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A Lattice of Homomorphs.

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Preliminary notes.

In this paper all groups are finite and soluble. The homomorph $h(\mathcal{B})$ for a boundary $\mathcal{B}$ consists of all «$\mathcal{B}$-perfect groups», namely all those groups that have no $\mathcal{B}$-groups among their epimorphic images. The boundary $b(\mathcal{X})$ for a homomorph $\mathcal{X}$ consists of all groups G such that $G \notin \mathcal{X}$ and if $1 \neq N \trianglelefteq G$, then $G/N \in \mathcal{X}$. The maps h and b are mutually inverse bijections between the set of non-empty homomorphs and the set of boundaries. Let $\mathcal{X}$ be a homomorph. We recall from [4] that the class $\boldsymbol{D}\mathcal{X}$ of $\mathcal{X}$ comprises all groups G such that $\mathrm{Cov}_{\mathcal{X}}(G) \neq \emptyset$ namely all those groups that have $\mathcal{X}$-covering subgroups. $\boldsymbol{D}\mathcal{X}$ is also a homomorph. We study in [6] the set

$$\mathbb{H}(\mathcal{U}) = \{\mathcal{X} | \boldsymbol{D}\mathcal{X} = \mathcal{U}\}, \quad \text{where } \mathcal{U} \text{ is a homomorph}.$$

Those homomorphs $\mathcal{X}$ such that $\boldsymbol{D}\mathcal{X} = \mathcal{U}$ behave with regard to $\mathcal{U}$ in a somewhat similar way to the Schunck classes with regard to the whole universe of soluble groups. The class $\mathcal{A}(\mathcal{U})$ (see (2.1) of [6]) is introduced in order to characterize the homomorphs $\mathcal{X}$ of $\mathbb{H}(\mathcal{U})$, when $\mathbb{H}(\mathcal{U}) = \emptyset$ or $|\mathbb{H}(\mathcal{U})| = 1$ and to study the relation of usual containment in $\mathbb{H}(\mathcal{U})$. The class $\mathcal{A}(\mathcal{U})$ consists of those primitive groups G in $\mathcal{U}$ that satisfy:

If $M \lessdot X$ and $X/\mathrm{core}_X M \cong G$, we have $M \in \mathcal{U}$ if and only if $X \in \mathcal{U}$.

Let $\mathcal{P}$ denote the class of finite soluble primitive groups.

If $\mathbb{H}(\mathcal{U}) \neq \emptyset$, the minimum in $\mathbb{H}(\mathcal{U})$ with regard to the relation of containment is $\mathcal{M} = h((b(\mathcal{U}) - \mathcal{P}) \cup \mathcal{A}(\mathcal{U}))$ (see [6], (3.3)).

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In this paper we study the relation of strong containment in $\mathbb{H}(\mathcal{U})$ given by

1 DEFINITION. Let $\mathcal{U}$ be a homomorph. Let $\mathcal{X}, \mathcal{Y} \in \mathbb{H}(\mathcal{U})$. We say that $\mathcal{X}$ is strongly contained in $\mathcal{Y}$, and write $\mathcal{X} \ll \mathcal{Y}$ if, for each $G \in \mathcal{U}$ an $\mathcal{X}$-covering subgroup of G is contained in some $\mathcal{Y}$-covering subgroup of G.

For a homomorph $\mathcal{H}$, we denote $\tilde{\mathcal{H}} := h(b(\mathcal{H}) \cap \mathcal{P})$. For every group $G \in \boldsymbol{D}\mathcal{H}$ we have: $\mathrm{Cov}_{\mathcal{H}}(G) = \mathrm{Cov}_{\tilde{\mathcal{H}}}(G)$ (see [6], (1.8)).

2 LEMMA. Let $\mathcal{H}$ be a homomorph. We denote

$$a(\mathcal{H}) := \{G \in \boldsymbol{D}\mathcal{H} | \quad \text{if} \quad H \in \mathrm{Cov}_{\mathcal{H}}(G), H \cap \mathrm{Soc}\, G = 1\}.$$

We have:

a) $a(\mathcal{H}) = a(\tilde{\mathcal{H}}) \cap \boldsymbol{D}\mathcal{H}$.

b) $\tilde{\mathcal{H}} = h(a(\mathcal{H}))$.

