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Solvability of Linear and Semilinear Eigenvalue Problems with L^1 Data.

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ABSTRACT - We study the equation

$$\begin{cases} A(u) = \lambda u + f & \text{in } \Omega , \\ u = 0 & \text{on } \partial\Omega , \end{cases}$$

where A is a linear elliptic operator in divergence form, λ is a real number and f is a function belonging to $L^1(\Omega)$. We find existence results similar to those obtained in the case $f \in L^2(\Omega)$. Furthermore, we study the Landesmann-Lazer, Dolph and Ambrosetti-Prodi problems for the operator A, always with $L^1(\Omega)$ data.

1. Introduction.

In this paper we give an existence result in $L^1(\Omega)$ for the problem

$$\begin{cases} A(u) = \lambda u + f & \text{in } \Omega , \\ u = 0 & \text{on } \partial\Omega , \end{cases} \tag{1.1}$$

where Ω is a bounded open set in $\mathbb{R}^N$, $N \geq 2, f \in L^1(\Omega)$ and A is a linear, selfadjoint elliptic operator in divergence form with compact resolvent, acting between $H_0^1(\Omega)$ and its dual space $H^{-1}(\Omega)$.

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Our result will be comparable with the results obtained if f belongs to $L^2(\Omega)$; we will actually find that (1.1) has a solution if λ is not an eigenvalue of A in $L^2(\Omega)$ and, if λ is an eigenvalue, has a family of solutions that can be written as $u = \overline{u} + \varphi$, with $\overline{u}$ fixed and φ belonging to the corresponding eigenspace. Differently from the $L^2(\Omega)$ case, the solution needs not be unique without regularity hypotheses on the coefficients of A; however, it is unique in a sense that will be specified afterwards.

We state now the basic assumptions on (1.1).

Let Ω be a bounded open set in $\mathbb{R}^N$, $N \geq 2$.

Let $a_{ij} : \Omega \to \mathbb{R}$, $i, j = 1, \dots, N$ be measurable functions such that, if we define the matrix $\mathcal{A}(x) = (a_{ij}(x))_{i,j=1\dots N}$, we have:

$$\text{(1.2)} \qquad \exists \alpha > 0 : (\mathcal{A}(x)\xi, \xi) \geq \alpha |\xi|^2 \quad \forall \xi \in \mathbb{R}^N, \text{ for a.e. } x \text{ in } \Omega,$$

$$\text{(1.3)} \qquad \exists \beta > 0 : |a_{ij}(x)| \leq \beta \quad \forall i, j = 1, \dots, N, \text{ for a.e. } x \text{ in } \Omega,$$

$$\text{(1.4)} \qquad a_{ij}(x) = a_{ji}(x) \quad \forall i, j = 1, \dots, N, \text{ for a.e. } x \text{ in } \Omega,$$

We define the following differential operator in divergence form:

$$A(u) = -\operatorname{div}(\mathcal{A}(x)Du).$$

Thanks to the assumptions on $\mathcal{A}$, A is a linear elliptic operator acting between $H_0^1(\Omega)$ and its dual $H^{-1}(\Omega)$. Moreover, A is autoadjoint and has compact resolvent; then, it is well known that there exists a sequence $\{\lambda_i\}$ of real numbers such that for every $i \in \mathbb{N}$ it is possible to find at least a function φ belonging to $H_0^1(\Omega)$, with $\varphi \neq 0$, that satisfies

$$\text{(1.5)} \qquad A(\varphi) = \lambda_i \varphi.$$

Beside that, the eigenvalues λ_i form an increasing sequence of real positive numbers that diverges to $+\infty$.

Moreover, for every λ_i there is only a finite number of linearly independent functions, that we can consider orthonormal in $L^2(\Omega)$, that satisfy (1.5). So, for every i we can define the eigenspace E_i as the linear space of the eigenfunctions corresponding to the same eigenvalue λ_i. Furthermore, since eigenfunctions corresponding to different eigenvalues are orthogonal in $L^2(\Omega)$, we have a sequence of orthonormal eigenfunctions, that form a Hilbert base of $L^2(\Omega)$. We will rename eigenvalues, eigenfunctions and eigenspaces so that we have

$$A(\varphi_n) = \lambda_n \varphi_n, \qquad \varphi_n \in E_n$$

for every $n \in \mathbb{N}$.

It is well known that every φ_n belongs to $L^\infty(\Omega)$; moreover, the dimension of E_1 is one and the first eigenfunction φ_1 can be chosen positive almost everywhere.

DEFINITION 1.1. We say that a solution u of a problem with datum f belonging to $L^1(\Omega)$ is a solution obtained by approximation, if u is the limit of a sequence u_n of solutions of the same problem with $L^2(\Omega)$ data f_n that approximate f in $L^1(\Omega)$ (see [1]).

We are going to prove the following result that, as announced, will be similar to the one obtained in the $L^2(\Omega)$ theory.

THEOREM 1.1. *Suppose the* (1.2), (1.3) *and* (1.4) *hold.*

Let f be a function in $L^1(\Omega)$.

Let q be a real number with $1 \leq q < N/N-1$. Let λ be a real number; we distinguish between two cases.

A) $\lambda \neq \lambda_i$ for every i. Then (1.1) *has a unique solution u obtained by approximation, with u in $W_0^{1,q}(\Omega)$;*

B) $\lambda = \lambda_i$ for some i. Then, if

$$\int_\Omega f\varphi\, dx = 0 \qquad \forall \varphi \in E_i,$$

(1.1) *has a family of solutions in $W_0^{1,q}(\Omega)$ that can be written as $u = \bar{u} + \varphi$, with $\bar{u}$ uniquely obtained by approximation and $\varphi \in E_i$; moreover*

$$\int_\Omega \bar{u}\varphi\, dx = 0 \qquad \forall \varphi \in E_i. \tag{1.6}$$

This paper will be divided into six sections. The second section will contain some technical tools that are needed in order to prove Theorem 1.1, that will be proved in section three.

In the fourth section, we will study the nonlinear problem with resonance

$$\begin{cases} A(u) + g(u) = \lambda_1 u + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where $g(s)$ is a bounded function of s that has limit both at $+\infty$ and $-\infty$; we will see that this equation has at least one solution if $f \in L^1(\Omega)$ satisfies the same condition that is sufficient to have existence of solutions in the case $f \in L^2(\Omega)$.

The fifth section will be devoted to the study of the problem

$$
(1.8) \qquad \begin{cases} A(u) = g(u) + f & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}
$$

where $g(s)$ is a differentiable function such that the quantities $(g(t) - g(s))/(t - s)$ (for every t and s in $\mathbb{R}$) and $\lim_{s \to \pm\infty} g(s)/s$ are strictly between two consecutive eigenvalues of A. As in the case $f \in L^2(\Omega)$, we will obtain the existence of a solution under no additional hypotheses on $f \in L^1(\Omega)$.

Aim of the sixth section will be the study of the «Ambrosetti-Prodi» equation

$$
(1.9) \qquad \begin{cases} A(u) = g(u) + f + t\varphi_1 & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}
$$

with g a twice continuously differentiable, strictly convex function with bounded derivative «crossing» the first eigenvalue of A. We will see again that there is no difference between the result that holds when f is an $L^2(\Omega)$ function and our result, with f belonging to $L^1(\Omega)$.

2. Some preliminary results.

This section contains some technical results that will be used to prove Theorem 1.1. The first one is based on ideas contained in [2]; for the sake of completeness, we will rewrite the proof.

