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Butler Groups Cannot be Classified by Certain Invariants.

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SUMMARY - In this paper, we give a necessary and sufficient condition for two B_0-groups to be quasi-isomorphic. This new characterization of quasi-isomorphism can be interpreted as an equivalence theorem for quasi-isomorphism, which extends the isomorphism equivalence theory that has heretofore proved fruitful in a variety of different settings. In the present context, this result enables us to prove that quasi-isomorphism invariants which have recently been used to classify certain strongly indecomposable Butler groups do not suffice in general. In particular, the quasi-isomorphism invariants that classify strongly indecomposable groups that are imbeddable as corank 1 pure subgroups of finite rank completely decomposable groups are not adequate in the corank 2 case.

A *Butler group* is a torsion-free abelian group that can be imbedded as a pure subgroup of a finite rank completely decomposable group. An alternative characterization is that Butler groups are precisely those that are torsion-free homomorphic images of finite rank completely decomposable groups [BUT]. A Butler group G is said to be a B_0*-group* provided, for each type τ, $G(\tau^*) = \langle G(\sigma) : \sigma > \tau \rangle$ is a pure subgroup. In this paper, we give a characterization of when two B_0-groups are quasi-isomorphic. Recall that finite rank torsion-free groups G and G' are quasi-isomorphic if and only if there exists monomorphisms $\phi : G \to G'$ and $\phi' : G' \to G$. More insightful is the observation that G and G' are quasi-isomorphic if and only if G' is isomorphic to a subgroup H of G

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with G/H finite. Our main result will rest on the following theorem, which can be thought of as an equivalence theorem for quasi-isomorphism.

THEOREM 1. Let G and G' be B_0-groups, and suppose that

$$(1) \qquad 0 \to B \to A \xrightarrow{\pi} G \to 0$$

and

$$(2) \qquad 0 \to B' \to A' \xrightarrow{\pi'} G' \to 0$$

are balanced exact sequences where A and A' are isomorphic finite rank completely decomposable groups. Then G and G' are quasi-isomorphic if and only if the following condition is satisfied:

(3) There exist monomorphisms $\psi: A \to A'$ and $\psi': A' \to A$ such that $B'/\psi(B)$ and $B/\psi'(B')$ are finite groups.

Perhaps surprisingly, the sufficiency of condition (3) is essentially trivial and does not depend on the hypothesis that the resolutions are balanced. Indeed given a monomorphism ψ as in (3), a routine diagram chase yields a homomorphism $\phi: G \to G'$ such that $\phi\pi = \pi'\psi$. Furthermore, because $B'/\psi(B)$ is finite, each element of ker ϕ necessarily has finite order. Since, however, G is torsion-free, ϕ is monic. By symmetry, there is an induced monomorphism $\phi': G' \to G$, and thus (3) implies that G and G' are quasi-isomorphic. Obviously then we do not put Theorem 1 forward primarily as a method for establishing the quasi-isomorphism of two given groups. Instead its greatest potential resides in proving two groups not quasi-isomorphic that are so closely related that other methods fail to distinguish them up to quasi-isomorphism. The demonstration that G and G' being quasi-isomorphic implies condition (3) is fairly involved and our proof relies heavily on an equivalence theorem established in [HM3].

For the convenience of the reader, before proceeding further, we now establish the notational conventions that will be in effect throughout this paper and recollect certain relevant definitions. All groups considered will be additively written abelian groups and, with few exceptions, torsion-free of finite rank. By a *height* we understand a sequence $s = (s_p)_{p \in \mathbb{P}}$ where $\mathbb{P}$ denotes the set of rational primes and each s_p is either a nonengative integer or the symbol ∞. Two heights s and t are *equivalent*, $s \sim t$, provided (i) $s_p = t_p$ for almost all p and (ii) $s_p = \infty$. if and only if $t_p = \infty$. An equivalence class of heights will be called a *type*. As functions to an ordered set, heights are ordered pointwise and this ordering of heights induces an order on the set of types: $\sigma \leq \tau$ if and only if there exist heights $s \in \sigma$ and $t \in \tau$ such that $s \leq t$. Since $\omega \cup \cup \{\infty\}$ is linearly ordered, the ordered sets of heights and types are distributive lattices. We shall employ the symbols $\wedge$ and $\vee$ for the lattice operations in all three of these ordered sets. If x is an element of the torsion-free group G, then we associate with it the height $|x| = = (|x|_p)_{p \in \mathbb{P}}$ where $|x|_p$ is the ordinary p-height of x computed in G. We let $\text{type}_G\,(x)$ denote the type determined by $|x|$. If $\text{rank}(G) = 1$, then all nonzero elements of G have the same type which we indicate as type (G). If H is a pure subgroup of the torsion-free group G, then we say that H is *balanced* in G provided each coset $x + H$ contains an element y with $|y| \geq |x + h|$ for all $h \in H$. In terms of types rather than heights, there is an equivalent formulation of balanceness that is usually easier to apply; namely, each coset $x + H$ contains an element y such that $\text{type}_G\,(y) = \text{type}_{G/H}\,(x + H)$ [AR2, pp. 4-5]. In Theorem 4 below, we give another characterization of balanced subgroups that involves both types and heights but avoids computations in the quotient group G/H. An epimorphism $\phi: G \to K$ is said to be *balanced* if $\ker \phi$ is a balanced subgroup of G. If H is an arbitrary subgroup of the torsion-free group G, then H_* will denote the minimal pure subgroup containing H.