PROOF. *a)* It is evident by the definition.

b) Since $b(\tilde{\mathcal{H}}) = b(\mathcal{H}) \cap \mathcal{P}$, we have $b(\tilde{\mathcal{H}}) \subseteq a(\tilde{\mathcal{H}}) \cap \boldsymbol{D}\mathcal{H} = a(\mathcal{H})$ and therefore $h(a(\mathcal{H})) \subseteq h(b(\tilde{\mathcal{H}})) = \tilde{\mathcal{H}}$. Since $\tilde{\mathcal{H}} = h(a(\tilde{\mathcal{H}}))$ (see [2], VI (1.4)) and $a(\mathcal{H}) \subseteq a(\tilde{\mathcal{H}})$, we have $\tilde{\mathcal{H}} = h(a(\tilde{\mathcal{H}})) \subseteq h(a(\mathcal{H}))$.

Let us recall now the following

3 DEFINITION ([5] and [3] (8.2)). Let $\mathcal{B} \subset \mathcal{P}$. We define $\mathcal{B}_0 = \mathcal{B}$, and if $\mathcal{B}_i$ has already been defined, let

$$\mathcal{B}_{i+1} = \{(X/C_X(V))[V] | H \leqslant X \leqslant K < G = KF(G) \in \mathcal{B}_i,$$

$$H \in \mathrm{Cov}_{h(\mathcal{B}_i)}(K), \quad \text{is } X\text{-composition factor of } F(G)\}.$$

We denote by $\mathcal{B}^\infty$ the union of all class $\mathcal{B}_i$ previously defined.

In a similar way to (8.3) from [3] we have

4 PROPOSITION. Let $\mathcal{H}$ be a homomorph and $\mathcal{B} \subset \mathcal{P}$ such that $\mathcal{B} \subseteq a(\mathcal{H})$. We have that $\mathcal{B}^\infty \subseteq a(\mathcal{H})$ (in particular $a(\mathcal{H})^\infty = a(\mathcal{H})$).

PROOF. Let us prove that $\mathcal{B}_i \subseteq a(\mathcal{H})$ for every $i \in \mathbb{N}$. We proceed by induction on i. We have that $\mathcal{B} = \mathcal{B}_0 \subseteq a(\mathcal{H})$. Suppose $\mathcal{B}_i \subseteq a(\mathcal{H})$. Let $B \in \mathcal{B}_{i+1}$. There exists $G \in \mathcal{B}_i \subseteq a(\mathcal{H})$, $Y \leqslant X \leqslant K$, K complement of $F(G)$, $H \in \mathrm{Cov}_{h(\mathcal{B}_i)}(K)$, V, W, X-subgroups of $F(G)$, V/W, X-composition of $F(G)$ such that $B = X/C_X(V/W)[(V/W)]$. Since $\mathcal{B}_i \subseteq a(\mathcal{X}) \subseteq a(\tilde{\mathcal{X}})$, by [1] (2.2), we have $\tilde{\mathcal{X}} \ll h(\mathcal{B}_i)$, hence there exists $H \in \mathrm{Cov}_{\tilde{\mathcal{X}}}(K)$ such that $H \leq Y$. As $G \in a(\mathcal{X}) \subseteq \boldsymbol{D}\mathcal{X}$, we have $H \in \mathrm{Cov}_{\mathcal{X}}(K) \subseteq \mathrm{Cov}_{\mathcal{X}}(G)$. Besides, it can be confirmed that

$$B = X/C_X(V/W)[(V/W)] \cong XV/C_X(V/W)W.$$

By the properties of covering subgroups $H \in \mathrm{Cov}_{\mathcal{X}}(XV)$ and

$$HC_X(V/W)W/C_X(V/W)W \in \mathrm{Cov}_{\mathcal{X}}(XV/C_X(V/W)W),$$

therefore $B \in \boldsymbol{D}\mathcal{X}$. We know from [3] (8.3), that $B \in a(\tilde{\mathcal{X}})$, so we can deduce that $B \in a(\tilde{\mathcal{X}}) \cap \boldsymbol{D}\mathcal{X} = a(\mathcal{X})$.

Below we study the relation «$\ll$» in $\mathbb{H}(\mathcal{U})$.

5 Proposition. Let $\mathcal{X}, \mathcal{Y} \in \mathbb{H}(\mathcal{U})$. We have $\mathcal{X} \ll \mathcal{Y}$ if and only if $\tilde{\mathcal{X}} \ll \tilde{\mathcal{Y}}$.

Proof. $\Leftarrow$) It is evident from that comment before Lemma 2.

