\, G| = p(p-1).$$

— *If* $m > l - n + s$, $s < k$ *and* $m + n - k > l$:

$$|\mathrm{Aut}\, G| = (p-1)\, p^{3l+m-n+s+2k-1}, \qquad |\mathrm{Inn}\, G| = p^{2n},$$
$$|O_p(\mathrm{Aut}\, G)| = p^{3l+m-n+s+2k-1}, \qquad |\mathrm{Aut}_l\, G| = p-1.$$

— *If* $m > l - n + s$ *and* $s \geq k$:

$$|\mathrm{Aut}\, G| = (p-1)\, p^{3l+m-n+2s+k-1}, \qquad |\mathrm{Inn}\, G| = p^{2n},$$
$$|O_p(\mathrm{Aut}\, G)| = p^{3l+m-n+2s+k-1}, \qquad |\mathrm{Aut}_l\, G| = p-1.$$

4. In this section we retain the general hypotheses and the notation established at the beginning of section 3, and we still assume that $C_G(G')/G'$ contains a direct factor $\langle bG'\rangle$ of G/G', but we now suppose that for all complements $\langle aG'\rangle$ of $\langle bG'\rangle$ in G/G' we have $\langle a\rangle \wedge G' \neq 1$.

The easy case in this context is when $\langle b\rangle \wedge G' = 1$. G has then a presentation $G = \langle a, b, u \mid a^{p^l} = u^{p^h},\ b^{p^m} = u^{p^n} = 1,\ b^a = bu,\ u^a = u^{1+p^s}, u^b = u\rangle$ with the usual inequalities $m \geq n$, $0 < s < n$; $[a^{p^l}, b] = 1$ gives $l \geq n$, and of course $n > h > 0$; moreover, $[u^{p^h}, a] = 1$ yields $h \geq n - s$. If $c = a^{\lambda p^{n-s}} b^{\mu} u^{\nu} \in C_G(G')$ and $\bar{a} = ac$ is in the Aut G-orbit of a, then $c^{p^l} = a^{\lambda p^{n-s+l}} b^{\mu p^l} = (u^{p^h})^{\lambda p^{n-s}} b^{\mu p^l} \in G'$, so that $b^{\mu p^l} = 1$ and $\bar{a}^{p^l} = u^{p^h(1+\lambda p^{n-s})}$. And if $\bar{b} = a^{xp^{n-s}} b^y u^z$ is in the Aut G-orbit of b then $y \not\equiv 0$ (p), and $\bar{b}^{p^m} = 1$ gives $a^{xp^{n-s+m}} = 1$. Also, $\bar{u} = [\bar{b}, \bar{a}] = u^{y+p^{n-s}(y\lambda - x\mu)+zp^s}$, and an automorphism θ of G mapping a to $\bar{a}$ and b to $\bar{b}$ exists if and only if $\bar{a}^{p^l} = \bar{u}^{p^h}$: we have to study the solutions of the congruence

$$1 + \lambda p^{n-s} + x\mu p^{n-s} \equiv y(1 + \lambda p^{n-s}) \qquad (p^{n-h})$$

with the conditions $\mu p^l \equiv 0\,(p^m)$ and $xp^{n-s+m} \equiv 0$ (p^{l+n-h}). This congruence has solutions in y (precisely p^{m-n+h} solutions mod p^m) for any given λ, μ, ν, x, z satisfying the conditions above. So we have

(D.1) $\quad G = \langle a, b, u \mid a^{p^l} = u^{p^h}, b^{p^m} = u^{p^n} = 1, b^a = bu, u^a = u^{1+p^s}, u^b = u\rangle$

with $l \geq n$, $m \geq n > h \geq n - s$, $0 < s < n$.

The effect of Aut G *on the generators* a, b *is*

$$\begin{cases} a \mapsto a^{1+\lambda p^{n-s}} b^{\mu} u^{\nu}, \\ b \mapsto a^{xp^{n-s}} b^y u^z, \end{cases}$$

where $\mu p^l \equiv 0$ (p^m), $xp^{n-s+m} \equiv 0$ (p^{l+n-h}) *and* $y(1 + \lambda p^{n-s}) \equiv 1 + (\lambda + x\mu)p^{n-s}$ (p^{n-h}).

— *If* $m \leq l$ *and* $m \geq l + s - h$:

$$|\mathrm{Aut}\, G| = p^{2l+2m-n+2s+h}, \qquad |\mathrm{Inn}\, G| = p^{2n}, \qquad |\mathrm{Aut}_l\, G| = p\,.$$

— *If* $m \leq l$ *and* $m < l + s - h$:

$$|\mathrm{Aut}\, G| = p^{l+3m-n+s+2h}, \qquad |\mathrm{Inn}\, G| = p^{2n}, \qquad |\mathrm{Aut}_l\, G| = p\,.$$

— *If* $m > l$ *and* $m \geq l + s - h$:

$$|\mathrm{Aut}\, G| = p^{3l+m-n+2s+h}, \qquad |\mathrm{Inn}\, G| = p^{2n}, \qquad |\mathrm{Aut}_l\, G| = 1\,.$$

— *If $m > l$ and $m < l + s - h$:*

$$|\mathrm{Aut}\, G| = p^{2l+2m-n+s+2h}, \qquad |\mathrm{Inn}\, G| = p^{2n}, \qquad |\mathrm{Aut}_l G| = 1 .$$

5. Suppose now that $C_G(G')/G'$ contains a direct factor of G/G', but for all generating pairs a, b (with $b \in C_G(G')$) for which $G/G' = \langle aG'\rangle \times \langle bG'\rangle$ one has $\langle a\rangle \wedge G' \neq 1$, $\langle b\rangle \wedge G' \neq 1$. With our usual notation ($u = [b, a]$ of order p^n, $u^a = u^{1+p^s}$ etc.) we find that G has a presentation $G = \langle a, b, u \,|\, u^{p^n} = 1,\ b^a = bu,\ u^a = u^{1+p^s},\ u^b = u,\ a^{p^l} = u^{p^h},\ b^{p^m} = u^{rp^k}\rangle$, with $0 < h < n \leqslant l$, $n - s \leqslant h$, $r \not\equiv 0$ (p), $0 < k < n$, $k < m$ and $p^m \equiv rp^{k+s}$ (p^n). Some cases have to be excluded. If $k < h$ then $l + k > m + h$, otherwise $(b^{tp^{m+h-k-l}})^{p^l} = (b^{tp^m})^{p^{h-k}} = (u^{p^k})^{p^{h-k}} = a^{p^l}$ (for the inverse t of $r \bmod p^n$); hence with $a_0 = ab^{-tp^{m+h-k-l}}$ we get $a_0^{p^l} = 1$ and the pair a_0, b satisfies $G/G' = \langle a_0 G'\rangle \times \langle bG'\rangle$ and $\langle a_0\rangle \wedge G' = 1$. Similarly, if $k \geqslant h$ we must have $m + n - s > l + k - h$, since otherwise $a^{rp^{l+k-h}} = (u^{p^h})^{rp^{k-h}} = b^{p^m} = (a^{rp^{l+k-h-m}})^{p^m}$ with $a^{rp^{l+k-h-m}} \in \langle a^{p^{n-s}}\rangle \leqslant C_G(G')$ and we could substitute $b_0 = a^{-rp^{l+k-h-m}} b$ for b; but $\langle b_0\rangle \wedge G' = 1$.

