RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

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Rendiconti del Seminario Matematico della Università di Padova, tome 90 (1993), p. 53-66
<http://www.numdam.org/item?id=RSMUP_1993__90__53_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 90 (1993)

Quasi-Basic Submodules over Valuation Domains.

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SUMMARY - We define two kinds of new invariants for arbitrary modules over valuation domains. The first kind of invariants form a complete and independent set of invariants for direct sums of uniserial modules, including the nonstandard ones, and they are related to U-quasi-basic submodules, i.e. maximal pure direct sums of uniserial submodules isomorphic to a fixed uniserial U. The second kind of invariants is related to quasi-basic submodules, i.e. maximal pure direct sums of arbitrary uniserial submodules. Uniqueness up to isomorphism of these submodules is also investigated.

Introduction.

This paper can be viewed as a partial refinement of the paper [FS1], written a decade ago by L. Fuchs and the second author, and is motivated by the discovery of nonstandard uniserial modules by Shelah [S].

In the paper [FS1] the notions of heights, indicators, Ulm-Kaplansky invariants and basic submodules have been generalized from the context of abelian groups to the more general setting of modules over valuation domains.

At that time the existence of nonstandard uniserial R-modules, for suitable valuation domains R, was still an open problem. After the first existence proof by Shelah, both the existence problem and structural problems have been extensively investigated (see [BFS] and references there).

Since the notions of prebasic and basic submodules introduced in [FS] deal exclusively with standard uniserials, it is natural to try to

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Lavoro eseguito con il contributo del M.U.R.S.T.

generalize these notions modifying them by including the nonstandard uniserials as well. By doing so, we obtain a more general notion of quasi-basic submodule, which is the main object of this paper.

Quasi-basic submodules are connected with some new invariants, introduced in the third section, and are in general not unique up to isomorphisms.

Actually, uniqueness is recaptured if we consider a «local» quasi-basic submodule; here by «local» we just mean «defined by means of a single uniserial module U». In this way we have U-quasi-basic submodules, which are connected with some other invariants, investigated in section 2; these invariants form a complete and independent set of invariants for arbitrary direct sums of uniserial modules.

These facts remind the situation occurring in abelian groups, when one passes from pure independence and basic subgroups in the local theory of primary groups, to the analogous concepts of quasi-pure independence and maximal completely decomposable pure subgroups in global torsionfree abelian groups (see [G]).

1. Preliminaries.

We denote by R a valuation domain, by P its maximal ideal, and by Q its field of quotients, that we always assume different from R. We recall some notions introduced in [FS1] and developed in [FS2]. Let M be an R-module and $x \in M$ an arbitrary element; then the submodule of Q containing R

$H_M(x) = \{r^{-1} | r \in R, x \in rM\}$ is called the *height ideal* of x in M;

$$h_M(x) = \begin{cases} J/R & \text{if } \exists \phi : J \to M \text{ such that } \phi 1 = x \\ (J/R)^- & \text{otherwise} \end{cases} \qquad (\text{where } J = H_M(x))$$

is the *height* of x in M. In the first case we say that x has a *non limit* height, in the second case x has a *limit* height. The set Σ of all the heights is obviously totally ordered, if we agree that $J/R > (J/R)^-$.

We will always denote by σ a non limit height, and by σ^- the corresponding limit height, slightly modifying the notation in [FS1] and [FS2].

The following are fully invariant submodules of M, for each $\sigma = J/R$

(non limit):

$$M^{\sigma^-} = \{x \in M \mid h_M(x) \geqslant \sigma^-\},$$
$$M^{\sigma} = \{x \in M \mid h_M(x) \geqslant \sigma\} = \{x \in M \mid h_M(x) > \sigma^-\},$$
$$M^{\sigma^+} = \{x \in M \mid h_M(x) > \sigma\}.$$

A submodule N of M is called *equiheight* if $h_N(x) = h_M(x)$ $\forall x \in N$, equivalently if $M^\sigma \cap N = N^\sigma$ and $M^{\sigma^-} \cap N = N^{\sigma^-}$ $\forall \sigma$; N is *pure* if $H_N(x) = H_M(x)$ $\forall x \in N$, equivalently, if $M^{\sigma^-} \cap N = N^{\sigma^-}$ $\forall \sigma^-$; notice that, if N is pure in M, then $M^{\sigma^+} \cap N = N^{\sigma^+}$ $\forall \sigma$. For an arbitrary R-module M and ideal $I < R$, the following fully invariant submodules of M and a prime ideal of R are defined