Let G be a fixed torsion-free group. With each height s we associate the fully invariant subgroups $G(s) = \{x \in G: |x| \geq s\}$ and $G(s^*) = = \langle G(t): t \geq s \text{ and } t \nsim s\rangle$. We shall write $G(s^*, p)$ for the subgroup $G(s^*) + pG(s)$. A nonzero element $x \in G$ is said to be *primitive* provided $x \notin G(s^*, p)$ whenever $|x| \sim s$ and $|x|_p \neq \infty$. If σ is a type, then the fully invariant subgroup $G(\sigma) = \{x \in G: \text{type}_G\,(x) \geq \sigma\} = \bigcup_{s \in \sigma} G(s)$ is a pure subgroup; but, in general, $G(\sigma^*) = \langle G(\tau): \tau > \sigma\rangle$ need not be pure in G. If G has finite rank, then it is a Butler group if and only if it satisfies the following three conditions: (i) G has finite *typeset* $\{\text{type}_G\,(x): 0 \neq \neq x \in G\}$; (ii) $G(\sigma^*)_* / G(\sigma^*)$ is finite for each type σ; (iii) for each type σ, there is a completely decomposable subgroup G_σ such that $G(\sigma) = G_\sigma \oplus \oplus G(\sigma^*)_*$. See [AV1] for a proof of this fundamental characterization of Butler groups, Clearly each nonzero element of G_σ has type σ and, in fact, each such element is primitive in G [HM1]. Accordingly, we refer to $P_G = \{\sigma: G(\sigma) \neq G(\sigma^*)_*\}$ as the set of *primitive types* of the Butler group G. If $C = \sum_{\sigma \in P_G} C_\sigma$ where $G(\sigma) = C_\sigma \oplus G(\sigma^*)_*$ for each σ, then C is called a *regulating subgroup* of the Butler group G. If G is *almost completely decomposable* in the sense that it is quasi-isomorphic to a completely decomposable group, then each regulating subgroup is completely decomposable [AR2, Corollary 3] and quasi-isomorphic to G.

PROPOSITION 2. Let $\pi: A \to G$ be a balanced epimorphism where A is a finite rank almost completely decomposable group and G is a B_0-

group. Then A contains a regulating subgroup C such that $\pi_1 : C \to G$ is a balanced epimorphism where $\pi_1 = \pi | C$.

PROOF. Since G is a B_0-group, $G = \sum_{\sigma \in P_G} G_\sigma$ where $G(\sigma) = G_\sigma \oplus G(\sigma^*)$ for each $\sigma \in P_G$. We begin by considering an arbitrary regulating subgroup $C = \bigoplus_{\sigma \in P_G} C_\sigma$ of A. Now assume that we have constructed a subgroup $B = \bigoplus_{\sigma \in P_G} B_\sigma$ of A that satisfies the following conditions for each $\sigma \in P_G$:

(i) B_σ is a direct summand of C_σ.

(ii) $\pi(B_\sigma)$ is a direct summand of G_σ.

(iii) $B_\sigma \cap \ker \pi = 0$.