To determine $\mathrm{Aut}\, G$, the point is to find all pairs $\bar{a}, B$ with $\bar{a} \in aC_G(G')$, B cyclic, $B \leqslant C_G(G')$ such that $B^{p^m} = (G')^{p^k}$, $\langle \bar{a}^{p^l}\rangle = (G')^{p^h}$, $G = \langle \bar{a}, B\rangle$ and $g^{p^m} = [g, \bar{a}]^{rp^k}$ for some generator g of B: in fact, if this is true, then for every generator g^i of B we have $[g^i, \bar{a}]^{rp^k} = (g^i)^{p^m}$, and it is clear that there is one particular generator $\bar{b}$ satisfying $\bar{a}^{p^l} = [\bar{b}, \bar{a}]^{p^h}$. Moreover, we can then easily determine all the «good» generators: $\bar{b}^j$ $(j \not\equiv 0$ $(p))$ is one if and only if $[\bar{b}^j, \bar{a}]^{p^h} = [\bar{b}, \bar{a}]^{p^h}$, *i.e.* $[\bar{b}^{p^h(j-1)}, \bar{a}] = 1$. This means $\bar{b}^{p^h(j-1)} \in \langle \bar{b}\rangle \wedge Z(G) = \langle \bar{b}^{p^n}\rangle$, $j \equiv 1$ (p^{n-h}), so there are precisely p^{m+h-k} «good» generators of B.

Now, take $\bar{a} = a^{1+\lambda p^{n-s}} b^{\mu} u^{\nu}$; we have $\bar{a}^{p^l} = a^{p^l} a^{\lambda p^{n-s+l}} b^{\mu p^l}$. $\bar{a}^{p^l} \in G'$ implies $\mu p^l \equiv 0$ (p^m) which in turn gives $\langle \bar{a}^{p^l}\rangle = \langle a^{p^l}\rangle = (G')^{p^h}$ if $k > h$; but if $k \leqslant h$ we have $l > m + h - k$, and then $b^{\mu p^l} = (u^{r\mu p^k})^{p^{l-m}} \in \langle u^{p^{h+1}}\rangle$ and again $\langle \bar{a}^{p^l}\rangle = (G')^{p^h}$.

If $g = a^{xp^{n-s}} b^y u^z$ $(y \not\equiv 0$ $(p))$, we find $g^{p^m} = a^{xp^{n-s+m}} b^{yp^m} u^{zp^m}$ (since $m \geqslant s$). In case $h \leqslant k$ we have $n - s + m > l + k - h$, and so $a^{xp^{n-s+m}} \in (G')^{p^{k+1}}$, which implies $\langle g^{p^m}\rangle = \langle b^{p^m}\rangle = (G')^{p^k}$. If instead $k < h$, then $g^{p^m} \in G'$ forces $a^{xp^{n-s+m}} \in G'$, *i.e.* $xp^{n-s+m} \equiv 0$ (p^l), and again $\langle g^{p^m}\rangle = (G')^{p^k}$.

Finally, we study the condition $g^{p^m} = [g, \bar{a}]^{rp^k}$ for $g, \bar{a}$ as above, with $\mu p^l \equiv 0$ (p^m) and $xp^{n-s+m} \equiv 0$ (p^l). The condition is

$$a^{xp^{n-s+m}} = u^{rp^{n-s+k}(y\lambda - x\mu)} .$$

At this point, we have found all pairs $\bar{a}, g$: $g = a^{xp^{n-s}} b^y u^z$ with $a^{xp^{n-s+m}} \in \langle u^{p^h}\rangle \wedge \langle u^{p^{n-s+k}}\rangle$ and $y \not\equiv 0$ (p), $\bar{a} = a^{1+\lambda p^{n-s}} b^{\mu} u^{\nu}$ where $\mu p^l \equiv 0$

(p^m) and λ is determined (modulo the order of $u^{p^{n-s+k}}$) by

$$(*) \qquad (u^{ryp^{n-s+k}})^{\lambda} = a^{xp^{n-s+m}} u^{rx\mu p^{n-s+k}}.$$

As we expected, the conditions on λ, μ depend only on B, and not on the particular generator g. The number of subgroups B is then $|A/\langle a^{p^l}\rangle| p^n \phi(p^m)/\phi(p^{m+n-k}) = p^k |A/\langle a^{p^l}\rangle|$, where ϕ is the Euler function and $A = \langle c \in \langle a^{p^{n-s}}\rangle | c^{p^m} \in \langle u^{p^{n-s+k}}\rangle\}$ (since $h + m \geq n - s + m > n$ we have $a^{p^{l+m}} = u^{p^{h+m}} = 1$, so that $a^{p^l} \in A$).

The possibilities for $\alpha = |A/\langle a^{p^l}\rangle|$ are the following (we use the fact that $|a^{p^{n-s}}| = p^{l-h+s}$):

— if $k \geq s$ then $A = \Omega_m(\langle a^{p^{n-s}}\rangle)$: if $l - h + s < m$ then $\alpha = p^{l-n+s}$, while if $l - h + s \geq m$ then $\alpha = p^{m+h-n}$;

— if $n > n - s + k \geq h$ we have $u^{p^{n-s+k}} = (u^{p^h})^{p^{n-s+k-h}} = a^{p^{l+n-s+k-h}}$, so that $A/\langle a^{p^{l+n-s+k-h}}\rangle = \Omega_m(\langle a^{p^{n-s}}\rangle/\langle a^{p^{l+n-s+k-h}}\rangle)$: if $l + k - h < m$ then again $\alpha = p^{l-n+s}$, while if $l + k - h \geq m$ then $\alpha = p^{m+h+s-n-k}$;

— if $h > n - s + k$, then $A = \{c \in \langle a^{p^{n-s}}\rangle | c^{p^m} \in \langle a^{p^l}\rangle\}$, so that $A/\langle a^{p^l}\rangle = \Omega_m(\langle a^{p^{n-s}}\rangle/\langle a^{p^l}\rangle)$: if $l - n + s < m$ then $\alpha = p^{l-n+s}$, while if $l - n + s \geq m$ then $\alpha = p^m$.