$\Rightarrow$) We have $b(\tilde{\mathcal{Y}}) = b(\mathcal{Y}) \cap \mathcal{P}$. By definition of $\ll$ and $a(\mathcal{X})$, we have that $b(\mathcal{Y}) \cap \mathcal{P} = b(\mathcal{Y}) \cap \boldsymbol{D}\mathcal{Y} \subseteq a(\mathcal{X})$. Moreover, $a(\mathcal{X}) \subseteq a(\tilde{\mathcal{X}})$, hence $b(\tilde{\mathcal{Y}}) \subseteq b(\tilde{\mathcal{X}})$ and by [1] (2.2), $\tilde{\mathcal{X}} \ll \tilde{\mathcal{Y}}$.

Since the mapping $\mathcal{X} \to \tilde{\mathcal{X}}$ from $\mathbb{H}(\mathcal{U})$ to the set of Schunck classes is injective (see [6], 3.1), $\mathbb{H}(\mathcal{U})$ can be considered a subset of the Schunck classes ordered by $\ll$.

In the examples described in [6] (1.9), (3.8), (3.9), $(\mathbb{H}(\mathcal{U}), \ll)$ has a lattice structure. In these examples we have $\mathfrak{a}(\mathcal{U}) = a(\mathcal{M})$. In this respect, we can say:

6 Proposition. Let $\mathcal{U}$ be a homomorph and $\mathcal{M}$ the minimum for $\subseteq$ in $\mathbb{H}(\mathcal{U})$. The following statements are equivalent:

a) $\mathfrak{a}(\mathcal{U}) = a(\mathcal{M})$;

b) $\mathfrak{a}(\mathcal{U})^\infty = \mathfrak{a}(\mathcal{U})$.

Proof. $a) \Rightarrow b)$ It follows immediately from Proposition 4.

$b) \Rightarrow a)$ By $b)$ we odviously have $\mathfrak{a}(\mathcal{U})^\infty \cap h(\mathfrak{a}(\mathcal{U})) = \emptyset$. By [3] (8.4), we have $\mathfrak{a}(\mathcal{U}) \subseteq a(h(\mathfrak{a}(\mathcal{U})))$. By [6] (3.3), $h(\mathfrak{a}(\mathcal{U})) = \tilde{\mathcal{M}}$ and therefore $\mathfrak{a}(\mathcal{U}) \subseteq a(\tilde{\mathcal{M}})$.

Besides, $\mathcal{A}(\mathcal{U}) \subseteq \mathcal{U} = \boldsymbol{D}\mathcal{M}$ implies $\mathcal{A}(\mathcal{U}) \subseteq a(\tilde{\mathcal{M}}) \cap \boldsymbol{D}\mathcal{M} = a(\mathcal{M})$. By [6] (1.7), we have $a(\mathcal{M}) \subseteq \mathcal{A}(\mathcal{U})$ and therefore the equality.

7 THEOREM. Let $\mathcal{U}$ be a homomorph such that $b(\mathcal{U}) \cap \mathcal{P} = \emptyset$. (These homomorphs are known as totally unsaturated).

$(\mathbb{H}(\mathcal{U}), \ll)$ is a lattice if and only if $\mathcal{A}(\mathcal{U})^{\infty} = \mathcal{A}(\mathcal{U})$.

PROOF. $\Rightarrow$) By the proposition above and [6] (1.7), it suffices to prove that $\mathcal{A}(\mathcal{U}) \subseteq a(\mathcal{M})$. Let $G \in \mathcal{A}(\mathcal{U})$. Let $\mathcal{H} = h(b(\mathcal{U}) \cup \{G\})$. By [6] (2.3), $\mathcal{H} \in \mathbb{H}(\mathcal{U})$. Since $\ll$ implies $\subseteq$, the infimum of $\{\mathcal{H}, \mathcal{M}\}$ must be $\mathcal{M}$. Thus $\mathcal{M} \ll \mathcal{H}$, therefore $\tilde{\mathcal{M}} \ll \tilde{\mathcal{H}}$ and consequently $b(\tilde{\mathcal{H}}) \subseteq a(\tilde{\mathcal{M}})$. As $\{G\} = b(\tilde{\mathcal{H}})$, we have that

$$G \in a(\tilde{\mathcal{M}}) \cap \mathcal{U} = a(\tilde{\mathcal{M}}) \cap \boldsymbol{D}\mathcal{M} = a(\mathcal{M}).$$

$\Leftarrow$) Let $\mathcal{X}, \mathcal{Y} \in \mathbb{H}(\mathcal{U})$. Recall from [5] Theorem A that

$$\tilde{\mathcal{X}} \wedge \tilde{\mathcal{Y}} = h((b(\tilde{\mathcal{X}}) \cup b(\tilde{\mathcal{Y}}))^{\infty}).$$