If it should furthermore happen that

(iv) $\pi(B_\sigma) = G_\sigma$ for each $\sigma \in P_G$,

then Theorem 2.2 of [AR2] implies that $\pi_1 = \pi | C$ is a balanced epimorphism. Suppose, however, that there is some $\tau \in P_G$ such that $\pi(B_\tau) \neq G_\tau$. In this case there is a primitive element y of G such that $\pi(B_\tau) \oplus \langle y \rangle_*$ is a direct summand of G_τ. Let $t = |y| \in \tau$. Since $\pi : A \to G$ is a balanced epimorphism, there is an $x \in A$ such that $|x| = t$ and $\pi(x) = y$. By (i), we have a direct decomposition $C_\tau = B_\tau \oplus D_\tau$ and hence we can write $x = b + d + z$ where $b \in B_\tau$, $d \in D_\tau$ and $z \in A(\tau^*)_*$. Replacing y by a nonzero multiple of itself, we may assume that $z \in A(\tau^*)$. Because $|x| = |b| \wedge |d| \wedge |z|$, each of the elements b, d and z lies in $A(t)$ and consequently z is in $A(\tau^*) \cap A(t) = A(t^*)$. We claim that $|d| = |x|$. Indeed if this were not the case, then we would have $x - b \in A(t^*, p)$ for some prime p with $t_p \neq \infty$. This, in turn would imply that $y - \pi(b) \in G(t^*, p)$, which is readily seen to contradict the fact that $\pi(B_\tau) \oplus \langle y \rangle_*$ is a direct summand of G_τ. Having now confirmed that $|d| = |x|$, we note that Lemma 2.5 in [HM1] implies that $A(\tau) = B_\tau \oplus \langle x \rangle_* \oplus D_\tau' \oplus A(\tau^*)_*$ where D_τ' is any complement in D_τ of the rank 1 direct summand $\langle d \rangle_*$. If we take $B_\tau' = B_\tau \oplus \langle x \rangle_*$, $C_\tau' = B_\tau' \oplus D_\tau'$ and, for $\sigma \neq \tau$, $B_\sigma' = B_\sigma$ and $C_\sigma' = C_\sigma$, then $C' = \sum_{\sigma \in P_A} C_\sigma'$ is a new regulating subgroup of A. Furthermore B' is a direct summand of C' with conditions (i), (ii) and (iii) still satisfied. A finite number of repetitions of the foregoing argument will yield an appropriate C and B with condition (iv) also satisfied; and thus the proof is complete.

COROLLARY. If G is a B_0-group, then the natural map $\rho : \bigoplus_{\sigma \in P_G} G_\sigma \to G$ is a *balanced projective cover* for G; that is, given a balanced epimorphism $\pi : G \to A$ with A completely decomposable, there exists a balanced epimorphism $\theta : A \to \bigoplus_{\sigma \in P_G} G_\sigma$ such that $\rho\theta = \pi$.

PROOF. By the proof of Proposition 2, A contains a direct summand $B = \bigoplus_{\sigma \in P_G} B_\sigma$ such that, for each σ, π maps B_σ isomorphically onto G_σ. Thus we may write $A = B \oplus \bigoplus_{i \in I} \langle x_i \rangle_*$ where $x_i \neq 0$ for each i. Since $\pi \mid B$ is a balanced epimorphism by [AR2, Theorem 2.2], we have, for each $i \in I$, a $b_i \in B_i$ such that $\pi(b_i) = \pi(x_i)$ and $|b_i| = |\pi(x_i)|$. It then follows from [HM1, Lemma 2.5] that $A = B \oplus \bigoplus_{i \in I} D_i$ where $D_i = \langle x_i - b_i \rangle_*$ for all i. Clearly $D = \bigoplus_{i \in I} D_i$ is contained in ker π and therefore, identifying B with $\bigoplus_{\sigma \in P_G} G_\sigma$, the desired θ is that projection of A onto B with ker $\theta = D$.

We need one further technical result before we can finish the proof of Theorem 1.

LEMMA 3. Let (1) and (2) be as in the statement of Theorem 1 and assume that the B_0-groups G and G' are quasi-isomorphic. Then for all types σ,

$$\text{rank}\left(\frac{B \cap A(\sigma)}{B \cap A(\sigma^*)}\right) = \text{rank}\left(\frac{B \cap A'(\sigma)}{B \cap A'(\sigma^*)}\right). \tag{3'}$$

PROOF. Let P_G and the G_σ's be as in the proof of Proposition 2 and take ρ: $\bigoplus_{\sigma \in P_G} G_\sigma \to G$ to be the canonical map induced by the identity maps of the G_σ's. Then, by the preceding corollary, there is a balanced epimorphism θ: $A \to \bigoplus_{\sigma \in P_G} G_\sigma$ such that $\rho\theta = \pi$. Since completely decomposable groups are balanced projectives [FU, Theorem 86.2], it follows that there is a direct decomposition $A = C \oplus D$ where $C \cong \bigoplus_{\sigma \in P_G} G_\sigma$ and $D \subseteq \ker \pi = B$. Therefore $B = B_0 \oplus D$ where $B_0 = B \cap C = (\ker \pi) \cap C$. But, by Theorem 4.2 in [HM2], $B_0 \cap C(\sigma) = B_0 \cap C(\sigma^*)$ for all types σ, and consequently $B \cap A(\sigma)/B \cap A(\sigma^*) \cong D(\sigma)/D(\sigma^*)$ for all types σ. Similarly, we have a direct decomposition $A' = C' \oplus D'$ where $C' \cong \bigoplus_{\sigma \in P_G} G'_\sigma$ and $B' \cap A'(\sigma)/B' \cap A'(\sigma^*) \cong D'(\sigma)/D'(\sigma^*)$ for all types σ. Since, however, G and G' are quasi-isomorphic, $C \cong C'$. Therefore $A \cong A'$ implies $D \cong D'$. Clearly (3') follows from the foregoing observations.