LEMMA 2.1. *Let $N > 2$. Let m be a real number, with $1 < m < 2N/(N+2)$.*

Let $q = m^* = Nm/(N - m)$. Let $\{g_n\}$ be a sequence of functions of $L^2(\Omega)$, bounded in $L^m(\Omega)$; suppose that $\{w_n\} \subset H_0^1(\Omega)$ is a sequence of solutions of the problems

$$
(2.1) \qquad \begin{cases} A(w_n) = g_n & \text{in } \Omega\,, \\ w_n = 0 & \text{on } \partial\Omega\,. \end{cases}
$$

Then $\{w_n\}$ is bounded in $W_0^{1,q}(\Omega)$ and there exists a constant c, that depends only on q, α, m and Ω, such that

$$
(2.2) \qquad \|w_n\|_{W_0^{1,q}(\Omega)} \leqslant c \|g_n\|_{L^m(\Omega)} \qquad \forall n \in \mathbb{N}\,.
$$

PROOF. Let r be a real number such that $0 < r < 1$ and define

$$v_n = ((1 + |w_n|)^{1-r} - 1)\,\mathrm{sgn}\,(w_n).$$

Since v_n belongs to $H_0^1(\Omega)$, it can be chosen as test function in (2.1). We thus obtain, thanks to (1.2):

$$\text{(2.3)} \qquad c_1 \int_\Omega \frac{|\nabla w_n|^2}{(1 + |w_n|)^r}\,dx < \int_\Omega |g_n|\,|(1 + |w_n|)^{1-r} - 1|\,dx \leqslant$$

$$\leqslant c_2 \|g_n\|_{L^m(\Omega)} + \|g_n\|_{L^m(\Omega)} \left(\int_\Omega (1 + |w_n|)^{(1-r)m'}\,dx \right)^{1/m'}$$

Let now $q < 2$ be a real number that we will choose later. Then we have, thanks to Sobolev imbedding, Hölder inequality and (2.3):

$$\text{(2.4)} \qquad c_3 \left(\int_\Omega |w_n|^{q^*}\,dx \right)^{q/q^*} \leqslant \int_\Omega |\nabla w_n|^q\,dx =$$

$$= \int_\Omega \frac{|\nabla w_n|^q}{(1 + |w_n|)^{rq/2}} (1 + |w_n|)^{rq/2}\,dx \leqslant$$

$$\leqslant \left(\int_\Omega \frac{|\nabla w_n|^2}{(1 + |w_n|)^r}\,dx \right)^{q/2} \left(\int_\Omega (1 + |w_n|)^{rq/(2-q)}\,dx \right)^{1-q/2} \leqslant$$

$$\leqslant \left(\frac{c_2}{c_1} \|g_n\|_{L^m(\Omega)} + \frac{1}{c_1} \|g_n\|_{L^m(\Omega)} \left(\int_\Omega (1 + |w_n|)^{(1-r)m'}\,dx \right)^{1/m'} \right)^{q/2} \cdot$$

$$\cdot \left(\int_\Omega (1 + |w_n|)^{rq/(2-q)}\,dx \right)^{1-q/2}.$$

Now we choose r such that $rq/(2-q) = q^*$, and $(1-r)m' = q^*$; this implies that $r = (N(2-q))/(N-q)$ and that $q = m^*$. It is easy to see that if $1 < m < 2N/(N+2)$ then the conditions $0 < r < 1$ and $q < 2$ are fulfilled.

Thanks to this choice of r and q, we have

$$c_3 \|w_n\|^q_{L^{q^*}(\Omega)} \leqslant \|\nabla w_n\|^q_{L^q(\Omega)} \leqslant$$

$$\leqslant c_4 \|g_n\|^{q/2}_{L^m(\Omega)} \|(1 + |w_n|)\|^{((1-r)q)/2}_{L^{q^*}(\Omega)} \|(1 + |w_n|)\|^{rq/2}_{L^{q^*}(\Omega)},$$

that can be rewritten as

$$(2.5)\qquad c_5\|w_n\|^2_{L^{q^*}(\Omega)} \leqslant \|\nabla w_n\|^2_{L^q(\Omega)} \leqslant c_6\|g_n\|_{L^m(\Omega)}\|(1+|w_n|)\|_{L^{q^*}(\Omega)}.$$

From (2.5) it follows that w_n is bounded in $L^{q^*}(\Omega)$ by a constant. Again by (2.5) it follows that the norm of ∇w_n in $L^q(\Omega)$ is bounded, up to a constant, by the norm of g_n in $L^m(\Omega)$, and so we have (2.2). ∎

REMARK 2.1. We note explicitly that the condition $m < 2N/(N+2)$ implies that $m^* < 2$; i.e., we cannot have w_n bounded in $H_0^1(\Omega)$. The condition $N > 2$ is due to the fact that, for $N = 2$, we have $2N/(N+2) = 1$.

Now we give the same result of Lemma 2.1, but with $m = 1$; in this case, however, we do not have boundedness of the w_n in $W_0^{1,m^*}(\Omega)$; actually, we obtain that the gradients of w_n are bounded in a space greater than $L^{m^*}(\Omega)$. This fact is justified by Example 2.1 below.

Moreover, for purposes that will be explained afterwards, we do not give estimates on the solutions of (2.1), but on some solutions w_n of (1.1); we will also need an additional hypothesis on w_n.

LEMMA 2.2. *Let q be a real number with $1 \leqslant q < N/(N-1)$. Let $\{g_n\}$ be a sequence of functions in $L^2(\Omega)$ that is bounded in $L^1(\Omega)$; suppose that $\{w_n\} \subset H_0^1(\Omega)$ is a sequence of solutions of* (1.1) *with data g_n; i.e.,*

$$\begin{cases} A(w_n) = \lambda w_n + g_n & \text{in } \Omega, \\ w_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Suppose that $\{w_n\}$ is bounded in $L^1(\Omega)$. Then $\{w_n\}$ is bounded in $W_0^{1,q}(\Omega)$.

Moreover, if $\lambda = 0$, there exists a constant c, that depends only on q, α and Ω, such that

$$(2.6)\qquad \|w_n\|_{W_0^{1,q}(\Omega)} \leqslant c\|g_n\|_{L^1(\Omega)} \qquad \forall n \in \mathbb{N}.$$

PROOF. The proof is essentially the same of Lemma 2.1. We choose a real number r with $r > 1$, and define $v_n = (1-(1+|w_n|)^{1-r})\,\mathrm{sgn}(w_n)$. Choosing v_n as test function in (1.1) we have, since $|v_n| \leqslant 1$

$$c_1\int_\Omega \frac{|\nabla w_n|^2}{(1+|w_n|)^r}\,dx \leqslant \lambda\|w_n\|_{L^1(\Omega)} + \|g_n\|_{L^1(\Omega)} \leqslant c_2,$$

for the boundedness of both w_n and g_n in $L^1(\Omega)$. Thanks to this inequality, (2.4) becomes

$$\left(\int_\Omega |w_n|^{q^*}\,dx\right)^{q/q^*} \leq c_3\left(\int_\Omega (1+|w_n|)^{rq/(2-q)}\,dx\right)^{1-q/2}.$$

Now we choose r such that $rq/(2-q)=q^*$; this implies that $r = (N(2-q))/(N-q)$, and so the condition $r>1$ is fulfilled if and only if $q<N/(N-1)$. If $N>2$ we have, with this choice of r and q, that $q/q^*>1-q/2$ and so we obtain the boundedness of w_n in $L^{q^*}(\Omega)$, hence in $W_0^{1,q}(\Omega)$. If $N=2$, since $q/q^*=1-q/2$, we have boundedness of w_n in $W_0^{1,q}(\Omega)$ if we add the hypothesis $c_3<1$; this inequality is fulfilled if the norms of g_n and w_n in $L^1(\Omega)$ are small. But, since problem (1.1) is linear, we can always reduce ourselves to this case, by eventually dividing both w_n and g_n by a constant, so that we obtain our result also in the case $N=2$.

If $\lambda=0$ we obtain (2.6) observing that c_3 is a constant multiplied by the norm of g_n in $L^1(\Omega)$ and reasoning as in the proof of Lemma 2.1. ■

EXAMPLE 2.1. Let $\Omega = B_{1/2}(0)=\{x\in\mathbb{R}^N\colon |x|<1/2\}$.
Let θ be a real number; define

$$f(x)=\frac{1}{|x|^N(-\log|x|)^\theta},\qquad x\in\Omega,\ x\neq 0,\ f(0)=0\,.$$

It is easily seen that if $\theta>1$, then f belongs to $L^1(\Omega)$ but does not belong to $L^p(\Omega)$, for any $p>1$.