$$M[I] = \{x \in M \mid \mathrm{Ann}_R(x) \geqslant I\},$$

$$M[I^+] = \{x \in M \mid \mathrm{Ann}_R(x) > I\},$$

$$I^\# = \{r \in R \mid rI < I\},$$

where $\mathrm{Ann}_R(x)$ denotes the annihilator ideal of x. Let $\sigma^- = (J/R)^-$ be a limit height; given a proper ideal $I < R$, the following quotient modules are all vector spaces over the field R_L/L, where $L = I^\# \cup J^\#$ and R_L is the localization of R at L:

$$\frac{M^{\sigma^-}[I]}{M^{\sigma^-}[I^+] + M^{\sigma^+}[I]} \geqslant \frac{M^\sigma[I] + M^{\sigma^-}[I^+]}{M^{\sigma^-}[I^+] + M^{\sigma^+}[I]} \cong \frac{M^\sigma[I]}{M^\sigma[I^+] + M^{\sigma^+}[I]}.$$

The larger vector space is denoted in [FS1] by $\overline{\alpha}_M(\sigma^-, I)$, the smaller one by $\alpha_M(\sigma, I)$. Clearly, the invariants $\overline{\alpha}(\sigma^-, I)$ and $\alpha_M(\sigma, I)$ are additive. The following characterizations of the elements of $\overline{\alpha}_M(\sigma^-, I)$ and of its subspace $\alpha_M(\sigma, I)$ are in [FS1].

PROPOSITION 1. *For $\sigma = J/r$ and $I < R$, an element $a \in M$ represents a non zero element of $\overline{\alpha}_M(\sigma^-, I)$ ((respectively of $\alpha_M(\sigma, I)$)) if and only if it satisfies:*

(i) $\mathrm{Ann}_R(a) = I$,

(ii) $H_M(a) = J$(*resp.* $h_M(a) = \sigma$)

(iii) *for all* $r \notin I$, $H_M(ra) = r^{-1}J$(*resp.* $h_M(ra) = r^{-1}\sigma$).

A *uniserial* module U is a module whose submodules are linearly oredered by inclusion. The uniserial module U is said to be *standard* if it is isomorphic to J/I for some $0 \leqslant I \leqslant J \leqslant Q$, otherwise it is called *non standard*. U is said of *type* $[J/I]$ (the isomorphy class of the standard uniserial J/I, $0 < I < R \leqslant J$), if there exists $u \in U$ such that $H_U(u) = J$ and $\mathrm{Ann}_R(u) = I$.

Using Proposition 1 it is easy to see that, given a uniserial module U, for each $\sigma = J/R$ and $I < R$, the following equalities hold

$$(*) \qquad \dim \bar{\alpha}_U(\sigma^-, I) = \begin{cases} 1 & \text{if } t(U) = [J/I], \\ 0 & \text{otherwise}. \end{cases}$$

Following [FS1], a *prebasic* (or α-basic, in the terminology of [FS2]) submodule of a module M is a pure submodule B of M which is a direct sum of *standard* uniserials, such that for each *standard* uniserial submodule V of M, either $B \cap V \neq 0$, or $B \oplus V$ is not pure in M.

2. U-dimension and U-quasi-basic submodules.

Our goal in this section is to investigate more closely the structure of the vector space $\bar{\alpha}_M(\sigma^-, I)$ ($\sigma = J/R$), and its connection with the existence of different isomorphy classes $[U]$ of uniserials of type $[J/I]$, i.e. in case non standard uniserials U of type $[J/I]$ exist. Let $\mathcal{A}$ denote the class of all ordered pairs (U, u), with U uniserial and $0 \neq u \in U$. Consider the quotient set $\mathcal{A}/\sim$, where $\sim$ denotes the equivalence relation defined by

$$(U, u) \sim (V, v)$$

$$\text{if there exists an isomorphism } \phi\colon U \to V \text{ such that } \phi u = v.$$

The equivalence class containing (U, u) will be denoted by $[U, u]$. For every module M and every element $[U, u]$ in $\mathcal{A}/\sim$ we set

$$M^{[U,u]} = \{x \in M \mid \exists \phi \in \mathrm{Hom}_R(U, M) \text{ such that } \phi u = x\}.$$

It is straightforward to show that this set is well defined and that it is a fully invariant submodule of M. Moreover it is easy to see that

(*a*) if $0 \neq ru$, then $M^{[U, ru]} = rM^{[U,u]}$,

(*b*) if $H_U(u) = J$ and $\mathrm{Ann}_R(u) = I$, then $M^{[U,u]} \leqslant M^{\sigma^-}[I]$ (where $\sigma = J/R$),

(*c*) if $[U, u] = [J/I, 1 + I]$, then $M^{[U,u]} = M^{\sigma}[I]$.