For a given choice of B, the number of $\bar{a}$'s is $\beta p^n |\Omega_l(\langle bG'\rangle)|$ where β stands for the number of distinct cosets $a^{\lambda p^{n-s}}\langle a^{p^l}\rangle$ when λ satisfies $(*)$. In case $k \geq s$ λ is arbitrary, hence $\beta = p^{l-n+s}$; on the other hand, when $k < s$ the coset $\lambda + p^{s-k}\mathbb{Z}$ is fixed, which splits into p^{l-n+k} cosets $\bmod p^{l-n+s}$: in this case $\beta = p^{l-n+k}$. In the end we get:

$$\text{(D.2)} \qquad G = \langle a, b, u \,|\, u^{p^n} = 1,\ b^a = bu,\ u^a = u^{1+p^s},$$
$$u^b = u,\ a^{p^l} = u^{p^h},\ b^{p^m} = u^{rp^k}\rangle$$

where $0 < h < n \leq l$, $0 < s < n$, $n - s \leq h$, $r \not\equiv 0\ (p)$, $0 < k < n$, $k < m$, $p^m \equiv rp^{k+s}\ (p^n)$, *and if* $k < h$ *then* $l + k > m + h$, *while if* $k \geq h$ *then* $m + n - s > l + k - h$.

$\mathrm{Aut}\,G$ *is a p-group,* $|\mathrm{Inn}\,G| = p^{2n}$, $|\mathrm{Aut}_l G| = p$ *if* $l \geq m$ *and* $|\mathrm{Aut}_l G| = 1$ *if* $l < m$.

— *If* $k \geq s$ *and* $l - h + s < m$:

$$|\mathrm{Aut}\,G| = p^{2l+m-n+h+2s+min\{l,m\}}.$$

— *If* $k \geq s$ *and* $l - h + s \geq m$:

$$|\mathrm{Aut}\,G| = p^{l+2m-n+2h+s+min\{l,m\}}.$$

— *If* $k < s$, $n - s + k \geqslant h$ *and* $l + k < m + h$:

$$|\operatorname{Aut} G| = p^{2l+m-n+h+s+k+min\{l,m\}}.$$

— *If* $k < s$, $n - s + k \geqslant h$ *and* $l + k \geqslant m + h$:

$$|\operatorname{Aut} G| = p^{l+2m-n+2h+s+min\{l,m\}}.$$

— *If* $h > n - s + k$ *and* $l - n + s < m$:

$$|\operatorname{Aut} G| = p^{2l+m-n+h+s+k+min\{l,m\}}.$$

— *If* $h > n - s + k$ *and* $l - n + s \geqslant m$:

$$|\operatorname{Aut} G| = p^{l+2m+h+k+min\{l,m\}}.$$

6. From now on, we shall deal with the case when no nontrivial direct factor of G/G' is contained in $C_G(G')/G'$. With our usual symbols, G is generated by a and b, $b \in C_G(G')$, $u = [b, a]$ has order p^n, $u^a = u^{1+p^s}$ $(0 < s < n)$, $|bG'| = p^m$, $|a\langle b, G'\rangle| = p^l$ and $a^{p^l}G' = b^{p^h}G'$ for some h, $0 < h < m$. G/G' does not split over $\langle bG'\rangle$, hence $l > h$; and $[b, a^{p^l}] = 1$ implies $l \geqslant n$. We also have $n - s > l - h$, because from $n - s \leqslant l - h$ it would follow $a^{p^{l-h}} \in C_G(G')$ and $G/G' = \langle aG'\rangle \times \langle a^{p^{l-h}} b^{-1} G'\rangle$. And conversely, if $n - s > l - h$, then $\langle a^{\lambda p^{n-s}} bG'\rangle \wedge \langle ab^{\mu} G'\rangle \geqslant \langle a^{p^l} G'\rangle$ for all λ, μ and indeed $C_G(G')/G'$ does not contain any direct factor of G/G'.

In this section we further assume that $\langle b, G'\rangle = \langle b\rangle \times G'$, *i.e.* $b^{p^m} = 1$; the remaining cases will be discussed from section 7 onwards.

The simplest instance of this class is when a, b can be chosen so that $\langle a\rangle\langle b\rangle \cap G' = 1$. G has then a presentation $G = \langle a, b, u \,|\, u^{p^n} = b^{p^m} = 1, b^a = bu, u^a = u^{1+p^s}, u^b = u, a^{p^l} = b^{p^h}\rangle$ where $0 < s < n$, $0 < h < m$, $0 < l - h < n - s$, $h \geqslant n$ (the last inequality coming from the fact that $b^{p^h} \in Z(G)$). If we take $\bar{a} = a^{1+\lambda p^{n-s}} b^{\mu} u^{\nu} \in aC_G(G')$, $\bar{b} = a^{xp^{n-s}} b^y u^z \in C_G(G')$ with $y \not\equiv 0$ (p), and $\bar{u} = [\bar{b}, \bar{a}]$, then $G = \langle \bar{a}, \bar{b}\rangle$, $\bar{b}^{p^m} = 1$ because $|a^{p^{n-s}}| = p^{m+l-h-n+s} < p^m$, $\bar{u}^{p^n} = 1$; it only remains to check the relation $\bar{a}^{p^l} = \bar{b}^{p^h}$, *i.e.* $a^{(1+\lambda p^{n-s})p^l + \mu p^{2l-h}} = a^{xp^{n-s+h} + yp^l}$. In other terms, x, y, λ, μ have to be solutions of

$$(**) \qquad xp^{n-s+h-l} + y \equiv 1 + \lambda p^{n-s} + \mu p^{l-h} \ (p^{m-h}):$$

x, λ, μ can be taken at will, and then $y + p^{m-h}\mathbb{Z}$ is determined. In particular $y \equiv 1$ (p), so $\operatorname{Aut} G$ is a p-group. We have $|C_G(G')| = |\langle a^{p^{n-s}}\rangle/\langle a^{p^l}\rangle| p^{m+n} = p^{l+m+s}$ choices for $\bar{a}$. Once $\bar{a}$ is given, according to $(**)$ we can choose $\bar{b}$ arbitrarily in the coset