Now we consider the following problem

$$\begin{cases}\Delta u = f & \text{in } \Omega\,,\\ u=0 & \text{on } \partial\Omega\,.\end{cases}$$

Since f depends only on the modulus of x, we seek radial solutions u; that is, we suppose that $u(x)=v(|x|)=v(\rho)$; passing to polar coordinates, we have that v solves the equation

$$\frac{1}{\rho^{N-1}}\frac{d}{d\rho}(\rho^{N-1}v'(\rho))=\frac{1}{\rho^N(-\log\rho)^\theta}\qquad\text{in }\left(0,\frac{1}{2}\right).$$

If we define $w(\rho)=\rho^{N-1}v'(\rho)$, then

$$w'(\rho)=\frac{1}{\rho(-\log\rho)^\theta}.$$

Integrating this equation, we have

$$w(\rho) = \frac{c_1}{(-\log \rho)^{\theta-1}}$$

and so

$$v'(\rho) = \frac{c_1}{\rho^{N-1}(-\log \rho)^{\theta-1}}.$$

If we choose $\theta > 1$ such that $N/(N-1) < \theta - 1$, that is to say if $1 < < \theta < 2 - 1/N$, then ∇u does not belong to $L^{N/(N-1)}(\Omega)$ but is in $L^q(\Omega)$ for every $q < N/(N-1)$. So, we have found that Lemma 2.2 is sharp: f in $L^1(\Omega)$ implies that u belongs to $W_0^{1,q}(\Omega)$ for every $q < N/(N-1)$ but not for $q = N/(N-1)$.

Now we recall that if $f \in L^2(\Omega)$, then (1.1) has a unique solution if λ is not an eigenvalue of A. On the contrary, if $\lambda = \lambda_i$ for some i, then (1.1) can be solved if and only if $f \in E_i^\perp$, where $E_i^\perp$ is the linear space of the functions in $L^2(\Omega)$ that are orthogonal in $L^2(\Omega)$ to every function in E_i; in this case there exists a unique $\overline{u} \in E_i^\perp$ such that any solution of (1.1) can be written as $\overline{u} + \varphi$, with φ in E_i.

From now on, when we say that u is a solution of (1.1) with data in $L^2(\Omega)$, we will mean either a solution of (1.1) when λ is not an eigenvalue, or $\overline{u}$ (as defined before) when $\lambda = \lambda_i$ for some i.

The following result will be used in the proof of Theorem 1.1, where we will deal with approximated equations.

LEMMA 2.3. *Let q be a real number, with $1 \leq q < N/(N-1)$. Let $\{f_n\}$ be a sequence of functions in $L^2(\Omega)$ such that for every $n \in \mathbb{N}$ there exists a solution v_n of*

$$\begin{cases} A(v_n) = \lambda v_n + f_n & \text{in } \Omega\,, \\ v_n = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Suppose that f_n tends strongly to 0 in $L^1(\Omega)$ as n tends to $+\infty$ and that v_n is bounded in $L^1(\Omega)$. Then v_n tends to 0 weakly in $W_0^{1,q}(\Omega)$.

PROOF. Let $k \in \mathbb{N}$ be the (unique) number such that

$$2k \leq N < 2(k+1).$$

We note that with this choice of k we have that

$$\frac{N}{N-k+1} \leqslant \frac{2N}{N+2} < \frac{N}{N-k}.$$

Now we define, for $j \in \mathbb{N}$, $p^{(j)*}$ as the Sobolev imbedding exponent for $W_0^{j,p}(\Omega)$ when $jp < N$; i.e.

$$p^{(j)*} = \frac{Np}{N-jp};$$

we also define $p^{(0)*} = p$.

We choose now a real number p such that $1 \leqslant p < N/(N-1)$ and

$$\frac{2N}{N+2} \leqslant p^{(k-1)*} < \frac{N}{N-k}. \tag{2.7}$$

This inequality, by the choice of k, is fulfilled if and only if

$$\frac{2N}{N+2k} \leqslant p < \frac{N}{N-1};$$

we can find such a p since $N < 2(k+1)$ implies that $2N/(N+2k) < < N/(N-1)$. Moreover, if $k > 1$, since $p^{(k-1)*} < N/(N-k)$, we have

$$p^{(k-2)*} < \frac{N}{N-k+1} \leqslant \frac{2N}{N+2}.$$

If we define $v_n^{(1)} = v_n$, we have that $v_n^{(1)}$ solves the equation

$$A(v_n^{(1)}) = \lambda v_n^{(1)} + f_n. \tag{2.8}$$

Since $v_n^{(1)}$ is bounded in $L^1(\Omega)$, we can apply Lemma 2.2 (with $w_n = = v_n^{(1)}$ and $g_n = f_n$); it follows that $v_n^{(1)}$ is bounded in $W_0^{1,q}(\Omega)$ for every q such that $1 \leqslant q < N/(N-1)$. In particular, it is bounded in $W_0^{1,p}(\Omega)$ (where p is the number that has been chosen before) and so in $L^p(\Omega)$. Moreover, it weakly converges in $W_0^{1,p}(\Omega)$ (up to a subsequence that we will call again $v_n^{(1)}$) to a function $v^{(1)}$.

If $k > 1$ (i.e., if $N > 2$), we consider, for any $j = 2, \ldots, k$, a sequence $v_n^{(j)} \in H_0^1(\Omega)$ of solutions of the equation

$$A(v_n^{(j)}) = \lambda v_n^{(j-1)}.$$

Since $v_n^{(1)}$ is bounded in $L^p(\Omega)$ and $p < 2N/(N+2)$, we can apply Lemma 2.1 (where we choose $w_n = v_n^{(2)}$, $g_n = \lambda v_n^{(1)}$ and $m = p$); we obtain that $v_n^{(2)}$ is bounded in $W_0^{1,p^{(1)*}}(\Omega)$ and so in $L^{p^{(1)*}}(\Omega)$.

Now we go on with this method; we can repeatedly apply Lemma 2.1 since $v_n^{(j-1)}$ is bounded in $L^{p^{(j-2)*}}(\Omega)$ and $p^{(j-2)*} \leq p^{(k-2)*} < 2N/(N+2)$ by the hypotheses on p.

At the last step, we obtain that $v_n^{(k)}$ is bounded in $W_0^{1,p^{(k-1)*}}(\Omega)$ and so, thanks to Sobolev imbedding, in $L^{p^{(k)*}}(\Omega)$. But from the hypotheses on p we have $p^{(k)*} \geq 2$ and so $v_n^{(k)}$ is bounded in $L^2(\Omega)$.

Now we consider the sequence $v_n^{(k+1)} \in H_0^1(\Omega)$ of solutions of the equations

$$A(v_n^{(k+1)}) = \lambda v_n^{(k)};$$

choosing $v_n^{(k+1)}$ as test function in this equation, we easily obtain, by the Hölder and Poincaré inequalities and by the boundedness of $v_n^{(k)}$ in $L^2(\Omega)$, that

$$\|v_n^{(k+1)}\|_{H_0^1(\Omega)} \leq c_1 \tag{2.9}$$

and so $v_n^{(k+1)}$ is bounded in $H_0^1(\Omega)$; thus, it weakly converges in $H_0^1(\Omega)$, up to a subsequence that we will call again $v_n^{(k+1)}$, to a function $v^{(k+1)}$.