For every element $[U, u]$ in $\mathcal{A}/\sim$ such that $t(U) = [J/I]$, $H_U(u) = J$

and $\mathrm{Ann}_R(u) = I$, setting $\sigma = J/R$, we define the following subspace of $\alpha_M(\sigma^-, I)$:

$$\alpha_M(U, u) = \frac{M^{[U, u]} + M^{\sigma^-}[I^+] + M^{\sigma^+}[I]}{M^{\sigma^-}[I^+] + M^{\sigma^+}[I]}.$$

Clearly, $\alpha_M(J/I, 1 + I) = \alpha_M(\sigma, I)$, so the vector subspace $\alpha_M(U, u)$ of $\alpha_M(\sigma^-, I)$ coincides with the subspace denoted by $\alpha_M(\sigma, I)$ in [FS1] if U is standard of type $[J/I]$. From now on, we will write $\alpha_M(J/I, 1 + I)$ instead of $\alpha_M(\sigma, I)$.

We complete the result in Proposition 1 by the following result, which gives also another characterization of the elements in $\alpha_M(J/I, 1 + I)$.

PROPOSITION 2. *For U uniserial and $u \in U$ such that $H_U(u) = J$, $\mathrm{Ann}_R(u) = I$, an element $a \in M$ represent a non zero element of $\alpha_M(U, u)$ if and only if the following facts hold:*

i) *there exists an injective homomorphism $\phi: U \to M$ such that $\phi u = a$,*

ii) *the image ϕU of U is a pure submodule of M.*

PROOF. If $a \in M$ represents a non zero element of $\alpha_M(U, u)$, then there is a homomorphism $\phi: U \to M$ such that $\phi u = a$. Obviously, $\mathrm{Ann}_R(a) = I$ and $H_M(a) = J$, thus ϕ is monic. In order to show that ϕU is pure in M, it is enough to prove that, for every $r \in R \setminus I$, $H_M(ra) = H_U(ra) = r^{-1}J$. Assume, by way of contradiction, that $H_M(ra) > r^{-1}J$ for some $r \in R \setminus I$; then there is an element $b \in M$ such that $ra = rb$ and $H_M(b) > J$. Hence $a - b \in M^{\sigma^-}[I^+]$ and $b \in M^{\sigma^+}[I]$, a contradiction. Conversely, if $a \in M$ satisfieds (i) and (ii), then $a \in M^{[U, u]}$. If $a \in M^{\sigma^-}[I^+] + M^{\sigma^+}[I]$, then there is an $r \in R \setminus I$ such that $H_M(ra) > r^{-1}J = H_{\phi U}(ra)$, so ϕU cannot be pure in M, a contradiction. ■

An immediate consequence of Proposition 2 is the following

COROLLARY 3. *In the notation of Propositions* 1 *and* 2, *an element $x \in M$ represents a non zero element of $\alpha_M(\sigma^-, I)$ not in $\alpha_M(U, u)$ if and only if no multiple $rx \neq 0$ of x belongs to a pure uniserial submodule of M isomorphic to U.*

PROOF. Assume that $0 \neq rx \in V$ pure in M, with $V \cong U$. Then $rx = ry$ for some $y \in U$, so that y represents a non zero element of $\alpha_M(U, u)$. Then x and y represent the same element in $\alpha_M(\sigma^-, I)$, since $x - y \in M^{\sigma^-}[I^+]$. ■

The quotient vector space $\alpha_M(\sigma^-, I)/\alpha_M(J/I, 1+I)$ is isomorphic to

$$M^{\sigma^-}[I]/M^{\sigma^-}[I^+] + M^{\sigma}[I] = \alpha_M(\sigma^-, I);$$

hence, as noted in [FS1], we have the isomorphism

$$\bar{\alpha}_M(\sigma^-, I) \cong \alpha_M(J/I, 1+I) \oplus \alpha_M(\sigma^-, I).$$

In the notation of the preceding Proposition 1, let $r \in R \setminus I$ and consider the linear transformation induced by the multiplication by r

$$\mu_r: \bar{\alpha}_M(\sigma^-, I) \to \bar{\alpha}_M(r^{-1}\sigma^-, r^{-1}I).$$

From Proposition 1 it follows immediately that μ_r is a monomorphism. It is easy to find examples where μ_r is not epic, however, as it is proved in [FS1], μ_r induces an isomorphism between the two subspaces $\alpha_M(J/I, 1+I)$ and $\alpha_M(r^{-1}J/r^{-1}I, 1+r^{-1}I)$; note that $[r^{-1}J/r^{-1}I, 1+r^{-1}I] = [J/I, r(1+I)]$. More generally, we have the following