$b^{1+\lambda p^{n-s}+\mu p^{l-h}}\langle a^{p^{n-s}} b^{-p^{n-s+h-l}}, b^{p^{m-h}}\rangle G'$. Since $\langle a^{p^{n-s}} b^{-p^{n-s+h-l}} G'\rangle$ has order p^{l-n+s}, $|\langle b^{p^{m-h}} G'\rangle| = p^h$, and these subgroups of G/G' are independent, we have p^{l+s+h} choices for $\bar{b}$. We obtained

(E.1) $G = \langle a, b, u \mid u^{p^n} = b^{p^m} = 1,\ b^a = bu,\ u^a = u^{1+p^s},\ u^b = u,\ a^{p^l} = b^{p^h}\rangle$,

where $0 < s < n$, $0 < h < m$, $0 < l - h < n - s$, $h \geqslant n$.

$$|\mathrm{Aut}\, G| = p^{2l+m+2s+h}, \qquad |\mathrm{Inn}\, G| = p^{2n}, \qquad |\mathrm{Aut}_l\, G| = p\,.$$

In the setting recorded at the beginning of this section, we suppose now that the generators a, b satisfy $\langle a\rangle \wedge G' = \langle b\rangle \wedge G' = 1$, but $\langle a\rangle\langle b\rangle \cap G' \neq 1$. G has now a presentation $G = \langle a, b, u \mid u^{p^n} = b^{p^m} = 1$, $b^a = bu$, $u^a = u^{1+p^s}$, $u^b = u$, $a^{p^l} = b^{p^h} u^{tp^j}\rangle$, where $t \not\equiv 0\ (p)$, $0 \leqslant j < n$, and the usual conditions $0 < s < n$, $l \geqslant n$, $0 < h < m$, $0 < l - h < n - s$, $m \geqslant n$ hold, and $p^h + tp^{s+j} \equiv 0\ (p^n)$ (which means $b^{p^h} u^{tp^j} \in Z(G)$). We also assume $j < h$, since otherwise $b_0 = bu^{tp^{j-h}}$ would satisfy $b_0^{p^h} = a^{p^l}$ and $\langle a\rangle\langle b_0\rangle \cap G' = 1$. We also have $|aG'| = p^{l+m-h}$, $\langle a\rangle \wedge G' = \langle a^{p^{l+m-h}}\rangle = \langle (b^{p^h} u^{tp^j})^{p^{m-h}}\rangle = \langle u^{tp^{j+m-h}}\rangle$, and $\langle a\rangle \wedge G' = 1$ gives the further condition $j + m - h \geqslant n$.

Here (and in the rest of the paper) it is expedient to use the direct decomposition $C_G(G')/G' = \langle cG'\rangle \times \langle bG'\rangle$, where $c = a^{p^{n-s}} b^{-p^{n-s-l+h}}$. The order $|cG'| = p^{l-n+s}$ is easily computed. The natural candidates for a^θ, b^θ ($\theta \in \mathrm{Aut}\, G$) can be written as $ac^\lambda b^\mu u^\nu$ and $c^x b^y u^z$, respectively ($y \not\equiv 0\ (p)$). Some tedious but elementary calculations give:

$$[c^x b^y u^z, ac^\lambda b^\mu u^\nu] = u^{p^{n-s}(y\lambda - x\mu) - xp^{n-s-l+h} + y + zp^s}. \tag{1}$$

If we take the relation $a^{p^l} = b^{p^h} u^{tp^j}$ into account, we can also compute

$$(ac^\lambda b^\mu u^\nu)^{p^l} = b^{p^h(1+\mu p^{l-h})} u^{tp^j(1+\lambda p^{n-s})}, \tag{2}$$

$$(c^x b^y u^z)^{p^h} = b^{yp^h} u^{xtp^{j+n-s+h-l} + zp^h}. \tag{3}$$

We now set $\bar{a} = ac^\lambda b^\mu u^\nu$, $\bar{b} = c^x b^y u^z$, $\bar{u} = [\bar{b}, \bar{a}]$. From (3) we get $\bar{b}^{p^m} = b^{yp^m} u^{xtp^{j+n-s+m-l} + zp^m}$; in our case $m \geqslant n$ and $j + n - s + m - l > j + m - h \geqslant n$, so $\bar{b}^{p^m} = 1$. It remains to check the relation $\bar{a}^{p^l} = \bar{b}^{p^h} \bar{u}^{tp^j}$: by (1), (2) and (3) the condition reads

$$b^{p^h(1+\mu p^{l-h})} u^{tp^j(1+\lambda p^{n-s})} = b^{yp^h} u^{zp^h} u^{tp^j(p^{n-s}(y\lambda - x\mu) + y + zp^s)}.$$

Now $\langle b, u\rangle = \langle b\rangle \times \langle u\rangle$ and $p^h + tp^{s+j} \equiv 0\ (p^n)$; so, we are looking for the solutions of the system

$$y \equiv 1 + \mu p^{l-h} \quad (p^{m-h}), \tag{I}$$

$$1 + \lambda p^{n-s} \equiv p^{n-s}(y\lambda - x\mu) + y \quad (p^{n-j}). \tag{II}$$

We also have $n-j \leqslant m-h$. Substituting y from (I) into (II) gives

$$\mu p^{l-h}(1+\lambda p^{n-s}-xp^{n-s-l+h}) \equiv 0 \ (p^{n-j}):$$

if (λ, μ, x, y) is a solution of the system, then $\mu p^{l-h} \equiv 0 \ (p^{n-j})$ and $y \equiv \equiv 1+\mu p^{l-h} \ (p^{m-h})$; and conversely any 4-tuple $(\lambda, \mu, x, 1+\mu p^{l-h}+ + \eta p^{m-h})$ with $\mu p^{l-h} \equiv 0 \ (p^{n-j})$ is a solution.

To determine $|\operatorname{Aut} G|$, we need to compute the orders of $\{c^\lambda b^\mu u^\nu \mid \mu p^{l-h} \equiv 0 \ (p^{n-j})\}$, which is p^{l+m+s} if $l-h \geqslant n-j$ and $p^{2l+m-n+s+j-h}$ if $l-h<n-j$, and of $\langle c, b^{p^{m-h}}, G'\rangle$, which is p^{l+s+h}. We conclude:

(E.2) $\quad G=\langle a, b, u \mid u^{p^n}=b^{p^m}=1, b^a=bu,$

$$u^a=u^{1+p^s}, u^b=u, a^{p^l}=b^{p^h}u^{tp^j}\rangle$$

with $0<s<n$, $l \geqslant n$, $0<h<m$, $0<l-h<n-s$, $m \geqslant n$, $t \not\equiv 0 \ (p)$, $0 \leqslant j<h$, $j<n$, $p^h+tp^{s+j} \equiv 0 \ (p^n)$ *and* $n \leqslant j+m-h$.