Now we substract the equations satisfied by $v_n^{(j)}$ as j ranges from 1 to $k+1$. We obtain

$$A(v_n^{(2)} - v_n^{(1)}) = -f_n,$$

$$A(v_n^{(3)} - v_n^{(2)}) = \lambda(v_n^{(2)} - v_n^{(1)}),$$

and, going on until the k-th step,

$$A(v_n^{(k+1)} - v_n^{(k)}) = \lambda(v_n^{(k)} - v_n^{(k+1)}).$$

Now we can apply Lemma 2.2 to the first of these equations: from (2.6) it follows that $v_n^{(2)} - v_n^{(1)}$ tends to 0 strongly in $W_0^{1,p}(\Omega)$, and so in $L^p(\Omega)$, since f_n tends to 0 in $L^1(\Omega)$; then, from the second equation and from Lemma 2.1 it follows that $v_n^{(3)} - v_n^{(2)}$ tends to 0 strongly in $W_0^{1,p^{(1)*}}(\Omega)$ and so in $L^{p^{(1)*}}(\Omega)$. Going on, we obtain that $v_n^{(k+1)} - v_n^{(k)}$ tends to 0 strongly in $W_0^{1,p^{(k-1)*}}(\Omega)$. This means that every sequence $v_n^{(j)}$ has the same limit in $W_0^{1,p}(\Omega)$, that is obviously $v^{(1)}$. So $v^{(k+1)} = v^{(1)}$; this implies that $v^{(1)}$ itself is in $H_0^1(\Omega)$.

Passing to the limit in (2.8), we obtain

$$A(v^{(1)}) = \lambda v^{(1)}, \qquad v^{(1)} \in H_0^1(\Omega). \tag{2.10}$$

Now we have two possibilities; if λ is not an eigenvalue then $v^{(1)} = 0$, since (2.10) has a unique solution in $H_0^1(\Omega)$.

If $\lambda = \lambda_i$ for some i, then $v^{(1)} \in E_i$; now we recall that we have chosen all the $v_n^{(1)}$ in $E_i^{\perp}$, and since the orthogonality relation passes to the limit by Lebesgue theorem, by the strong convergence of $v_n^{(1)}$ to $v^{(1)}$ in $L^p(\Omega)$ and by the fact that φ_i belongs to $L^{\infty}(\Omega)$, we have that also $v^{(1)}$ belongs to $E_i^{\perp}$; these two facts imply again that $v^{(1)} = 0$.

So, up to a subsequence, $v_n^{(1)} = v_n \to 0$ weakly in $W_0^{1,p}(\Omega)$; now we recall that v_n is bounded in $W_0^{1,q}(\Omega)$ for every q such that $1 \leqslant q < N/(N-1)$ and so it weakly converges, up to a subsequence, to a function v that is obviously 0. Since any converging subsequence of v_n has the same limit, that is zero, we can conclude that the whole sequene v_n tends to zero in $W_0^{1,q}(\Omega)$, for every q such that $1 \leqslant q < N/(N-1)$. ■

REMARK 2.2. We note explicitly that (2.10) has been found using only the fact that v_n was bounded in $L^1(\Omega)$ and the convergence of f_n to 0 in $L^1(\Omega)$. This means that we can conclude that equation (2.10) holds not only in $W_0^{1,q}(\Omega)$ but also in $H_0^1(\Omega)$ every time we obtain it passing to the limit in equations that satisfy the hypotheses of Lemma 2.3.

REMARK 2.3. If A is an elliptic operator with regular coefficients (for example, $A = -\Delta$) the fact that $v^{(1)} = 0$ follows immediately from a passage to the limit in the equation satisfied by v_n, i.e. in (2.8). Actually, we obtain that $v^{(1)}$ satisfies the equation

$$A(v^{(1)}) = \lambda v^{(1)} \qquad \text{in } \mathcal{D}'(\Omega).$$

By means of «bootstrap» arguments, we have that $v^{(1)}$ belongs to $H_0^1(\Omega)$. Now we can make the same remarks as in the proof of Lemma 2.3 to obtain that $v^{(1)} = 0$.

If A is more general, as in the present case, this method cannot be used if the solution is not *a priori* in $H_0^1(\Omega)$; see [3] for a counterexample about the lack of uniqueness when the solution is not regular.

Now we need the last result, concerning the possibility of approximating an $L^1(\Omega)$ function with $L^2(\Omega)$ functions that satisfy some conditions.

LEMMA 2.4. *Let ρ be a real number. Let f be a function in $L^1(\Omega)$. Suppose that, for some i,*

$$\int_{\Omega} f\varphi_i^j \, dx = \rho \qquad \forall j = 1, \dots, k,$$

where $\varphi_i^1, \dots, \varphi_i^k$ is a base of E_i.

Then there exists a sequence $\{f_n\}$ of functions of $L^2(\Omega)$ such that:

$$\lim_{n\to\infty} f_n = f \quad \textit{in } L^1(\Omega),$$

$$\int_\Omega f_n \varphi_i^j dx = \rho \qquad \forall j = 1, \dots, k, \ \forall n \in \mathbb{N}. \tag{2.11}$$

PROOF. Let $\bar{f}_n$ be a sequence of functions in $L^2(\Omega)$ that tends to f in $L^1(\Omega)$. We define

$$t_n^j = \int_\Omega \bar{f}_n \varphi_i^j dx \qquad j = 1, \dots, k.$$

Let now $f_n = \bar{f}_n - \sum_{j=1}^k (t_n^j - \rho)\varphi_i^j$. From Lebesgue theorem and from the hypothesis on f, we have that $t_n^j \to \rho$ as $n \to \infty$, for every j. If we choose f_n as our approximating sequence, we have that (2.11) holds true (by the orthonormality of the φ_i^j in $L^2(\Omega)$); moreover

$$\int_\Omega |f_n - f|\, dx \le \int_\Omega |\bar{f}_n - f_n|\, dx + \int_\Omega |\bar{f}_n - f|\, dx \le$$

$$\le \sum_{j=1}^k |t_n^j - \rho|\, \|\varphi_i^j\|_{L^1(\Omega)} + \|\bar{f}_n - f\|_{L^1(\Omega)}$$

and the latter expression tends to 0 as n tends to infinity. ∎

3. Proof of Theorem 1.1.

We are going to give the proof of Theorem 1.1. We will use the tools introduced in the last section to show the boundedness of some approximating solutions of (1.1).

The idea is to approximate f with $L^2(\Omega)$ functions, solve the problems with these functions as data, and then show that these solutions converge to a solution of (1.1). As a first step we will choose the approximating functions; we distinguish between two cases.

A) Suppose that λ is not an eigenvalue. Let f_n be any sequence of functions in $L^2(\Omega)$ that approximates f in $L^1(\Omega)$. Then there exists a sequence $\{u_n\}$ of solutions of (1.1) with f_n as data.

B) If $\lambda = \lambda_i$ for some i, we approximate f in $L^1(\Omega)$ with the sequence f_n given by Lemma 2.4 with $\rho = 0$, so that $f_n \in E_i^\perp$ for every n. Then, again by the $L^2(\Omega)$ theory, there exists a sequence $\{u_n\}$ of solutions of (1.1) with f_n as data. We recall that we consider this sequence contained in $E_i^{\perp}$.

Now we are going to prove that u_n is bounded in $W_0^{1,q}(\Omega)$. We argue by contradiction, supposing as a first step that u_n is not bounded in $L^1(\Omega)$.

If we define $v_n = u_n/\|u_n\|_{L^1(\Omega)}$, we have that v_n solves the equation

$$A(v_n) = \lambda v_n + \frac{f_n}{\|u_n\|_{L^1(\Omega)}}.$$

Since v_n and $f_n/\|u_n\|_{L^1(\Omega)}$ satisfy the hypotheses of Lemma 2.3, we have that v_n tends to 0 weakly in $W_0^{1,q}(\Omega)$ and, by Rellich theorem, strongly in $L^1(\Omega)$; since $\|v_n\|_{L^1(\Omega)} = 1$ we have a contradiction.

So, u_n is bounded in $L^1(\Omega)$. Now, we apply Lemma 2.2 (with $w_n = u_n$ and $g_n = f_n$) to obtain that u_n is bounded in $W_0^{1,q}(\Omega)$. Then we extract from u_n a subsequence that converges weakly in $W_0^{1,q}(\Omega)$ to a function u that obviously solves the equation.

Moreover, if $\lambda = \lambda_i$ for some i, since we have chosen u_n in $E_i^{\perp}$ for every $n \in \mathbb{N}$, we obtain that also u is in $E_i^{\perp}$ and so it satisfies (1.6); furthermore, it is easy to see that $u + \varphi$ solves again the equation for every φ in E_i.