PROPOSITION 4. *In the above notation, μ_r induces as isomorphism between $\alpha_M(U, u)$ and $\alpha_M(U, ru)$.*

PROOF. If $a \in M$ represents a non zero element of $\alpha_M(U, u)$, then, by Proposition 2, there exists a pure monomorphism $\phi: U \to M$ such that $\phi u = a$, where $H_U(u) = J$ and $\mathrm{Ann}_R(u) = I$; since $r \in R \setminus I$, ra represents a non zero element of $\alpha_M(U, ru)$, with $\phi ru = ra$, thus μ_r induces a monomorphism from $\alpha_M(U, u)$ into $\alpha_M(U, ru)$. This map is onto: for, let $b \in M$ represent a non zero element of $\alpha_M(U, ru)$. Then there exists a pure monomorphism $\Psi: U \to M$ such that $\Psi ru = b$. If $a = \Psi u$, then a represents a non zero element of $\alpha_M(U, u)$, wich is sent by μ_r to that represented by b. ■

The preceding proposition allows us to define, for each uniserial module U of type $[J/I]$, the *U-dimension* of M as the cardinal number $\alpha_M[U]$, which is the common value of the dimensions of all vector spaces $\alpha_M(U, u)$ with $0 \neq u \in U$:

$$\alpha_M[U] = \dim \alpha_M(U, u) \qquad (0 \neq u \in U).$$

Notice that $\alpha_M[J/I] = \dim \alpha_M[\sigma, I]$ (where $\sigma = J/R$) is the old invariant defined in [FS1]. Clearly the invariants $\alpha_M[U]$ are additive; they take the expected values for a uniserial module $M = V$.

LEMMA 5. *Let V be a uniserial module and $[U]$ an isomorphy* *class of uniserials modules. Then for the U-dimension of V we have*

$$\alpha_V[U] = \begin{cases} 1 & \textit{if } V \cong U, \\ 0 & \textit{otherwise}. \end{cases}$$

PROOF. Clearly $\alpha_V[U] \leqslant 1$ and, if $V \cong U$, then the dimension equals 1. Assume now $\alpha_V[U] = 1$; then, by Proposition 2, there is an injective homomorphism $\phi: V \to U$ such that ϕV is pure in U; this implies that ϕ is epic, hence $V \cong U$. ■

The utility of the invariants $\alpha_M[U]$ is shown by the next result, which immediately follows from the additivity of the invariants and Lemma 5.

PROPOSITION 6. *The U-invariants $\alpha_M[U]$, with $[U]$ running over the isomorphy classes of uniserial modules, form a complete and independent set of invariants for the class of direct sums M of uniserial modules.* ■

We introduce now the following

DEFINITION. Given a uniserial module U, a submodule B of a module M is called a *U-quasi-basic submodule* if it satisfies the following properties:

(i) B is a direct sum of uniserial submodules of M isomorphic to U and it is pure in M,

(ii) given a uniserial submodule V of Misomorphic to U, either $V \cap B \neq 0$. or $V \oplus B$ is not pure in M.

In order to prove the uniqueness up to isomorphism of U-quasi-basic submodels, we need a technical lemma, which shows the connection between pure direct sums of copies of U contained in M and linearly independent sets of elements of the vector space $\alpha_M(U, u)$.

LEMMA 7. *Let M be a module and U a uniserial module. Assume that $\{\phi_\lambda \mid \lambda \in \kappa\}$ is a family of pure injections of U into M. Then the submodule $\sum_{\lambda \in \kappa} \phi_\lambda(U)$ is a direct sum and is pure in M if and only if, for each $0 \neq u \in U$, $\{\overline{\phi_\lambda(u)} \mid \lambda \in \kappa\}$ is a linearly independent set in the vector space $\alpha_M(U, u)$.*

PROOF. Assume first that $\{\overline{\phi_\lambda(u)} \mid \lambda \in \kappa\}$ is linearly independent in the vector space $\alpha_M(U, u)$ for each $0 \neq u \in U$. Suppose the sum $\sum_{\lambda \in \kappa} \phi_\lambda(U)$

is not direct. Then there exists a finite subset $F = \{1, 2, ..., n\}$ of κ, such that

$$\phi_1(u_1) + \phi_2(u_2) + \ldots + \phi_n(u_n) = 0$$

for some $0 \neq u_i \in U$. Without loss of generality, we can assume $Ru_1 \geq \geq Ru_i$ for all i, thus there are elements $r_i \in R$ $(i = 2, \ldots, n)$ such that