The effect of Aut G *on the generators* a, b *is*

$$\begin{cases} a \mapsto ac^\lambda b^\mu u^\nu, \\ b \mapsto c^x b^{1+\mu p^{l-h}+\eta p^{m-h}} u^z, \end{cases}$$

where $c=a^{p^{n-s}}b^{-p^{n-s-l+h}}$ *and* $\mu p^{l-h} \equiv 0 \ (p^{n-j})$.

— *If* $l-h \geqslant n-j$:

$$|\operatorname{Aut} G|=p^{2l+2s+m+h}, \qquad |\operatorname{Inn} G|=p^{2n}, \qquad |\operatorname{Aut}_l G|=p.$$

- *If* $l-h<n-j$:

$$|\operatorname{Aut} G|=p^{3l+2s+m-n+j}, \qquad |\operatorname{Inn} G|=p^{2n}, \qquad |\operatorname{Aut}_l G|=p.$$

To conclude this section, we now study the groups given by a presentation $G=\langle a, b, u \mid u^{p^n}=b^{p^m}=1, b^a=bu, u^a=u^{1+p^s}, u^b=u, a^{p^l}= =b^{p^h}u^{tp^j}\rangle$ as in (E.2), but with $n>j+m-h$ (we retain the other conditions on the parameters). This means that $C_G(G')/G'$ does not contain direct factors of G/G', there is $b \in C_G(G') \setminus \Phi(G)$ such that $\langle b, G'\rangle=\langle b\rangle \times G'$, and for all elements $a \in G$ which, together with b, generate G, we have $\langle a\rangle \wedge G' \neq 1$. As in the previous case, we set $\bar{a}= =ac^\lambda b^\mu u^\nu$, $\bar{b}=c^x b^y u^z$ where $c=a^{p^{n-s}}b^{-p^{n-s-l+h}}$, $\bar{u}=[\bar{b}, \bar{a}]$ and check whether they satisfy the relations. Since $\bar{b}^{p^m}=u^{xtp^{j+n-s+m-l}}$, we see that $\bar{b}^{p^m}=1$ for all choices of x if $j+m \geqslant l+s$; on the other hand, if $j+m<l+s$ we must take $x \equiv 0 \ (p^{l+s-m-j})$. Exactly as in the discus-

sion leading to (E.2), $\bar{a}^{p^l} = \bar{b}^{p^h}\bar{u}^{tp^j}$ is equivalent to the system

$$\text{(I)} \qquad y \equiv 1 + \mu p^{l-h} \quad (p^{m-h}),$$

$$\text{(II)} \qquad 1 + \lambda p^{n-s} \equiv p^{n-s}(y\lambda - x\mu) + y \quad (p^{n-j}),$$

but in this case $n - j > m - h$. If we multiply (I) through $1 + \lambda p^{n-s}$, write (II) in the form $y(1 + \lambda p^{n-s}) \equiv 1 + \lambda p^{n-s} + x\mu p^{n-s}$ (p^{n-j}), and substitute into (I) we get $\mu p^{l-h}(1 + \lambda p^{n-s} - xp^{n-s-l+h}) \equiv 0$ (p^{m-h}): if $l < m$ then μ must be $\equiv 0$ (p^{m-l}). And (II) may also be rewritten as $(y - 1)(1 + \lambda p^{n-s}) \equiv x\mu p^{n-s}$ (p^{n-j}), or $y \equiv 1 + x\mu p^{n-s}\sigma$ (p^{n-j}), where σ is the inverse of $1 + \lambda p^{n-s}$ in $\mathbb{Z}/p^m\mathbb{Z}$.

Hence, the solutions of our system are all the 4-tuples $(\lambda, \mu, x, 1 + x\sigma\mu p^{n-s} + \eta p^{n-j})$, where $\mu \equiv 0$ (p^{m-l}) if $l < m$ and $x \equiv 0$ $(p^{l+s-m-j})$ if $j + m < l + s$. To compute the order of $\operatorname{Aut} G$, we note that for $\mu \equiv 0$ (p^{m-l}) (if $l < m$; and for any μ if $l \geq m$) $|b^{\sigma\mu p^{n-s}}G'| \leq p^{l+s-n} = |cG'|$, so that $C_G(G')/G' = \langle cb^{\sigma\mu p^{n-s}}G'\rangle \times \langle bG'\rangle$ and the sets $\{(cb^{\sigma\mu p^{n-s}})^x b^{\eta p^{n-j}}\}G'$ and $\{(cb^{\sigma\mu p^{n-s}})^x b^{\eta p^{n-j}} | x \equiv 0 \ (p^{l+s-m-j})\}G'$ have orders $p^{l+m-n+s+j}$ and, respectively, $p^{2m-n+2j}$. We state our results:

$$\text{(E.3)} \qquad G = \langle a, b, u | u^{p^n} = b^{p^m} = 1, b^a = bu,$$

$$u^a = u^{1+p^s}, u^b = u, a^{p^l} = b^{p^h}u^{tp^j}\rangle$$

with $0 < s < n$, $l \geq n$, $0 < h < m$, $0 < l - h < n - s$, $m \geq n$, $t \not\equiv 0$ (p), $0 \leq j < h$, $j < n$, $p^h + tp^{s+j} \equiv 0$ (p^n) *and* $n > j + m - h$.

The effect of $\operatorname{Aut} G$ *on the generators* a, b *is*

$$\begin{cases} a \mapsto ac^\lambda b^\mu u^\nu, \\ b \mapsto b(cb^{\sigma\mu p^{n-s}})^x b^{\eta p^{n-j}} u^z, \end{cases}$$

where $c = a^{p^{n-s}} b^{-p^{n-s-l+h}}$, $\mu \equiv 0$ (p^{m-l}) *if* $l < m$, $x \equiv 0$ $(p^{l+s-m-j})$ *if* $j + m < l + s$, *and* σ *is the inverse of* $1 + \lambda p^{n-s}$ *in* $\mathbb{Z}/p^m\mathbb{Z}$; $|\operatorname{Inn} G| = p^{2n}$.