Now we approximate f in $L^1(\Omega)$ with two sequences $\{f_n^1\}$ and $\{f_n^2\}$ of functions in $L^2(\Omega)$; then, the corresponding sequences $\{u_n^1\}$ and $\{u_n^2\}$ of solutions have the same limit in $W_0^{1,q}(\Omega)$ (it suffices to choose $v_n = u_n^1 - u_n^2$ and $g_n = f_n^1 - f_n^2$ in Lemma 2.3 to obtain this). So we have shown the last part of Theorem 1.1: the solution u is uniquely obtained by approximation (if $\lambda = \lambda_i$ for some i, then its projection onto $E_i^{\perp}$ is uniquely obtained by approximation). ∎

REMARK 3.1. As it has been said in Remark 2.3, if A is an operator with regular coefficients, we can conclude that the solution is unique for every $f \in L^1(\Omega)$, since we can substract the equations satisfied by two solutions and find that their difference solves a homogeneous equation; again by means of bootstrap arguments, this solution is in $H_0^1(\Omega)$ and so it is zero (if λ is an eigenvalue, then its projection onto $E_i^{\perp}$ is zero).

Anyway this fact is not enough to conclude about uniqueness: we can only speak of uniqueness of approximated solutions; for instance, the counterexample of Serrin (see [3]) is about an equation with $f = 0$ that has both a zero solution and a non zero one; our method obtains only the former, that is, in a sense, «natural», while the other is «pathological».

REMARK 3.2. Without any change it is also possible to prove the following result, in which (as in [2] for the equation $A(u)=f$) we obtain more regularity on u if we increase the regularity on f.

THEOREM 3.1. *Suppose that* (1.2), (1.3) *and* (1.4) *hold.*
Let m be a real number with $1 < m < 2N/(N+2)$.
Let f be a function in $L^m(\Omega)$.
Let $q = m^* = Nm/(N-m)$. *Let λ be a real number; we distinguish between two cases.*

A) $\lambda \neq \lambda_i$ *for every i. Then* (1.1) *has a unique solutions u obtained by approximation, with u in* $W_0^{1,q}(\Omega)$;

B) $\lambda = \lambda_i$ *for some i. Then, if*

$$\int_\Omega f\varphi \, dx = 0 \qquad \forall \varphi \in E_i ,$$

(1.1) *has a family of solutions in* $W_0^{1,q}(\Omega)$ *that can be written as* $u = \overline{u} + \varphi$, *with* $\overline{u}$ *uniquely obtained by approximation and* $\varphi \in E_i$; *moreover*

$$\int_\Omega \overline{u}\varphi \, dx = 0 \qquad \forall \varphi \in E_i . \tag{3.1}$$

4. The «resonant» case.

In this section, we will study the so-called «Landesmann-Lazer» equation:

$$\begin{cases} A(u) + g(u) = \lambda_1 u + f & \text{in } \Omega , \\ u = 0 & \text{on } \partial\Omega , \end{cases} \tag{4.1}$$

with $g(s)$ a continuous, bounded function. We are going to show that problem (4.1) has at least one solution under the same hypotheses on f in $L^1(\Omega)$ that are needed in order to have existence in the case $f \in L^2(\Omega)$.

Actually, we will prove the following

THEOREM 4.1. *Let $g: \mathbb{R} \to \mathbb{R}$ be a continuous function; suppose that there exist*

$$\lim_{s \to \pm\infty} g(s) = g_\pm ,$$

that $g_{\pm} \in \mathbb{R}$ *and that*

$$g_- \leqslant g(s) \leqslant g_+ \qquad \forall s \in \mathbb{R}. \tag{4.2}$$

Let f *be an* $L^1(\Omega)$ *function such that*

$$g_- \int_\Omega \varphi_1 \, dx < \int_\Omega f\varphi_1 \, dx < g_+ \int_\Omega \varphi_1 \, dx. \tag{4.3}$$

Then there exists at least one solution u *of* (4.1)*; moreover,* u *belongs to* $W_0^{1,q}(\Omega)$*, for every real number* q *such that* $1 \leqslant q < N/(N-1)$.

PROOF. We proceed by approximation, as usual. We consider a sequence f_n of functions belonging to $L^2(\Omega)$ and converging to f in $L^1(\Omega)$, such that each f_n satisfies (4.3). Such a sequence can be found thanks to Lemma 2.4 (where we choose $\rho = \int_\Omega f\varphi_1 \, dx$).

As it is well known (see [4]), condition (4.3) for an $L^2(\Omega)$ function f_n is a sufficient condition for the existence of a solution u_n of the problem:

$$\begin{cases} A(u_n) + g(u_n) = \lambda_1 u_n + f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

Now we will show that the sequence u_n is bounded in $W_0^{1,q}(\Omega)$, for every real number q with $1 \leqslant q < N/(N-1)$.

Suppose that u_n is not bounded in $L^1(\Omega)$. Then, reasoning as in the proof of Theorem 1.1, we consider the functions

$$v_n = \frac{u_n}{\|u_n\|_{L^1(\Omega)}}$$

that satisfy the equations

$$A(v_n) = \lambda_1 v_n + \frac{f_n - g(u_n)}{\|u_n\|_{L^1(\Omega)}}.$$

Since $(f_n - g(u_n))/\|u_n\|_{L^1(\Omega)}$ tends to 0 in $L^1(\Omega)$ and $\|v_n\|_{L^1(\Omega)} = 1$ we have, by Remark 2.2, that v_n converges in $W_0^{1,q}(\Omega)$ to a function v that belongs to $H_0^1(\Omega)$ and solves the equations

$$A(v) = \lambda_1 v.$$

Since we do not know that v belongs to $E_1^\perp$, we cannot conclude directly that $v = 0$; we can only say that there exists $t \in \mathbb{R}$ such that

$$v = t\varphi_1.$$

Suppose that $t > 0$; then (since $\varphi_1 > 0$ and since v_n converges to v almost everywhere) we have that

$$u_n(x) = v_n(x)\|u_n\|_{L^1(\Omega)} \to +\infty \qquad \text{a.e. in } \Omega$$

and so, from the definition of g_+,

$$g(u_n(x)) \to g_+ \qquad \text{a.e. in } \Omega. \tag{4.5}$$

Now we multiply equation (4.4) by φ_1, and obtain

$$\lambda_1 \int_\Omega u_n \varphi_1 \, dx + \int_\Omega g(u_n) \varphi_1 \, dx = \lambda_1 \int_\Omega u_n \varphi_1 \, dx + \int_\Omega f_n \varphi_1 \, dx .$$

Passing to the limit, thanks to (4.5) and to the boundedness of g, we have

$$g_+ \int_\Omega \varphi_1 \, dx = \int_\Omega f \varphi_1 \, dx ,$$

that contradicts (4.3). With the same calculations we arrive at a contradiction also in the case $t < 0$. So we have $t = 0$, and this implies that $v = 0$. But since v_n has norm 1 in $L^1(\Omega)$ and converges strongly to v in the same space, we have again a contradiction.

So the sequence u_n is bounded in $L^1(\Omega)$. No we can apply Lemma 2.2 (where we choose $g_n = f_n - g(u_n)$) to obtain that u_n is bounded in $W_0^{1,q}(\Omega)$, and so, up to a subsequence that we will call again u_n, it converges (weakly in $W_0^{1,q}(\Omega)$, strongly in $L^q(\Omega)$ and a.e.) to a function u.

Since g is continuous and bounded, and u_n tends to u almost everywhere, we have that $g(u_n)$ tends to $g(u)$ in $L^p(\Omega)$, for every p, and so u solves the problem. ■

5. Dolph theorem.

In this section we will study a semi-linear equation whose nonlinear term is related to the eigenvalues of A. We will show that this equation possesses at least one solution.