We are now in position to complete the proof of Theorem 1. Assuming that G and G' are quasi-isomorphic, fix a monomorphism ϕ': $G' \to G$ and let $G_1 = \phi'(G')$ where G/G_1 is finite. By symmetry, it suffices to construct a monomorphism ψ': $A' \to A$ such that $B/\psi'(B')$ is finite. Since B is balanced in $A^* = \pi^{-1}(G_1)$ and the latter is almost completely decomposable because $A/A^* \cong G/G_1$, an application of Proposition 2 yields a balanced exact sequence

$$0 \to B_1 \to A_1 \xrightarrow{\pi_1} G_1 \to 0$$

where $A_1 \cong A$. By Proposition 1.7 in [HM2], B' and B_1 are *weakly $*$-pure* subgroups (see Definition 1.3 in [HM2]) of A' and A_1, respectively. As B' and B_1 are balanced subgroups, the isomorphism $G' \cong G_1$ can be construed as an isomorphism $\phi: A'/B' \to A_1/B_1$ that *respects heights* in the sense that

$$\phi((A'(s) + B')/B') = (A_1(s) + B_1)/B_1$$

for all heigths s. Finally an application of Lemma 3 shows that all the hypotheses of Theorem 1.5 in [HM2] (see [HM3] for a proof) are satisfied and hence that theorem implies the existence of an isomorphism $\psi': A' \to A_1$ with $\psi'(B') = B_1$. Noting that $B/B_1 = (\ker\pi)/(A_1 \cap \cap \ker\pi) \cong (\ker\pi + A_1)/A_1$ is finite since A^*/A_1 is, we can view ψ' as the desired monomorphism from A' to A with $B/\psi'(B')$ finite.

Given a torsion-free group G, there is a standard method for constructing a balanced epimorphism $\pi: A \to G$ with A completely decomposable. When G is a Butler group, this can be done in a quite explicit manner so as to insure that A also has finite rank (see Theorem 1.2 in [AV1]). On the other hand, the problem of determining when a particular pure subgroup B of a completely decomposable group A is balanced is less often dealt with in the literature. The difficulty in the latter situation is that we are not given, *a priori*, any specific information about the structure of the quotient group $G = A/B$. To be sure, heights in A/B are computable via the formula $|a + B| = \sup\{|a + b| : b \in B\}$, but it is the unwieldy nature of this formula which is the heart of the difficulty in determining $\text{type}_{A/B}(a + B)$. There is nevertheless a very elementary criterion for balanceness that can often be applied quite efficiently when one is given adequate information about the generators of B. We say that the pure subgroup B is *prebalanced* in the torsion-free group A provided for each $a \in A$ there exists a finite collection of elements $b_1, b_2, \dots b_n$ in B such that $\sup\{|a + b_k| : k = 1, 2, \dots, n\} = \sup\{|a + b| : b \in B\}$.

THEOREM 4. A pure subgroup B of the torsion-free group A is a balanced subgroup of A if and only if the following two conditions are satisfied.

(i) B is prebalanced in A.

(ii) For each $a \in A \setminus B$, the coset $a + B$ contains an element of maximum type.

PROOF. Clearly if B is balanced in A, then conditions (i) and (ii) are satisfied since there is a single $b_0 \in B$ with $|a + b_0| = \sup\{|a + b| : b \in B\} = |a + B|$ and hence $\text{type}_A(a + b_0) = \text{type}_{A/B}(a + B) \geq \text{type}_A(a + b)$ for all $b \in B$. Conversely, assume that conditions (i) and (ii) are satisfied, and consider an arbitrary $a \in A \setminus B$. Select $b_1, b_2, \ldots, b_n$ in B such that $|a + B| = \sup\{|a + b_k| : k = 1, 2, \ldots, n\}$ and observe that without loss of generality we may assume that $\tau = \text{type}_A(a) \geq \text{type}_A(a + b)$ for all $b \in B$. Consequently, each of the sets, for the various k's,

$$P_k = \{p \in \mathbb{P} : |a + b_k|_p \neq \infty \text{ and } |a + b_k|_p > |a|_p\}$$

is finite and $|a|_p = \infty$ whenever $|a + b_k|_p = \infty$. Obviously it suffices to show that $\text{type}_{A/B}(a + B) = \tau$. Since $|a + B| \geq |a|$, this amounts to verifying the following two facts;

(iii) If p is a prime such that $|a + B|_p = \infty$, then $|a|_p = \infty$.

(iv) The set $P = \{p \in \mathbb{P} : |a + B|_p \neq \infty \text{ and } |a + B|_p > |a|_p\}$ is finite.

Since $|a + B|_p = |a + b_1|_p \vee |a + b_2|_p \vee \ldots \vee |a + b_n|_p$, (iii) is a consequence of our observation that $|a + b_k|_p = \infty$ implies $|a|_p = \infty$ and (iv) holds because $P = P_1 \cup P_2 \cup \ldots \cup P_n$.