By Proposition 6 we have $\mathcal{A}(\mathcal{U})^{\infty} = \mathcal{A}(\mathcal{U}) = a(\mathcal{M})$. Since $b(\tilde{\mathcal{X}}) \cup b(\tilde{\mathcal{Y}}) \subseteq a(\mathcal{M})$, by Proposition 4, we have that $(b(\tilde{\mathcal{X}}) \cup b(\tilde{\mathcal{Y}}))^{\infty} \subseteq a(\mathcal{M})$ and therefore $b(\tilde{\mathcal{X}} \wedge \tilde{\mathcal{Y}}) \subseteq \mathcal{A}(\mathcal{U})$. By [6] (2.3), we have that $\mathcal{H} = h(b(\mathcal{U}) \cup b(\tilde{\mathcal{X}} \wedge \tilde{\mathcal{Y}})) \in \mathbb{H}(\mathcal{U})$ and it can easily be confirmed that $\mathcal{H} = \mathcal{X} \wedge \mathcal{Y}$.

Now let, $\mathcal{J} = h(a(\mathcal{X}) \cap a(\mathcal{Y}))$. Again by the characterization in [6] (2.3) and (3.1), of the homomorphs in $\mathbb{H}(\mathcal{U})$ we have that $\mathcal{Z} = \mathcal{J} \cap \mathcal{U} \in \mathbb{H}(\mathcal{U})$, and $\mathcal{J} = \tilde{\mathcal{Z}}$. It can be confirmed that $\mathcal{Z} = \mathcal{X} \vee \mathcal{Y}$.

8 PROPOSITION. Let $\mathcal{U}$ be a totally unsaturated homomorph such that $(\mathbb{H}(\mathcal{U}), \ll)$ is a lattice. For every $\mathcal{X}, \mathcal{Y} \in \mathbb{H}(\mathcal{U})$ we have:

a) $\widetilde{\mathcal{X} \wedge \mathcal{Y}} = \tilde{\mathcal{X}} \wedge \tilde{\mathcal{Y}}$.

b) $\mathcal{X} \ll \mathcal{Z} \neq \mathcal{U}$ implies $\mathcal{X} = \mathcal{Z}$ if and only if $|b(\mathcal{X}) \cap \mathcal{P}| = 1$.

PROOF. *a*) It is clear from the previous proof that

$$b(\mathcal{X} \wedge \mathcal{Y}) \cap \mathcal{P} = b(\tilde{\mathcal{X}} \wedge \tilde{\mathcal{Y}}).$$

b) $\Rightarrow$) If $|b(\mathcal{X}) \cap \mathcal{P}| \neq 1$, we can have $\emptyset \neq \mathcal{B} \subset b(\mathcal{X}) \cap \mathcal{P} \subseteq \mathcal{A}(\mathcal{U})$. Now $\mathcal{Z} = h(b(\mathcal{U}) \cup \mathcal{B}) \in \mathbb{H}(\mathcal{U})$, $\mathcal{Z} \neq \mathcal{X}$ and $\mathcal{X} \ll \mathcal{Z} \neq \mathcal{U}$ in contradiction with the hypothesis.

$\Leftarrow$) As $\tilde{\mathcal{X}} = h(b(\mathcal{X}) \cap \mathcal{P})$, $\tilde{\mathcal{X}}$ is maximal, hence $\tilde{\mathcal{X}} \ll \tilde{\mathcal{Z}} \neq \mathcal{S}$ implies $\tilde{\mathcal{X}} = \tilde{\mathcal{Z}}$ and by Proposition 5 we have the thesis.

REFERENCES

[1] K. DOERK, *Über Homomorph endlicher auflösbarer Gruppen*, J. Algebra, **30** (1974), pp. 12-30.
[2] K. DOERK - T. O. HAWKES, *Finite Soluble Groups*, Walter de Gruyter, Berlin, New York (1992).
[3] P. FÖRSTER, *Charakterisierungen einiger Schunckklassen endlicher aufösbarer Gruppen III*, J. Algebra, **62** (1980), pp. 124-153.
[4] P. FÖRSTER, *Über die iterierten Definitionsbereiche von Homomorphen endlicher auflöbarer Gruppen*, Arch. Math., **35** (1980), pp. 27-41.
[5] T. HAWKES, *The family of Schunck classes as a lattice*, J. Algebra, **39** (1976), pp. 527-550.
[6] P. JIMÉNEZ-SERAL, *Homomorphs with relative Schunck classes*, Communication in Algebra, **19**, no. 10 (1991), pp. 2687-2707.

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