— *If* $l \geq m$ *and* $j + m \geq l + s$:

$$|\operatorname{Aut} G| = p^{2l+2m-n+2s+j}, \qquad |\operatorname{Aut}_l G| = p.$$

— *If* $l \geq m$ *and* $j + m < l + s$:

$$|\operatorname{Aut} G| = p^{l+3m-n+s+2j}, \qquad |\operatorname{Aut}_l G| = p.$$

— *If* $l < m$ *and* $j + m \geq l + s$:

$$|\operatorname{Aut} G| = p^{3l+m-n+2s+j}, \qquad |\operatorname{Aut}_l G| = 1.$$

— *If* $l < m$ *and* $j + m < l + s$:

$$|\mathrm{Aut}\, G| = p^{2l+2m-n+s+2j}, \qquad |\mathrm{Aut}_l G| = 1 .$$

7. In order to complete our analysis, we still have to consider the following situation: for every $b \in C_G(G')\setminus\Phi(G)$ we have

— $\langle bG'\rangle$ is not a direct factor of G/G'; and

— $\langle b\rangle$ is not a direct factor of $\langle b, G'\rangle$.

We again use our standard notation: $G = \langle a, b\rangle$, $b \in C_G(G')$, $u = [b, a]$ has order p^n, $|bG'| = p^m$, $u^a = u^{1+p^s}$ with $0 < s < n$, $|a\langle b, G'\rangle| = p^l$. As we saw at the beginning of the previous section, the first condition is equivalent to: $a^{p^l} = b^{p^h} u^{tp^j}$ with $l > h$, $m > h$ and $n - s > l - h$, $t \not\equiv 0\ (p)$ (at least for the moment, we are not excluding the possibility that $j \geqslant n$); $[a^{p^l}, b] = 1$ implies $p^h + tp^{j+s} \equiv 0\ (p^n)$. And the second condition says $b^{p^m} = u^{rp^k}$ for some $r \not\equiv 0\ (p)$, $0 < k < n$; $[b^{p^m}, a] = [u^{rp^k}, a]$ then implies $p^m \equiv rp^{k+s}\ (p^n)$; so, in particular, $m > k$ and $m > s$.

Any cyclic subgroup of $C_G(G')$, not contained in $\Phi(G)$, is generated by some element $g = a^{xp^{n-s}} bu^z$; an easy calculation gives

$$g^{p^m} = a^{xp^{m+n-s}} b^{p^m} u^{zp^m} = (b^{p^h} u^{tp^j})^{xp^{m+n-s-l}} u^{rp^k} u^{zp^m} =$$

$$= u^{x(rp^{k+n-s-l+h} + tp^{j+m+n-s-l}) + rp^k + zp^m} .$$

If $j + m + n \leqslant l + s + k$, the congruence $x(rp^{k+n-s-l+h} + tp^{j+m+n-s-l}) + rp^k + zp^m \equiv 0\ (p^n)$ in the unknowns x, z has a solution $(x_1, 0)$, and then $b_1 = a^{x_1 p^{n-s}} b$ satisfies $\langle b_1\rangle \wedge G' = 1$. On the other hand, if $j + m + n > l + s + k$, then for all choices of x, z g as above satisfies $\langle g^{p^m}\rangle = \langle u^{p^k}\rangle \neq 1$. Hence our second condition is equivalent to $b^{p^m} = u^{rp^k}$, $r \not\equiv 0\ (p)$, $0 < k < n$, $p^m \equiv rp^{k+s}\ (p^n)$ and $j + m + n > l + s + k$.

Next, we determine $\langle a\rangle \wedge G'$. Since $|aG'| = p^{l+m-h}$, we have $\langle a\rangle \wedge G' = \langle a^{p^{l+m-h}}\rangle$, and $a^{p^{l+m-h}} = (b^{p^h} u^{tp^j})^{p^{m-h}} = b^{p^m} u^{tp^{j+m-h}} = u^{rp^k + tp^{j+m-h}}$.

In this section we study the special case in which $\langle a\rangle \wedge G' = 1$, *i.e.* $rp^k + tp^{j+m-h} \equiv 0\ (p^n)$. Since $k < n$, this implies $h + k = j + m$, and the inequality $j + m + n > l + s + k$ reduces to $n - s > l - h$. Once more, we set $\bar{a} = ac^\lambda b^\mu u^\nu$, $\bar{b} = c^x b^y u^z$ $(y \not\equiv 0\ (p))$, $\bar{u} = [\bar{b}, \bar{a}]$ (where $c = a^{p^{n-s}} b^{-p^{n-s-l+h}}$) and check the relations. Using (1), (2), (3), it is easily seen that $\bar{b}^{p^m} = \bar{u}^{rp^k}$ translates into

$$u^{yrp^k + xtp^{j+n-s+m-l} + zp^m} = u^{rp^k(p^{n-s}(y\lambda - x\mu) - xp^{n-s-l+h} + y) + zrp^{s+k}},$$

i.e.

$$(a^{p^{l+m-h}})^{xp^{n-s+h-l}} = u^{rp^{k+n-s}(y\lambda - x\mu)}. \tag{4}$$

Similarly, we may write $\overline{a}^{p^l} = \overline{b}^{p^h}\overline{u}^{tp^j}$ both as

$$b^{p^h(1+\mu p^{l-h}-y)} = u^{tp^j(p^{n-s}(y\lambda - x\mu)+y-1-\lambda p^{n-s})} \tag{5}$$

and as

$$a^{p^l(1+\mu p^{l-h}-y)} = u^{tp^j(p^{n-s}(y\lambda - x\mu) - \lambda p^{n-s} + \mu p^{l-h})}. \tag{6}$$

In our case $\langle a\rangle \wedge G' = \langle a^{p^{l+m-h}}\rangle = 1$, so these conditions are equivalent to the system

$$p^{k+n-s}(y\lambda - x\mu) \equiv 0 \quad (p^n), \tag{I}$$

$$1 + \mu p^{l-h} - y \equiv 0 \quad (p^{m-h}), \tag{II}$$

$$p^{n-s}(y\lambda - x\mu) - \lambda p^{n-s} + \mu p^{l-h} \equiv 0 \quad (p^{n-j}); \tag{III}$$

note that $m-h = k-j < n-j$.

Suppose first that $m-h \geqslant s-j$, so that $k-s = m-h+j-s \geqslant 0$: the first congruence is trivial. Since $n+s+m-h \geqslant n-j$, from (II) and (III) we get

$$p^{n-s}(\lambda(y-1) - x\mu) + \mu p^{l-h} \equiv \mu p^{l-h}(1 - xp^{n-s-l+h}) \equiv 0 \ (p^{n-j})$$

and then $\mu p^{l-h} \equiv 0 \ (p^{n-j})$, $y \equiv 1 \ (p^{m-h})$. Conversely, any 4-tuple (λ, μ, x, y) with $\mu p^{l-h} \equiv 0 \ (p^{n-j})$ and $y \equiv 1 \ (p^{m-h})$ is a solution of the system.