$$\phi_1(u_1) + r_2\phi_2(u_1) + \ldots + r_n\phi_n(u_1) = 0\,.$$

This relation implies that the set $\{\overline{\phi_\lambda(u_1)} \mid \lambda \in \kappa\}$ is linearly dependent, a contradiction; so $\sum_{\lambda \in \kappa} \phi_\lambda(U) = \oplus_{\lambda \in \kappa} \phi_\lambda(U)$. We must show now that this direct sum is pure in M. Let $x \in N = \oplus_{\lambda \in \kappa} \phi_\lambda(U)$; as before, we can assume that

$$x = \phi_1(u_1) + r_2\phi_2(u_1) + \ldots + r_n\phi_n(u_1)$$

for some elements $r_i \in R$ $(i = 2, \ldots, n)$ and $0 \neq u_1 \in U$; setting

$$\psi = \phi_1 + r_2\phi_2 + \ldots + r_n\phi_n$$

we have $x = \psi(u_1)$; notice that $\psi\colon U \to M$ is an embedding. Computing the height ideal of x in N we obtain

$$H_N(x) = \inf_{i=2,\ldots,n} H_U(r_i u_1) = H_U(u_1)\,.$$

Assume, by way of contradiction, that $H_M(x) > H_N(x)$; then $\psi(U)$ is not pure in M, hence, by Proposition 2, $\bar{x} = \bar{0}$ in $\alpha_M(U, u_1)$, equivalently, $\{\overline{\phi_\lambda(u_1)} \mid \lambda \in \kappa\}$ is linearly dependent, again a contradiction.

Conversely, suppose that $\oplus_{\lambda \in \kappa} \phi_\lambda(U)$ is a pure submodule of M, and assume, by way of contradiction, that for a fixed $0 \neq u \in U$ there exist elements $r_i \in R_L \setminus L$ $(i = 1, 2, \ldots, n)$ such that

$$(r_1 + L)\,\overline{\phi_1(u)} + (r_2 + L)\,\overline{\phi_2(u)} + \ldots + (r_n + L)\,\overline{\phi_n(u)} = \bar{0}\,.$$

Then $\overline{\psi(u)} = \bar{0}$ for $\psi = r_1\phi_1 + r_2\phi_2 + \ldots + r_n\phi_n$; ψ is a homomorphism from U into $\oplus_{i \leq n} \phi_i(U)$. Clearly

$$\operatorname{Ker}\psi = \bigcap_{i \leq n} \operatorname{Ker} r_i\phi_i\,,$$

and each $\operatorname{Ker} r_i\phi_i$ is zero, since $r_i \in R_L \setminus L$ induces an automorphism of U. Thus we have seen that ψ is monic. To reach the desired contradiction, *viz.* $\overline{\psi(u)} \neq \bar{0}$, it is enough to show that $\psi(U)$ is pure in M and to apply Proposition 2. Let $0 \neq v \in U$; then

$$H_M(\psi(v)) = H_{\oplus_{i \leq n}\phi_i(U)}(\psi(v))\,.$$

Without loss of generality we can assume $Rr_1 \geq Rr_i$ for all i, so that

$$H_{\oplus_{i \leq n}\phi_i(U)}(\psi(v)) = H_{\phi_1(U)}(r_1 \phi(v)) = H_U(r_1 v);$$

the last heigh ideal equals $H_U(v)$, since $r_1 \in R_L \setminus L$, hence

$$H_M(\psi(v)) = H_{\psi(U)}(\psi(v))$$

and $\psi(U)$ is pure in M. ■

We apply now Lemma 7 to prove the following main result.

THEOREM 8. *Let M be an R-module and U a uniserial R-module. Then every U-quasi-basic submodule B of M is isomorphic to $\oplus \alpha U$, with $\alpha = \alpha_M[U]$, the U-dimension of M.*

PROOF. Let $B = \oplus_{i \in I} U_i$ ($U_i \cong U$ for all $i \in I$) be a U-quasi-basic submodule of M. We must show that $|I| = \alpha_M[U]$. For each $i \in I$, let $\phi_i: U \to U_i$ be an isomorphism. By Lemma 7 we know that the set $\{\overline{\phi_i(u)} \mid i \in I\}$ is linearly independent in $\alpha_M(U, u)$ for each $0 \neq u \in U$. We shall prove that this set is indeed a basis of $\alpha_M(U, u)$. Let $\bar{0} \neq \bar{x} \in \alpha_M(U, u)$; then, by Proposition 2, there exists a pure injection $\psi: U \to M$ such that $\psi(u) = x$. The maximality of B implies that either $B \cap \psi(U) \neq 0$, or $B \oplus \psi(U)$ is not pure in M. In both cases, Lemma 7 ensures that the set $\{\overline{\psi(u)}, \overline{\phi_i(u)} \mid i \in I\}$ is linearly dependent, hence $\bar{x}$ is contained in the subspace generated by $\{\overline{\phi_i(u)} \mid i \in I\}$, as desired. ■