We now state the result:

THEOREM 5.1. *Let $g: \mathbb{R} \to \mathbb{R}$ be a differentiable function.*

Suppose that $g(0) = 0$ and that there exist $k \in \mathbb{N}$ and $\delta > 0$ such that:

$$\lambda_k + \delta < \frac{g(t) - g(s)}{t - s} < \lambda_{k+1} - \delta, \qquad \forall s, t \in \mathbb{R}, \ s \neq t; \tag{5.1}$$

moreover, suppose that there exist

$$g_- = \lim_{s \to -\infty} \frac{g(s)}{s}, \qquad g_+ = \lim_{s \to +\infty} \frac{g(s)}{s}.$$

Let f be a function in $L^1(\Omega)$.

Then there exists a unique solution obtained by approximation for the equation

$$\text{(5.2)} \qquad \begin{cases} A(u) = g(u) + f & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Moreover, u belongs to $W_0^{1,q}(\Omega)$ for every q such that $1 \leq q < N/(N-1)$.

PROOF. Let f_n be a sequence of functions in $L^2(\Omega)$ that approximate f in $L^1(\Omega)$. Then, the problem

$$\text{(5.3)} \qquad \begin{cases} A(u_n) = g(u_n) + f_n & \text{in } \Omega\,, \\ u_n = 0 & \text{on } \partial\Omega\,, \end{cases}$$

has a unique solution u_n in $H_0^1(\Omega)$ for every $n \in \mathbb{N}$ (see [5]).

We will prove that u_n is bounded in $W_0^{1,q}(\Omega)$, for every real number q, $1 \leq q < N/(N-1)$. As a first step, we suppose that u_n is not bounded in $L^1(\Omega)$.

If we define $v_n = u_n / \|u_n\|_{L^1(\Omega)}$, then v_n solves the equation

$$A(v_n) = \frac{g(u_n) + f_n}{\|u_n\|_{L^1(\Omega)}}.$$

It is easily seen that from the hypotheses on g it follows that

$$\text{(5.4)} \qquad \lambda_k + \delta \leq g_-\,, g'(0)\,, g_+ \leq \lambda_{k+1} - \delta\,.$$

Now we define

$$h_n(x) = \begin{cases} \dfrac{g(u_n(x))}{u_n(x)} & \text{if } u_n(x) \neq 0\,, \\ g'(0) & \text{if } u_n(x) = 0\,, \end{cases}$$

so that the former equation can be written as

$$\text{(5.5)} \qquad A(v_n) = h_n v_n + \frac{f_n}{\|u_n\|_{L^1(\Omega)}}.$$

Since h_n is bounded in $L^\infty(\Omega)$ and v_n is bounded in $L^1(\Omega)$, we obtain, as in Lemma 2.2, that v_n is bounded in $W_0^{1,q}(\Omega)$ and so we can extract from it a subsequence that converges (weakly in $W_0^{1,q}(\Omega)$, strongly in $L^q(\Omega)$ and a.e.) to a function v.

From the boundedness of h_n in $L^\infty(\Omega)$ it follows that we can extract a subsequence that converges, $L^\infty(\Omega)$-weak-$*$, to a function h. Moreover, since $\lambda_k + \delta < h_n(x) < \lambda_{k+1} - \delta$, then

$$\lambda_k + \delta \leqslant h(x) \leqslant \lambda_{k+1} - \delta, \qquad \text{a.e. in } \Omega.$$

Passing to the limit, we have that v solves the equation

$$A(v) = hv, \qquad v \in W_0^{1,q}(\Omega). \tag{5.6}$$

Since h is a bounded function, we can reason as in Lemma 2.3 and obtain that v is not only in $W_0^{1,q}(\Omega)$, but also in $H_0^1(\Omega)$ (see Lemma 6.3 below for a proof of this claim).

If a is a bounded measurable function, we can define the weighted eigenvalue problem as follows: we search real numbers $\lambda(a)$ and measurable functions $\varphi(a) \neq 0$ such that

$$\begin{cases} A(\varphi(a)) = \lambda(a)\, a\varphi(a) & \text{in } \Omega, \\ \varphi(a) \in H_0^1(\Omega). \end{cases}$$

Then, it is seen that, as in the case $a \equiv 1$, it is possible to find a sequence $\lambda_n(a)$ of eigenvalues, with corresponding eigenfunctions and eigenspaces; moreover, the following properties hold

$$a(x) = \lambda_i \text{ a.e. in } \Omega \Rightarrow \lambda_j(a) = \frac{\lambda_j}{\lambda_i} \qquad \forall i, j \in \mathbb{N}, \tag{5.7}$$

$$a(x) < b(x) \text{ a.e. in } \Omega \Rightarrow \lambda_i(a) > \lambda_i(b) \qquad \forall i \in \mathbb{N} \tag{5.8}$$

(see [6]).

Now we consider equation (5.6). If v is not identically zero, then, since it is in $H_0^1(\Omega)$, it can be seen as an eigenfunction of a weighted problem with weight h and eigenvalue 1. So, there exists i such that $\lambda_i(h) = 1$. Since $\lambda_k < h(x) < \lambda_{k+1}$ for almost every $x \in \Omega$, from (5.7) and (5.8) we have:

$$\frac{\lambda_i}{\lambda_k} = \lambda_i(\lambda_k) > \lambda_i(h) = 1 > \lambda_i(\lambda_{k+1}) = \frac{\lambda_i}{\lambda_{k+1}},$$

so that

$$\lambda_k < \lambda_i < \lambda_{k+1}.$$

Since λ_k and λ_{k+1} are two consecutive eigenvalues of A, we have a contradiction, and so $v = 0$. But this is again impossible, because v_n has norm equal to 1 in $L^1(\Omega)$ and converges strongly to v in the same space.

So, u_n is bounded in $L^1(\Omega)$; since (5.1) implies that there exists a constant c_1 such that

$$|g(s)| \leq c_1 |s|, \qquad \forall s \in \mathbb{R}, \tag{5.9}$$

we have, applying Lemma 2.2 with $g_n = g(u_n)$, that u_n is bounded in $W_0^{1,q}(\Omega)$; so, we can extract from u_n a subsequence (that we will call again u_n) that converges (weakly in $W_0^{1,q}(\Omega)$, strongly in $L^q(\Omega)$ and a.e.) to a function u.

Thanks to the continuity of g, to (5.9) and to the strong convergence of u_n to u in $L^q(\Omega)$, it follows that $g(u_n)$ converges to $g(u)$ in the same $L^q(\Omega)$; so, we can pass to the limit in the approximate equation, obtaining that u solves (5.3).

Let now u_n^1 and u_n^2 be two sequences of solutions of (5.3) corresponding to two sequences of $L^2(\Omega)$ functions, f_n^1 and f_n^2, that approximate f in $L^1(\Omega)$. If we define v_n as the difference between u_n^1 and u_n^2, then v_n solves the equation

$$A(v_n) = h_n v_n + f_n^1 - f_n^2,$$

where h_n has been defined as

$$h_n = \begin{cases} \dfrac{g(u_n^1) - g(u_n^2)}{u_n^1 - u_n^2} & \text{if } u_n^1 \neq u_n^2, \\ g'(u_n^2) & \text{if } u_n^1 = u_n^2. \end{cases}$$

Again, h_n is bounded in $L^\infty(\Omega)$, and so it converges in $L^\infty(\Omega)$-weak-$*$, up to a subsequence, to a function h. Passing to the limit we obtain that v_n tends to a function v that belongs to $H_0^1(\Omega)$ (see Lemma 6.3 below) and that solves the equation

$$A(v) = hv.$$

Since the hypotheses on g imply that $\lambda_k < h(x) < \lambda_{k+1}$ almost everywhere, we can reason as before and obtain that $v = 0$; hence, we have a unique solution obtained by approximation. ■

6. The Ambrosetti-Prodi theorem.

In this section we are going to study the «Ambrosetti-Prodi» equation with $L^1(\Omega)$ data; i.e.

$$\begin{cases} A(u) = g(u) + f + t\varphi_1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

where g is a twice continuosly differentiable, strictly convex function, with the property that $g(s)/s$ «crosses» the first eigenvalue of A as s varies on $\mathbb{R}$, f is an $L^1(\Omega)$ function such that $\int_\Omega f\varphi_1\,dx = 0$ and t is a real number (see [7]). Again, our result will be similar to the one obtained when f belongs to $L^2(\Omega)$.