In applying Theorem 4, it is helpful to have a more concrete formulation of condition (i). This is supplied by the following essentially trivial observation.

PROPOSITION 5. *Let B be a pure subgroup of the torsion-free group A and suppose that to each $a \in A \setminus B$ there corresponds a finite subset Q of P that satisfies the following two conditions:*

(α) *If $p \notin Q$, then $|a + b|_p \leq |a|_p$ for all $b \in B$.*

(β) *If $p \in Q$, then $\sup\{|a + b|_p : b \in B\} = |a + b'|_p$ for some particular $b' \in B$.*

Then B is a prebalanced subgroup of A.

PROOF. Let $\{p_1, p_2, \ldots, p_n\} = Q$ and choose for $k = 1, 2, \ldots, n$ an element $b_k \in B$ such that $|a + b_k|_{p_k} = \sup\{|a + b|_{p_k} : b \in B\}$. Then taking $b_0 = 0$, we see that $\sup\{|a + b_k| : k = 0, 1, 2, \ldots, n\} = \sup\{|a + b| : b \in B\}$.

Using Theorem 1, we can show that there exist strongly indecomposable B_0-groups G and G' which are not quasi-isomorphic in spite of having in common all the quasi-isomorphism invariants that have figured in recent classifications of special classes of strongly indecomposable Butler groups. The tecnique used to construct balanced subgroups in the following existence theorem is quite general and lends itself to many variations. (Previously unexplained notation and terminology appearing in the statement of our final theorem, as well as its relevance the literature, will be discussed in the ensuing proof.)

THEOREM 6. There exist two strongly indecomposable Butler groups G and G' that are not quasi-isomorphic, but which satisfy the following conditions:

(1) $\operatorname{rank}(G) = \operatorname{rank}(G')$.

(2) The typesets of G and G' are equal.

(3) The cotypesets of G and G' are equal.

(4) G and G' have isomorphic endomorphism rings.

(5) G and G' have the same Richman type.

(6) $\operatorname{rank} G(M) = \operatorname{rank} G'(M)$ for all subsets M of the typesets of G and G'.

(7) $\operatorname{rank} G[M] = \operatorname{rank} G'[M]$ for all subsets M of the cotypesets of G and G'.

(8) $r_G(\tau, \sigma) = r_{G'}(\tau, \sigma)$ for all types τ and σ.

Select four rank 1 groups A_1, A_2, A_3, A_4 of incomparable idempotent type $\tau_1, \tau_2, \tau_3, \tau_4$, respectively, such that $\tau_i \wedge \tau_j = \tau_0 = \operatorname{type}(\mathbb{Z})$ whenever $i \neq j$. Then fix elements $a_i \in A_i$ for $i = 1, 2, 3, 4$ such that each $|a_i|$ involves only 0's and ∞'s. We also assume, for reasons that will shortly be evident that $|a_3|_p = \infty$ for $p = 2$, 3 and 5. In order to construct our first group G, we let B denote the pure subgroup of $A = A_1 \oplus A_2 \oplus \oplus A_3 \oplus A_4$ generated by $b_1 = a_1 + a_2 + a_3$ and $b_2 = a_2 + 3a_3 + a_4$. We shall represent an arbitrary element $a \in A$ in the form $a = s_1 a_1 + s_2 a_2 + + s_3 a_3 + s_4 a_4$ where the s_i's are appropriate rational numbers. Indeed we may write, for each $i = 1, 2, 3, 4$, $s_i = n_i/m_i$ where $\gcd(n_i, m_i) = 1$ and m_i is a positive integer with prime factors from $P_i = \{p \in \mathbb{P}\colon |a_i|_p = = \infty\}$. By considering the equation $na = t_1 b_1 + t_2 b_2$, a simple computation establishes the following:

(0) $a \in B$ if and only if $s_2 - s_1 - s_4 = 0$ and $s_3 - s_1 - 3s_4 = 0$.

Furthermore, since the sets P_1, P_2, P_3, P_4 are pairwise disjoint, elementary number theoretical arguments show that each of s_1, s_2, s_3 and s_4 is an integer provided the two equations in condition (0) are satisfied. Thus when $a \in B$, it is easily seen that $a = s_1 b_1 + s_4 b_2$, and therefore $B = \langle b_1 \rangle \oplus \langle b_2 \rangle$.

The next four observations are almost equally routine, but require our assumption that 2 and 3 are in P_3. Assuming that $a \notin B$, then the following hold:

(1) $(a + B) \cap A_1 \neq \emptyset$ if and only if $s_3 - s_2 - 2s_4 = 0$.

(2) $(a + B) \cap A_2 \neq \emptyset$ if and only if $s_3 - s_1 - 3s_4 = 0$.

(3) $(a + B) \cap A_3 \neq \emptyset$ if and only if $s_2 - s_1 - s_4 = 0$.