3. The invariants $\beta_M(\sigma^-, I)$ and quasi-basic submodules.

It is easy to find examples of modules M such that, for non isomorphic uniserials U's of the same type $[J/I]$, the subspaces $\alpha_M(U, u)$ of $\bar{\alpha}_M(\sigma^-, I)$ have non trivial intersections ($\sigma = J/R$, $H_U(u) = J$ and $\mathrm{Ann}_R(u) = I$). It is natural to ask how the subspace generated by all the $\alpha_M(U, u)$'s is related to M. This suspace is the object of the investigation in this section, so it requires a new notation:

$$\sum \alpha_M(U, u) = \beta_M(\sigma^-, I)$$

(in the summation (U, u) ranges over all pairs such that $H_U(u) = J$ and $\mathrm{Ann}_R(u) = I$).

In order to answer this question, it is useful to introduce the following definition, which extends the notion of pre-basic submodule introduced in [FS1].

DEFINITION. A submodule B of a module M is called a *quasi-basic submodule* if it satisfies the following properties:

(i) B is a direct sum of uniserial submodules of M and it is pure in M,

(ii) given a uniserial submodule U of M, either $U \cap B \neq 0$, or $U \oplus B$ is not pure in M.

REMARKS. 1) If there are no non-standard uniserial modules, the notion of quasi-basic submodule coincides with that of pre-basic submodule introduced in [FS1], and subsequently called α-basic in [FS2].

2) The terminology «quasi-basic» is borrowed from abelian group theory; in fact, Griffith [G, pg. 93] calls «quasi-pure independent» a subset S of a torsionfree abelian group G if it is independent and $\oplus_{s \in S} R_s$ is pure in G, where R_s denotes the rank one pure subgroup of G generated by s. Extending the terminology, one could call «quasi-basic subgroup» of G a completely decomposable pure subgroup of G obtained by a maximal quasi-pure independent subset. It is shown in [G] that quasi-basic subgroups are not unique up to isomorphism and, more surprisingly, they can have also different cardinalities (whenever small).

LEMMA 9. *If N is a pure submodule of the module M, then there are canonical embeddings*

(i) $\bar{\alpha}_N(\sigma^-, I) \to \bar{\alpha}_M(\sigma^-, I)$ *for all* $\sigma \in \Sigma$ *and ideals* I,

(ii) $\alpha_N(U, u) \to \alpha_M(U, u)$ *for all* $(U, u) \in \mathcal{A}$.

PROOF. The two embedding follow from the first isomorphism theorem and the following equalities, which obviously hold by the purity of N in M:

$$N^{\sigma^-}[I] \cap (M^{\sigma^-}[I^+] + M^{\sigma^+}[I]) = N^{\sigma^-}[I^+] + N^{\sigma^+}[I],$$

$$N^{[U,u]} \cap (M^{\sigma^-}[I^+] + M^{\sigma^+}[I]) = N^{[U,u]} \cap (N^{\sigma^-}[I^+] + N^{\sigma^+}[I]). \qquad \blacksquare$$

In the following theorem we consider the canonical embedding of Lemma 9 as inclusions, and we show that the subspace $\beta_M(\sigma^-, I)$ of $\bar{\alpha}_M(\sigma^-, I)$ coincides with the (image through the canonical embedding of the) subspace $\bar{\alpha}_B(\sigma^-, I)$ for any quasi-basic submodule B of M.

THEOREM 10. *Let B be any quasi-basic submodule of M. Then, for all $\sigma \in \Sigma$ and all ideals I, the following equality holds:* $\beta_M(\sigma^-, I) = \bar{\alpha}_B(\sigma^-, I)$.