We are going to prove the following theorem.

THEOREM 6.1. *Let $g:\mathbb{R}\to\mathbb{R}$ be a strictly convex function belonging to $C^2(\mathbb{R})$; suppose that $g(0) = 0$ and that there exist two real numbers λ and μ, with*

$$0 < \lambda < \lambda_1 < \mu < \lambda_2, \tag{6.2}$$

such that

$$\lambda < \frac{g(s) - g(t)}{s - t} < \mu \qquad \forall s, t \in \mathbb{R},\ s \neq t, \tag{6.3}$$

$$\lim_{s\to-\infty} \frac{g(s)}{s} = \lambda, \qquad \lim_{s\to+\infty} \frac{g(s)}{s} = \mu. \tag{6.4}$$

Let f be an $L^1(\Omega)$ function such that

$$\int_\Omega f\varphi_1\,dx = 0.$$

Then there exists a unique real number $\bar{t} = \bar{t}(f)$ such that the following holds

A) if $t > \bar{t}$ then (6.1) *has no solutions obtained by approximation,*

B) if $t = \bar{t}$ then (6.1) *has exactly one solution obtained by approximation,* *C) if* $t < \bar{t}$ *then* (6.1) *has exactly two solutions obtained by approximation.*

Moreover, any solution of (6.1) *belongs to* $W_0^{1,q}(\Omega)$, *for every real number* q *such that* $1 \leq q < N/N - 1$.

REMARK 6.1. The main difference with the Ambrosetti-Prodi theorem lies in the fact, when $f \in L^2(\Omega)$, then equation (6.1) has no solution when $t > \bar{t}$ and exactly one and two solutions when $t = \bar{t}$ or $t < \bar{t}$ respectively.

Since we proceed by approximation, we can only say that our solutions are uniquely obtained by approximation, and nothing more.

Before giving the proof of Theorem 6.1, we need some technical results.

LEMMA 6.1. *Let* u *be an* $L^1(\Omega)$ *function, and define, for* $y \in \Omega$,

$$P(u)(y) = u(y) - \left(\int_\Omega u(x)\,\varphi_1(x)\,dx\right)\varphi_1(y). \tag{6.5}$$

Then, if u *belongs to* $L^p(\Omega)$ *and* v *belongs to* $L^{p'}(\Omega)$, *we have*

$$\int_\Omega P(u)\,P(v)\,dx = \int_\Omega uv\,dx - \int_\Omega u\varphi_1\,dx \int_\Omega v\varphi_1\,dx. \tag{6.6}$$

PROOF. The result follows from easy computations. ■

REMARK 6.2. We note explicitly that, if u belongs to $L^2(\Omega)$, then $P(u)$ is nothing but the projection of u onto $E_1^\perp$.

Now we recall the necessary steps in order to achieve the proof of the Ambrosetti-Prodi theorem when f belongs to $L^2(\Omega)$ (see [8] and [9]).

A solution u of (6.1) can be written in the form

$$u = u(s, f) + s\varphi_1,$$

with $u(s, f) \in E_1^\perp$ and $s = \int_\Omega u\varphi_1\,dx$. Then it is seen that $u(s, f)$ solves the equation

$$A(u(s, f)) = P(g(u(s, f) + s\varphi_1)) + f, \qquad \text{in } E_1^\perp, \tag{6.7}$$

while t and s satisfy the identity

$$t = \lambda_1 s - \int_\Omega g(u(s, f) + s\varphi_1)\varphi_1 dx = \Gamma(s). \tag{6.8}$$

Since $A - P$ is coercive on $E_1^\perp$, equation (6.7) has a unique solution for every $s \in \mathbb{R}$, so that, for t fixed, the number of solutions of (6.1) is equal to the number of roots of the equation $\Gamma(s) = t$. The hypotheses on g imply that $\Gamma(s)$ is a concave function that diverges to $-\infty$ both at $+\infty$ and $-\infty$ and has a unique maximum; if we define $\bar{t}$ to be the maximum of $\Gamma(s)$ on $\mathbb{R}$, then we find 0, 1 or 2 solutions if $t > \bar{t}$, $t = \bar{t}$ or $t < \bar{t}$ respectively.

Now we begin to study equation (6.7) when f belongs to $L^1(\Omega)$. The first result is similar to Lemma 2.2.

LEMMA 6.2. *Let q be a real number, with $1 \leq q < N/(N-1)$.*

Let s be a real number. Let $\{f_n\}$ be a sequence of $L^2(\Omega)$ functions that is bounded in $L^1(\Omega)$ and such that $\int_\Omega f_n \varphi_1 dx = 0$ for every $n \in \mathbb{N}$.

Let $\{h_n\}$ be a sequence of measurable functions that is bounded in $L^\infty(\Omega)$.

Suppose that there exists a sequence $\{w_n\} \subset E_1^\perp$ of solutions of the problem:

$$A(w_n) = P(h_n w_n) + f_n, \qquad \text{in } E_1^\perp. \tag{6.9}$$

Suppose that w_n is bounded in $L^1(\Omega)$. Then there exists a constant c, that does not depend on n, such that

$$\|w_n\|_{W_0^{1,q}(\Omega)} \leq c. \tag{6.10}$$

PROOF. Let r be a real number, with $r > 1$.

Let $v_n = ((1 + |w_n|)^{1-r} - 1)\operatorname{sgn}(w_n)$ and choose $P(v_n)$ as test function in (6.9). Since φ_1 and w_n are orthogonal in $H_0^1(\Omega)$ for every n, we obtain, by (6.6),

$$c_1 \int_\Omega \frac{|\nabla w_n|^2}{(1 + |w_n|)^r} dx \leq \int_\Omega h_n w_n v_n dx - \int_\Omega h_n w_n \varphi_1 dx \int_\Omega v_n \varphi_1 dx - \int_\Omega f_n v_n dx.$$

Since $|v_n| \leqslant 1$, we have

$$c_1 \int_\Omega \frac{|\nabla w_n|^2}{(1+|w_n|)^r}\,dx \leqslant \|h_n\|_{L^\infty(\Omega)}\|w_n\|_{L^1(\Omega)} +$$

$$+ c_2\|h_n\|_{L^\infty(\Omega)}\|w_n\|_{L^1(\Omega)} + \|f_n\|_{L^1(\Omega)} \leqslant c_3,$$

by the hypotheses on w_n, f_n and h_n. The end of the proof is achieved exactly as in Lemma 2.2. ■

The following result is the anlogous of Lemma 2.3. Again, if the data tend to zero in $L^1(\Omega)$, the limit solution is zero if we add some hypotheses on the sequence h_n.

LEMMA 6.3. *Let q be a real number, with $1 \leqslant q < N/(N-1)$.*

Let $\{f_n\}$ be a sequence of $L^2(\Omega)$ functions that tends to zero in $L^1(\Omega)$ and such that $\int_\Omega f_n \varphi_1\,dx = 0$ for every $n \in \mathbb{N}$.

Let $\{h_n\}$ be a sequence of measurable functions that is bounded in $L^\infty(\Omega)$; so we can extract a subsequence that converges, $L^\infty(\Omega)$-weak-$*$, to a function h. We suppose that

$$\|h\|_{L^\infty(\Omega)} \leqslant \mu .$$

Suppose that there exists a sequence $\{v_n\} \subset E_1^\perp$ of solutions of the problem

$$A(v_n) = P(h_n v_n) + f_n, \qquad \text{in } E_{1\perp}. \tag{6.11}$$

Suppose that v_n is bounded in $L^1(\Omega)$. Then v_n tends weakly in $W_0^{1,q}(\Omega)$ to zero.