(4) $(a + B) \cap A_4 \neq \emptyset$ if and only if $s_3 + 2s_1 - 3s_2 = 0$.

For example, if $a + t_1 b_1 + t_2 b_2$ is an element of A_1, then the equation $s_3 - s_2 - 2s_4 = 0$ follows from the fact that $s_2 + t_1 + t_2 = s_3 + t_1 + 3t_2 = s_4 + t_2 = 0$. Conversely suppose that $s_3 - s_2 - 2s_4 = 0$. Then another simple number theoretical argument exploiting the fact that $2 \notin P_4$ allows us to conclude that s_2, s_3 and s_4 are integers. Taking $t_1 = s_4 - s_2$ and $t_2 = -s_4$, we see that $a + t_1 b_1 + t_2 b_2$ is in A_1. Since $s_3 + 2s_1 - 3s_2 = (s_3 - s_1 - 3s_4) - 3(s_2 - s_1 - s_4)$ and $s_3 - s_2 - 2s_4 = (s_3 - s_1 - 3s_4) - (s_2 - s_1 - s_4)$, it quickly follows from (0)-(4) that if $a \in A \setminus B$, then there is at most a single $i = 1, 2, 3, 4$ such that $(a + B) \cap A_i \neq \emptyset$. Consequently, condition (ii) of Theorem 4 is satisfied.

To complete the proof that B is balanced in A, we need only verify that B is prebalanced. We shall accomplish this by showing that conditions (α) and (β) of Proposition 5 are satisfied. Towards this end, we note that we have a partition $\{P_0, P_1, P_2, P_3, P_4\}$ of $\mathbb{P}$ where

$$P_0 = \{p \in \mathbb{P}\colon |a_1|_p = |a_2|_p = |a_3|_p = |a_4|_p = 0\}$$

and, whenever s is a rational number and p is a prime, we write $|s|_p = k$ to indicate that $s = p^k(m/m)$ where neither n nor m is divisible by p. We shall require the following crucial facts:

(0′) If $p \in P_0$ and $|a + b|_p \geqslant k$ for some $b \in B$, then $|s_2 - s_1 - s_4|_p \geqslant k$ and $|s_3 - s_1 - 3s_4|_p \geqslant k$.

(1′) If $p \in P_1$ and $|a + b|_p \geqslant k$ for some $b \in B$, then $|s_3 - s_2 - 2s_4|_p \geqslant k$.

(2′) If $p \in P_2$ and $|a + b|_p \geqslant k$ for some $b \in B$, then $|s_3 - s_1 - 3s_4|_p \geqslant k$.

(3′) If $p \in P_3$ and $|a + b|_p \geqslant k$ for some $b \in B$, then $|s_2 - s_1 - s_4|_p \geqslant k$.

(4′) If $p \in P_4$ and $|a + b|_p \geqslant k$ for some $b \in B$, then $|s_3 + 2s_1 - 3s_2|_p \geqslant k$.

We shall verify (0′) and (1′). Write $b = t_1 b_1 + t_2 b_2$. If $p \in P_0$, then

$$|a + b|_p = |s_1 + t_1|_p \wedge |s_2 + t_1 + t_2|_p \wedge |s_3 + t_1 + 3t_2|_p \wedge |s_4 + t_2|_p .$$

So if $|a + b|_p \geqslant k$, then

$$|s_2 - s_1 - s_4|_p = |(s_2 + t_1 + t_2) - (s_1 + t_1) - (s_4 + t_2)|_p \geqslant$$

$$\geqslant |s_2 + t_1 + t_2|_p \wedge |s_1 + t_1|_p \wedge |s_4 + t_2|_p \geqslant k$$

and similarly

$$|s_3 - s_1 - 3s_4|_p \geqslant |s_3 + t_1 + 3t_2|_p \wedge |s_1 + t_1|_p \wedge |3(s_4 + t_2)|_p \geqslant k .$$

On the other hand, if $p \in P_1$, then

$$|a + b|_p = |s_2 + t_1 + t_2|_p \wedge |s_3 + t_1 + 3t_2|_p \wedge |s_4 + t_2|_p .$$

Hence, in this case, $|a + b|_p \geqslant k$ implies

$$|s_3 - s_2 - 2s_4|_p \geqslant |s_3 + t_1 + 3t_2|_p \wedge |s_2 + t_1 + t_2|_p \wedge |2(s_4 + t_2)|_p \geqslant k .$$