If, on the other hand, $m-h < s-j$ (*i.e.* $k < s$), we proceed as follows. If (λ, μ, x, y) is a solution, then $y \equiv 1 \ (p)$; let y' be the inverse of y in $\mathbb{Z}/p^n\mathbb{Z}$. From (I) and (II) we can write $\lambda = y'x\mu + \sigma p^{s-k}$, $y = 1 + \mu p^{l-h} + \rho p^{m-h}$ for some σ, ρ and then (III) becomes

$$\mu p^{l-h}(1 + \lambda p^{n-s} + \rho y' xp^{n-s+m-l} - xp^{n-s+h-l}) \equiv 0 \ (p^{n-j}),$$

forcing $\mu p^{l-h} \equiv 0 \ (p^{n-j})$ (we used the equality $(n-s)+(s-k)+(m-h) = = n-j$). And now (II) implies $y \equiv 1 \ (p^{m-h})$. Moreover, $n-j-l+h > s-j > s-j-m+h = s-k > 0$, so $\mu \equiv 0 \ (p^{n-j-l+h})$ yields $\lambda \equiv 0 \ (p^{s-k})$. *Vice versa*, it is clear that any 4-tuple (λ, μ, x, y) with $\lambda \equiv 0 \ (p^{s-k})$, $\mu \equiv 0 \ (p^{n-j-l+h})$, $y \equiv 1 \ (p^{m-h})$ is a solution of the system. So, we have

(F.1) $G = \langle a, b, u \mid u^{p^n} = 1, b^a = bu, u^a = u^{1+p^s},$

$$u^b = u, b^{p^m} = u^{rp^k}, a^{p^l} = b^{p^h} u^{tp^j} \rangle$$

with $0 < s < n$, $0 < k < n$, $m > k$, $m > s$, $p^m \equiv rp^{k+s}$ (p^n), $r \not\equiv 0$ (p), $l > h$, $m > h$, $n - s > l - h$, $t \not\equiv 0$ (p), $l \geqslant n$, $p^h + tp^{j+s} \equiv 0$ (p^n), $rp^k + tp^{j+m-h} \equiv 0$ (p^n).

The effect of Aut G *on the generators* a, b *is*

$$\begin{cases} a \mapsto ac^{\lambda} b^{\mu} u^{\nu}, \\ b \mapsto bc^{x} b^{\eta p^{m-h}} u^{z}, \end{cases}$$

where $c = a^{p^{n-s}} b^{-p^{n-s-l+h}}$, $\mu p^{l-h} \equiv 0$ (p^{n-j}), *and* $\lambda \equiv 0$ (p^{s-k}) *in case* $k < s$.

— *If* $s \leqslant k$ *and* $l - h \geqslant n - j$:

$$|\mathrm{Aut}\, G| = p^{2l+m+n+h+s}, \qquad |\mathrm{Inn}\, G| = p^{2n}, \qquad |\mathrm{Aut}_l G| = p.$$

— *If* $s \leqslant k$ *and* $l - h < n - j$:

$$|\mathrm{Aut}\, G| = p^{3l+m-n+j+s}, \qquad |\mathrm{Inn}\, G| = p^{2n}, \qquad |\mathrm{Aut}_l G| = 1.$$

— *If* $s > k$:

$$|\mathrm{Aut}\, G| = p^{3l-n+h+s+2k}, \qquad |\mathrm{Inn}\, G| = p^{2n}, \qquad |\mathrm{Aut}_l G| = 1.$$

8. In this final section, we will use the notation established in section 7 to study the only case left, namely: for all $b \in C_G(G') \setminus \Phi(G)$, $\langle bG' \rangle$ is not a direct factor of G/G' and $\langle b \rangle$ is not a direct factor of $\langle b, G' \rangle$, and also $\langle a \rangle \wedge G' \neq 1$ for every $a \in G \setminus < b, \Phi(G) >$. We saw that G has a presentation $G = \langle a, b, u \mid u^{p^n} = 1$, $b^a = bu$, $u^a = u^{1+p^s}$, $u^b = u$, $b^{p^m} = u^{rp^k}$, $a^{p^l} = b^{p^h} u^{tp^j} \rangle$ (with some conditions on the numbers n, s, l, h, t, j, m, r, k, for which we refer to the previous section). And $\langle a \rangle \wedge G' = \langle a^{p^{l+m-h}} \rangle$, where $a^{p^{l+m-h}} = u^{rp^k + tp^{j+m-h}}$, so that $rp^k + tp^{j+m-h} \not\equiv 0$ (p^n). We set $\langle a \rangle \wedge G' = \langle u^{p^i} \rangle$; we have $0 < i < n$. Of course, $i = k$ in case $k < j + m - h$, and $i = j + m - h$ in case $k > j + m - h$. If $k = j + m - h$ then $i = k + i'$, where $r + t \equiv 0$ $(p^{i'})$, $r + t \not\equiv 0$ $(p^{i'+1})$. We claim that $i < k + l - h$. This is obvious in the first two cases. Suppose $k = j + m - h$; if $i \geqslant k + l - h$, then $r + t \equiv 0$ (p^{l-h}), and the congruence $r + t + p^{l-h} r\mu \equiv 0$ (p^{n-k}) has a solution μ_0: using (2) we get $(ab^{\mu_0})^{p^{l+m-h}} = (b^{p^h(1+\mu_0 p^{l-h})} u^{tp^j})^{p^{m-h}} =$

$= u^{rp^k(1+\mu_0 p^{l-h})+tp^{j+m-h}} = u^{p^k(r+t+r\mu_0 p^{l-h})} = 1$, contradicting an earlier assumption.