PROOF. In order to show that $\beta_M(\sigma^-, I) \leq \bar{\alpha}_B(\sigma^-, I)$, we must prove that $\alpha_M(U, u) \leq \bar{\alpha}_B(\sigma^-, I)$ for each pair (U, u) such that $H_U(u) = J$ $(\sigma = J/R)$ and $\mathrm{Ann}_R(u) = I$. It is enough to show that

$$M^{[U, u]} \leq B(\sigma^-, I) + (M^{\sigma^-}[I^+] + M^{\sigma^+}[I]).$$

So, choose $a \in M^{[U, u]} \setminus (M^{\sigma^-}[I^+] + M^{\sigma^+}[I])$; thus $H_M(a) = J$, $\mathrm{Ann}_R(a) = I$ and a is contained in a pure uniserial submodule $U' \cong U$ of M. Since B is quasi-basic, either $U \cap B \neq 0$, or $U \oplus B$ is not pure in M. Now the proof goes as in Theorem 15 and its Remark in [FS1], taking into account the following facts:

(*a*) if $rx = a \in M$ is solvable in B, then there is a solution in B with an element of height ideal $rH_M(a)$,

(*b*) if $b \in B$, then $b \in B[I] + B^{\tau^-}$, where $\tau = \inf\{rh_B(rb)\colon 0 \neq r \in I\}$ (this is a weaker version of the «smoothness» used in [FS1]).

To prove the converse inclusion: $\bar{\alpha}_B(\sigma^-, I) \leq \sum \alpha_M(U, u)$, pick an element $b \in B$ representing a non zero element of $\bar{\alpha}_B(\sigma^-, I)$; so, by Proposition 1, $\mathrm{Ann}_R(b) = I$, $H_B(b) = J$ and $H_B(rb) = r^{-1}J$ for all $r \notin I$. Let $B = \bigoplus_{i \in I} U_i$, with each U_i uniserial. If $b = u_1 + \ldots + u_n$ $(u_j \in U_{i_j})$, then we can assume, without loss of generality, that $\mathrm{Ann}_R(b) = \mathrm{Ann}_R(u_i)$ and $H_B(b) = H_B(u_i)$ for all i (if one of the two equalities fails for some i, then $u_i \in B^{\sigma^-}[I^+] + B^{\sigma^+}[I]$). Since obviously u_i represents an element of $\alpha_M(U_i, u_i)$, b represents an element of $\sum \alpha_M(U, u)$. ■

Proposition 2 has the following immediate consequence.

COROLLARY 11. *For a module M, $\beta_M(\sigma^-, I) = 0$ (resp. $\alpha_M(\sigma, I) = 0$) for all non limit heights σ and ideals I if and only if M does not cointain any pure (standard) uniserial submodule.* ■

Given a module M, we define the *uniserial typeset* $\mathcal{C}(M)$ of M as the set of types $t(U)$ of those uniserial modules U such that M contains a pure uniserial submodule isomorphic to U; therefore, owing to the preceding results, we have:

$$\mathcal{C}(M) = \{t(U) \mid \alpha_M[U] \neq 0\}.$$

Since purity is a transitive property, it is obvious that, given a quasi-basic submodule B of M, $\mathcal{C}(B) \subseteq \mathcal{C}(M)$. The converse inclusion follows from Theorem 10; in fact, if U is a pure uniserial submodule of

M of type $[J/I]$, then $\alpha_M[U] \neq 0$ implies $\bar{\alpha}_B(\sigma^-, I) \neq 0$ $(\sigma = J/R)$, hence, by the additivity of $\bar{\alpha}(\sigma^-, I)$ and (*) in section 1, B contains a summand of type $[J/I]$. Thus we have proved the first statement in

COROLLARY 12. 1) *For a module M and any quasi-basic submodule B of M, we have $\mathcal{T}(M) = \mathcal{T}(B)$.*

2) *If $B = \bigoplus_{\iota \in I} U_i$, with the U_i's uniserials, then $\mathcal{T}(B) = \{t(U_i)\}_{\iota \in I}$ is independent of the representation of B as a direct sum of uniserials.*

PROOF. For part 2 it is enough to recall that any two decompositions of B as direct sum of uniserials are isomorphic. ■

EXAMPLES. 1) In [FS2], Chapter X.4, a cohesive module M (i.e. without elements of limit heights) is constructed such that $\bar{\alpha}_M(\sigma^-, I) = 0$ for all heights σ and ideals I.

2) In [SZ] a module S (all whose non zero elements have limit height) is constructed satisfying $\beta_S(\sigma^-, I) = 0$ for all heights σ and ideals I, and with the spaces $\alpha_S(\sigma^-, I)$ of countable dimension for certain σ and I.