We can now show each $a \in A \setminus B$ satisfies the conditions of Proposition 5. First consider the situation where $(a + B) \cap A_i = \emptyset$ for $i = 1, 2, 3, 4$. In this instance, (0)-(4) imply that all four of the rationals $r_1 = s_3 - s_2 - 2s_4$, $r_2 = s_3 - s_1 - 3s_4$, $r_3 = s_2 - s_1 - s_4$, and $r_4 = s_3 + 2s_1 - 3s_2$ are nonzero, and furthermore $|a|_p$ is finite for each prime p since $\tau_i \wedge \tau_j = \tau_0$ whenever $i \neq j$. Then, by (0′)-(4′), the set Q of those primes p for which there is some $b \in B$ with $|a + b|_p > |a|_p$ must be finite, since any such p must have the property that $|r_i|_p \neq 0$ for some $i = 1$, 2, 3 or 4. (Note that $s \neq 0$ implies $|s|_p = 0$ for all but finitely many primes p.) But even if $p \in Q$, $\sup\{|a + b|_p : b \in B\}$ is attained for some $b' \in B$ because this supremum cannot exceed the maximum of $|r_1|_p$, $|r_2|_p$, $|r_3|_p$, $|r_4|_p$. The argument where $(a + B) \cap A_i \neq \emptyset$ for some i is only slightly different. For definiteness, consider the case where $(a + B) \cap A_1 \neq \emptyset$. Without loss of generality we may assume that $a = s_1 a_1 \neq 0$, so that $s_2 = s_3 = s_4 = 0$. In this case, we take $Q = \{p \in \mathbb{P} : p \notin P_1 \text{ and } |s_1|_p \neq 0\}$, which is certainly finite since $s_1 \neq 0$. When $p \in Q$, $\sup\{|a + b|_p : p \in B\} \leqslant |s_1|_p$ and hence the supremum is attained for some $b' \in B$. On the other hand, if $p \notin Q$, then either $|a|_p = \infty$ or else $|a + b|_p = 0$ for all $b \in B$. Thus taking $G = A/B$, we see that the canonical short exact sequence

$$0 \to B \to A \xrightarrow{\pi} G \to 0$$

is balanced exact and, by Proposition 1.7 in [HM2] (or from more elementary considerations), G is a B_0-group.

To construct the companion group $G' = A'/B'$, we take $A' = A = A_1 \oplus A_2 \oplus A_3 \oplus A_4$ and let B' be the pure subgroup generated by $b_1' = b_1 = a_1 + a_2 + a_3$ and $b_2' = a_2 + 5a_3 + a_4$. The proof that $B' = \langle b_1' \rangle \oplus \langle b_2' \rangle$ is balanced in A' is substantially the same as that given above for B in A, except in this case $a' = s_1 a_1 + s_2 a_2 + s_3 a_3 + s_4 a_4$ lies in B' if and only if $s_2 - s_1 - s_4 = 0$ and $s_3 - s_1 - 5s_4 = 0$. Obviously $B \cong B'$, and since both 3 and 5 are in P_3, b_2 and b_2' even have the same associated height vector in the sense of [HM2]. Nevertheless, Theorem 1 implies that the B_0-groups G and G' fail to be quasi-isomorphic. In fact, there cannot exist any monomorphism $\psi: A \to A'$ mapping B into B'. Indeed since the A_i's have incomparable types, if $\psi: A \to A'$ is a monomorphism, then there must exist nonzero rational numbers k_1, k_2, k_3 and k_4 such that $\psi(a_i) = k_i a_i$ for $i = 1, 2, 3, 4$. But then the above conditions for $\psi(b_1) = k_1 a_1 + k_2 a_2 + k_3 a_3$ to lie in B' yield $k_1 = k_2 = k_3$; while the requirement that $\psi(b_2) = k_2 a_2 + 3k_3 a_3 + k_4 a_4$ be an element of B' implies $k_4 = k_2$ and $5k_4 = 3k_3$. Since the k_i's are nonzero, these equations contradict the fact the $5 \neq 3$.

Inspite of the fact that G and G' are not quasi-isomorphic, the two groups are quite alike in structure. Clearly $\operatorname{rank}(G) = \operatorname{rank}(G') = 2$, and the groups have the same typeset $\{\tau_0, \tau_1, \tau_2, \tau_3, \tau_4\}$. By Theorem 3.2 in [AR1], it follows that both G and G' are strongly indecomposable with endomorphisms rings and quasi-endomorphism rings isomorphic, respectively, to $\mathbb{Z}$ and $\mathbb{Q}$. Obviously $G(\tau_i) = \langle a_i + B \rangle_*$ for $i = 1, 2, 3, 4$ and $G(\tau_0) = G$. Since the corresponding observations hold for G', $\operatorname{rank} G(\tau) = \operatorname{rank} G'(\tau)$ for all types τ. Recall that the type μ is said to be a *cotype* of G provided there exists a pure subgroup H such that $\operatorname{rank}(G/H) = 1$ and $\operatorname{type}(G/H) = \mu$. By Remark (4) on page 108 of [AV1] and the fact that $\operatorname{rank}(G/G(\tau_i)) = 1$ for $i = 1, 2, 3, 4$, $\operatorname{type}(G/G(\tau_i)) = \sup\{\operatorname{type}_G(x): x \notin G(\tau_i)\}$ and hence each of the four types