Set once again $\overline{a} = ac^{\lambda} b^{\mu} u^{\nu}$, $\overline{b} = c^x b^y u^z$, $\overline{u} = [\overline{b}, \overline{a}]$, where $c = a^{p^{n-s}} b^{-p^{n-s-l+h}}$ and $y \not\equiv 0$ (p). There exists $\theta \in \operatorname{Aut} G$ such that $a^{\theta} = \overline{a}$, $b^{\theta} = \overline{b}$ if and only if $\overline{b}^{p^m} = \overline{u}^{rp^k}$ and $\overline{a}^{p^l} = \overline{b}^{p^h} \overline{u}^{tp^j}$, *i.e.* if and only if (4) and (6) hold. Suppose first that $\theta \in C_{\operatorname{Aut} G}(a)$; then (4) and (6) with $\lambda, \mu = 0$ are, respectively:

$$(a^{p^{l+m-h}})^{xp^{n-s+h-l}} = 1 ; \qquad a^{p^l(1-y)} = 1$$

whose solutions are: $x \equiv 0$ (p^{s+l-h}) if $i < s + l - h$ (and x arbitrary if $i \geqslant s + l - h$), $y \equiv 1$ $(p^{m-h+n-i})$. In this way we determine $|C_{\operatorname{Aut} G}(a)| = p^{l-n+h+i}$ if $i \geqslant s + l - h$, $|C_{\operatorname{Aut} G}(a)| = p^{2h+i-n}$ if $i < s + l - h$ (notice that $u^{p^i} \in Z(G)$, hence $i \geqslant n - s$ and $h + i - n \geqslant h - s > 0$)).

We will now use (4) and (6) again, in order to find the Aut G-orbit of a. If $(\overline{a}, \overline{b}) = (a^{\theta}, b^{\theta})$, $\theta \in \operatorname{Aut} G$, then from (6) we get $a^{p^l(1+\mu p^{l-h}-y)} \in \langle a \rangle \wedge \langle u \rangle$, hence $1 + \mu p^{l-h} - y \equiv 0$ (p^{m-h}) and $y - 1 = \mu p^{l-h} - \rho p^{m-h}$ for some ρ. Note that $j + n - s + m - h > l + s + k - s - h = k + l - h > i$ implies $u^{tp^{j+n-s}\rho p^{m-h}} \in \langle a \rangle \wedge \langle u \rangle$, so that again (6) yields $u^{tp^j(p^{n-s}(\lambda\mu p^{l-h} - x\mu) + \mu p^{l-h})} \in \langle a \rangle \wedge \langle u \rangle$, *i.e.* $t\mu p^{j+l-h}(\lambda p^{n-s} - xp^{n-s-l+h} + 1) \equiv 0$ (p^j). And we have shown that if $\overline{a}$ is in the Aut G-orbit of a, then $\mu p^{j+l-h} \equiv 0$ (p^i).

For the converse, suppose $\mu p^{j+l-h} = \sigma p^i$ for some σ, and set $\rho p^{m-h} = 1 + \mu p^{l-h} - y$ and $\xi = xp^{n-s-l+h}$. Then (4) and (6) translate into

$$(4^*) \quad qp^i xp^{n-s+h-l} \equiv rp^{k+n-s}(\lambda(1 + \mu p^{l-h} - \rho p^{m-h}) - x\mu) \ (p^n),$$

$$(6^*) \quad qp^i \rho \equiv tp^j(p^{n-s}((\mu p^{l-h} - \rho p^{m-h})\lambda - x\mu) + \mu p^{l-h}) \ (p^n),$$

where $q \not\equiv 0$ (p) is such that $a^{p^{l+m-h}} = u^{qp^i}$.

Since $k + n - s > k + l - h > i$, $k + n - s - i - (n - s - l + h) = k - i - h - l > 0$, $j + n + m - s - h > l + k - h > i$, all the coefficients are divisible by p^i; hence (4) and (6) are equivalent to the system Σ of congruences (in the unknowns ξ, ρ)

$$\begin{cases} (q + \mu r p^{k+l-i-h})\xi + r\lambda p^{k+n-s-i+m-h}\rho \equiv r\lambda p^{k+n-s-i}(1 + \mu p^{l-h}) \ (p^{n-i}), \\ t\sigma\xi + (q + t\lambda p^{j+n-s+m-h-i})\rho \equiv t\sigma(\lambda p^{n-s} + 1) \quad (p^{n-i}). \end{cases}$$

The determinant of Σ is invertible in $\mathbb{Z}/p^{n-i}\mathbb{Z}$ for any choice of λ and μ (with $\mu p^{j+l-h} = \sigma p^i$). Moreover, $k + n - s - i > n - s - l + h$ implies that the solution for ξ (which is unique in $\mathbb{Z}/p^{n-i}\mathbb{Z}$, for given λ and μ) is divisible by $p^{n-s-l+h}$; so we can solve for x, and take $y = 1 + \mu p^{l-h} - \rho p^{m-h}$. And then we conclude that the Aut G-orbit of a is the set

$\{ac^{\lambda}b^{\mu}u^{\nu} \mid \mu p^{j+l-h} \equiv 0 \ (p^i)\}$, whose cardinality is p^{l+m+s} if $j + l - h \geq i$, and $p^{2l+m+s+j-h-i}$ otherwise. We can now state

$$\text{(F.2)} \qquad G = \langle a, b, u \mid u^{p^n} = 1,\ b^a = bu,\ u^a = u^{1+p^s},$$

$$u^b = u,\ a^{p^l} = b^{p^h} u^{tp^j},\ b^{p^m} = u^{rp^k} \rangle$$

where $0 < s < n$, $0 < k < n$, $m > k$, $m > s$, $p^m \equiv rp^{k+s}$ (p^n), $r \not\equiv 0$ (p), $l > h$, $m > h$, $n - s > l - h$, $t \not\equiv 0$ (p), $l \geq n$, $p^h + tp^{j+s} \equiv 0$ (p^n), $rp^k + tp^{j+m-h} \not\equiv 0$ (p^n), $j + m + n > l + s + k$.

Put p^i = *the p-part of* $rp^k + tp^{j+m-h}$.

— *If* $i \geq s + l - h$ *and* $i \leq j + l - h$:

$$|\mathrm{Aut}\, G| = p^{2l+m-n+s+h+i}, \qquad |\mathrm{Inn}\, G| = p^{2n}, \qquad |\mathrm{Aut}_l\, G| = p\,.$$

— *If* $i \geq s + l - h$ *and* $i > j + l - h$:

$$|\mathrm{Aut}\, G| = p^{3l+m-n+s+j}, \qquad |\mathrm{Inn}\, G| = p^{2n}, \qquad |\mathrm{Aut}_l\, G| = 1\,.$$

— *If* $i < s + l - h$ *and* $i \leq j + l - h$:

$$|\mathrm{Aut}\, G| = p^{l+m-n+s+2h}, \qquad |\mathrm{Inn}\, G| = p^{2n}, \qquad |\mathrm{Aut}_l\, G| = p\,.$$

— *If* $i < s + l - h$ *and* $i > j + l - h$:

$$|\mathrm{Aut}\, G| = p^{2l+m-n+s+h+j}, \qquad |\mathrm{Inn}\, G| = p^{2n}, \qquad |\mathrm{Aut}_l\, G| = 1\,.$$

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