3) The pure-injective hull M of a uniserial module of type J/I has the following invariants: if $\sigma = J/R$, then

$$\bar{\alpha}_M(\sigma^-, I) = \beta_M(\sigma^-, I) = \alpha_M(U, u)$$

with dimension 1, for each U of type $[J/I]$,

$$\bar{\alpha}_M(\rho^-, L) = 0 \quad \text{if } (\rho, L) \neq (\sigma, I).$$

For, a pure injective module M is cohesive, by [FS2, XI.4.3], thus for each $\rho \in \Sigma$ and $L < R$ we have:

$$\bar{\alpha}_M(\rho^-, L) = \alpha_M(\rho, L).$$

Clearly, $\alpha_M(\rho, L) = 0$ if $\rho = H/R$ and $H/L \not\equiv J/I$, hence we have to consider only the invariant $\alpha_M(\alpha, I)$. By [FS1], Theorem 17, $\dim \alpha_M(\sigma, I) = 1$. Since every uniserial module U of type $[J/I]$ can be embedded as a pure submodule in M, and M is the pure-injective hull of each of these submodules, the claim is clear. Observe that any uniserial of type $[J/I]$ is a quasi-basic submodule M. This example shows that a quasi-basic submodule is very far from being the direct sum of U-quasi-basic submodules, U running over the class of the uniserial modules.

Let $\mathcal{T}$ denote the set of all types, i.e. of all isomorphy classes of standard uniserials; if $B = \bigoplus U_i$, with the U_i's uniserials, for each $\tau \in \mathcal{T}$

we set

$$B_\tau = \oplus\{U_i \mid t(U_i) = \tau\}.$$

B_τ is called the *homogeneous component of type* τ of B, or, more briefly, the τ*-component* of B. Clearly $B = \oplus_{\tau \in \mathcal{T}} B_\tau$.

The preceding Example 3 shows that the isomorphy class of B_τ is not an invariant for M, for a quasi-basic submodule B of M. But its Goldie dimension $g(B_\tau)$ is an invariant of M, as the next main consequence of Theorem 10 shows.

COROLLARY 13. *Let B be a quasi-basic submodule of M and $\tau \in \mathcal{T}$ a type. Then the Goldie dimension $g(B_\tau)$ of the τ-component of B does not depend on the choice of B.*

PROOF. It is enough to note that, for $\tau = [J/I]$ and $\sigma = J/R$,

$$g(B_\tau) = \dim \bar{\alpha}_B(\sigma^-, I).$$

By Theorem 10, this dimension depends on M only. ■

Corollary 13 allows us to define, for each type $\tau \in \mathcal{T}$, the τ*-dimension* of a module M as the cardinal $g_\tau(M)$ which is the common value of $g(B_\tau)$, B running over the set of all the quasi-basic submodules of M.

We close the paper by proving that pure-injective modules satisfy a particular property, that can be expressed in terms of the τ-dimension, showing that they are in a certain sense «homogeneous» with respect to uniserial modules.

THEOREM 14. *Let M be a pure-injective module and $\tau \in \mathcal{T}$ a type. Then, for each uniserial U of type τ, the τ-dimension $g_\tau(M)$ of M equals the U-dimension $\alpha_M[U]$.*

PROOF. In view of [FS2, XI.4.9], we can assume that M is the pure-injective hull of its basic submodule $B = \oplus U_i$, with U_i standard uniserial for each i. Let $0 \neq u \in U$ such that $H_U(u) = J$ and $\mathrm{Ann}_R(u) = I$. Since B is pure-essential in M, hence quasi-basic, we have $g_\tau(M) = g(B_\tau)$; thus it is enough to prove that $g(B_\tau) = \alpha_M[U]$. Note that the pure-injective hull $PE(B_\tau)$ of B_τ is a summand in M, and it coincides with the pure-injective hull of $\oplus_{g(B_\tau)} U$, since $PE(J/I) = PE(U)$. There follows that $g(B_\tau) \leq \alpha_M[U]$. The converse inequality is always true for arbitrary modules. ■

REFERENCES

[BFS] S. Bazzoni - L. Fuchs - L. Salce, *The hierarchy of uniserial modules over valuation domains*, preprint.

[FS1] L. Fuchs - L. Salce, *Prebasic submodules over valuation rings*, Ann. Mat. Pura Appl., (IV), 32 (1982), pp. 257-274.

[FS2] L. Fuchs - L. Salce, *Modules over Valuation Domains*, Lecture Notes Pure Appl. Math. N.97, Marcel Dekker, New York (1985).

[G] P. Griffith, *Infinite Abelian Groups*, Chicago Lectures in Math., Chicago, London (1970).

[SZ] L. Salce - P. Zanardo, *A Kulikov's result that does not extend to modules over general valuation domains*, Proc. Amer. Math. Soc., **111** (1991), pp. 643-649.

[S] S. Shelah, *Nonstandard uniserial module over a uniserial domain exists*, Lecture Notes Math. N. 1182, Springer-Verlag (1986), pp. 135-150.

Manoscritto pervenuto in redazione il 19 marzo 1992.