$$\mu_1 = \tau_2 \vee \tau_3 \vee \tau_4, \qquad \mu_2 = \tau_1 \vee \tau_3 \vee \tau_4, \qquad \mu_3 = \tau_1 \vee \tau_2 \vee \tau_4,$$
$$\text{and} \qquad \mu_4 = \tau_1 \vee \tau_2 \vee \tau_3$$

are cotypes of G. Similarly, these μ_i's are cotypes of G', and since the set of cotypes of a Butler group is closed under supremums, $\mu_0 = \tau_1 \vee \tau_2 \vee \tau_3 \vee \tau_4$ is also a cotype of both G and G'. For an arbitrary torsion-free group G and type μ, the fully invariant pure subgroup $G[\mu]$ is defined as the intersection of the kernels of all homomorphisms of G into a rank 1 group of type μ; and when G is a Butler groups, it is known that $G[\mu] = \langle G(\tau): \tau \not\leq \mu \rangle_*$ [AV1, Proposition 1.9]. Consequently, for our particular rank 2 group G, $G[\mu_0] = 0$ and $G[\mu_i] = G(\tau_i)$ for $i = 1, 2, 3, 4$; and similarly for G'. Notice, because $\operatorname{rank}(G) = 2$, that

$$G[\tau_1 \vee \tau_2] = \langle G(\tau): \tau \nleq \tau_1 \vee \tau_2 \rangle_* = \langle G(\tau_3), G\tau_4) \rangle_* = G,$$

and likewise for each $\tau_i \vee \tau_j$ with $i \neq j$. Thus none of the $\tau_i \vee \tau_j$'s can be cotypes of G; nor for that matter can any of the τ_i's. Since the cotypes of any Butler group lie in the lattice generated by its typeset [AV1, Corollary 1.5], it follows that $\{\mu_0, \mu_1, \mu_2, \mu_3, \mu_4\}$ is the set of cotypes of G and also of G'. Consequently, we have $\operatorname{rank} G[\mu] = \operatorname{rank} G'[\mu]$ for all types μ (see Remark (1) on page 107 of [AV1]). The groups G and G' also have the same *Richman type* (see page 12 of [AR1]). Indeed $F = = \langle a_1 + B \rangle \oplus \langle a_2 + B \rangle$ is a free subgroup of G and, from the definition of B, it follows that $\pi^{-1}(F) = \langle a_1, a_2, a_3, a_4 \rangle$. Therefore G/F is isomorphic to the divisible torsion group $D = \bigoplus_{1=1}^{4} A_i / \langle a_i \rangle$. Similarly, $G'/F' \cong D$ where $F' = \langle a_1 + B' \rangle \oplus \langle a_2 + B' \rangle$, and so G and G' are both *quotient-divisible* [BP1].

Recently, strongly indecomposable Butler groups G that can be realized by epimorphisms $\pi: A \to G$, where A is a finite rank completely decomposable group and $\operatorname{rank}(\ker \pi) = 1$, have been classified [AV2] up to quasi-isomorphism by the invariants $\operatorname{rank} G(M)$ where $G(M) = = \langle G(\tau): \tau \in M \rangle$ and M is an arbitrary subset of the typeset of G [AV4]. Dually, strongly indecomposable Butler groups G that can be imbedded as a corank 1 pure subgroup of a finite rank completely decomposable group have been classified up to quasi-isomorphism by the invariants rank $G[M]$ where $G[M] = \bigcap \{G[\mu]: \mu \in M\}$ and M is an arbitrary set of cotypes of G. Another class of corank 1 Butler groups, the so-called *CT-groups* of [AV3], have been classified up to quasi-isomorphism by the invariants $r_G(\tau, \sigma) = \operatorname{rank}((G(\tau) + G[\sigma])/G[\sigma])$. From the analysis given in the preceding paragraph for our particular rank 2 groups G and G', it is immediate that $r_G(\tau, \sigma) = r_{G'}(\tau, \sigma)$ for all types τ and σ, and that $\operatorname{rank} G(M) = \operatorname{rank} G'(M)$ and $\operatorname{rank} G[M] = \operatorname{rank} G'[M]$ for every set of types M.

Theorem 6 suggests, as do earlier observations by Arnold and Vinsonhaler, that any further progress in the classification of strongly indecomposable Butler groups will require the introduction of new invariants. Indeed by the example constructed above and the duality theory of [AV4], invariants defined in terms of the $G(\tau)$'s and $G[\mu]$'s will not suffice to classify those strongly indecomposable groups G that are imbeddable as corank 2 pure subgroups of finite rank completely decomposable groups. There are, of course, earlier clasifications up to quasi-isomorphism of quotient-divisible groups [BP1] and of general rank 2 groups [BP2]. But these older invariants have proved rather intractable and have consequently fallen out of favor.

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