PROOF. The proof is essentially the same of Lemma 2.3; we fix k and p and construct the sequences $v_n^{(1)} = v_n,\ v_n^{(2)}, \dots, v_n^{(k+1)}$ that solve the equations

$$A(v_n^{(1)}) = P(h_n v_n^{(1)}) + f_n,$$

$$A(v_n^{(2)}) = P(h_n v_n^{(1)}),$$

$$\dots\dots\dots\dots\dots\dots$$

$$A(v_n^{(k+1)}) = P(h_n v_n^{(k)}).$$

The regularity estimates on the $v_n^{(j)}$ are still valid thanks to the boundedness of h_n, to Lemma 6.2 and to the fact that P is continuous between $L^p(\Omega)$ and $L^p(\Omega)$ for any $p \geqslant 1$ (this is an easy consequence of the definition of P). As in the proof of Lemma 2.3, we have that $v_n^{(k+1)}$ is bounded in $H_0^1(\Omega)$ and so it weakly converges, up to a subsequence that we will call again $v_n^{(k+1)}$, to a function $v^{(k+1)}$ that belongs to $H_0^1(\Omega)$.

Substracting the equations satisfied by the $v_n^{(j)}$, we have

$$A(v_n^{(2)} - v_n^{(1)}) = -f_n,$$

$$A(v_n^{(3)} - v_n^{(2)}) = P(h_n(v_n^{(2)} - v_n^{(1)})),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$A(v_n^{(k+1)} - v_n^{(k)}) = P(h_n(v_n^{(k)} - v_n^{(k-1)})),$$

so that all the sequences have the same limit in $W_0^{1,p}(\Omega)$, thanks to the fact that f_n tends to zero in $L^1(\Omega)$ and by the boundednes of h_n; this limit is obviously $v^{(k+1)}$. So $v_n = v_n^{(1)}$ weakly converges to $v^{(k+1)}$ in $W_0^{1,p}(\Omega)$, and $v = v^{(k+1)}$ solves the equation

$$A(v) = P(hv), \qquad v \in H_0^1(\Omega)$$

(we can pass to the limit since $v_n^{(k+1)}$ converges strongly to $v^{(k+1)}$ in $L^2(\Omega)$).

Since all the v_n are in $E_1^\perp$, then also v belongs to $E_1^\perp$. Choosing v as test function, we have

$$\int_\Omega (\mathcal{a}(x)\nabla v, \nabla v)\,dx = \int_\Omega h|v|^2\,dx \leq \mu \int_\Omega |v|^2\,dx.$$

Since

$$\int_\Omega (\mathcal{a}(x)\nabla v, \nabla v)\,dx - \mu\int_\Omega |v|^2\,dx \geq (\lambda_2 - \mu)\int_\Omega |v|^2\,dx \tag{6.12}$$

on $E_1^\perp$, we obtain $\int_\Omega |v|^2\,dx \leq 0$ and so $v = 0$. This means that the sequence v_n weakly converges to 0 in $W_0^{1,p}(\Omega)$. Then we can end the proof as in Lemma 2.3. ∎

In the following Lemma, we state the properties of the solutions of equation (6.7) when the datum f is in $L^1(\Omega)$.

LEMMA 6.4. *Let q be a real number such that $1 \leq q < N/(N-1)$. Let s be a real number. Let f be a function in $L^1(\Omega)$ such that* $\int_\Omega f\varphi_1\,dx = 0$.

Then there exists $u = u(s, f)$, unique solution obtained by approxi-

mation of the equation

$$A(u) = P(g(u + s\varphi_1)) + f, \qquad \int_\Omega u\varphi_1\,dx = 0\,, \tag{6.13}$$

in the sense that

$$\int_\Omega (\mathcal{A}(x)\nabla u, \nabla\varphi)\,dx = \int_\Omega P(g(u + s\varphi_1))\,\varphi\,dx + \int_\Omega f\varphi\,dx\,, \tag{6.14}$$

for every test function φ *in* $\mathcal{D}(\Omega)$ *such that* $\int_\Omega \varphi\varphi_1\,dx = 0$.

Moreover, u *belongs to* $W_0^{1,q}(\Omega)$.

Furthermore, we have that $u(s,f) - u(t,f)$ *belongs to* $H_0^1(\Omega)$ *for every* s *and* t *in* $\mathbb{R}$ *and there exists a constant* c *such that*

$$\|u(s,f) - u(t,f)\|_{H_0^1(\Omega)} \leq c\,|s - t|\,, \qquad \forall s, t \in \mathbb{R}\,. \tag{6.15}$$

PROOF. Let f_n be a sequence of $L^2(\Omega)$ functions that approximate f in $L^1(\Omega)$ and such that $\int_\Omega f_n\varphi_1\,dx = 0$ for every $n \in \mathbb{N}$ (such a sequence is given by Lemma 2.4 with $\rho = 0$). Consider the sequence $u_n = u_n(s, f_n)$ of solutions of equation (6.7). We are going to show that u_n is bounded in $W_0^{1,q}(\Omega)$.

As usual, we begin the proof arguing by contradiction and supposing that u_n is not bounded in $L^1(\Omega)$. Then, we define $v_n = u_n/\|u_n\|_{L^1(\Omega)}$, so that v_n solves the equation

$$A(v_n) = P\left(\frac{g(u_n + s\varphi_1)}{\|u_n\|_{L^1(\Omega)}}\right) + \frac{f_n}{\|u_n\|_{L^1(\Omega)}}\,.$$

Now we define

$$h_n(x) = \begin{cases} \dfrac{g(u_n(x) + s\varphi_1(x))}{u_n(x) + s\varphi_1(x)} & \text{if } u_n(x) + s\varphi_1(x) \neq 0\,, \\[2ex] g'(0) & \text{if } u_n(x) + s\varphi_1(x) = 0\,. \end{cases}$$

So, if we define

$$\tilde{f}_n = \frac{f_n + sP(h_n\varphi_1)}{\|u_n\|_{L^1(\Omega)}}\,,$$

then v_n solves the equation

$$A(v_n) = P(h_n v_n) + \tilde{f}_n .$$

Since we have that h_n is bounded in $L^\infty(\Omega)$ by the hypotheses on g, that v_n has norm equal to 1 in $L^1(\Omega)$ and that $\tilde{f}_n$ is bounded in $L^1(\Omega)$, we can apply Lemma 6.2 to obtain the boundedness of v_n in $W_0^{1,q}(\Omega)$. Thus, we can extract a subsequence (that we will still call v_n), that converges (weakly in $W_0^{1,q}(\Omega)$, strongly in $L^1(\Omega)$ and a.e.) to a function v.

Moreover h_n is bounded in $L^\infty(\Omega)$ and so it converges in L^∞-weak-$*$, up to a subsequence, to a function h such that $\|h\|_{L^\infty(\Omega)} \leq \mu$. Since $\tilde{f}_n$ tends to 0 in $L^1(\Omega)$, we can apply Lemma 6.3 and have that $v = 0$. But this is a contradiction, since $\|v\|_{L^1(\Omega)} = 1$ by the strong convergence of v_n to v in $L^1(\Omega)$.

So, u_n is bounded in $L^1(\Omega)$. Now we can apply Lemma 6.2 to equation (6.7) choosing $h_n(x)$ as before; since $\tilde{f}_n\|u_n\|_{L^1(\Omega)}$ is bounded in $L^1(\Omega)$ we obtain that u_n is bounded in $W_0^{1,q}(\Omega)$ and so it converges (up to a subsequence) to a function $u = u(s, f)$ that solves equation (6.13).

To show that this solution is uniquely obtained by approximation, we suppose to have two sequences f_n^1 and f_n^2 of $L^2(\Omega)$ functions that approximate f in $L^1(\Omega)$. Then we consider the corresponding sequences u_n^1 and u_n^2 of solutions of (6.7). If we substract the equations satisfied by these functions, we have

$$A(u_n^1 - u_n^2) = P(g(u_n^1 + s\varphi_1) - g(u_n^2 + s\varphi_1)) + f_n^1 - f_n^2 . \tag{6.16}$$

Now we define

$$h_n(x) = \begin{cases} \dfrac{g(u_n^1(x) + s\varphi_1(x)) - g(u_n^2(x) + s\varphi_1(x))}{u_n^1(x) - u_n^2(x)} & \text{if } u_n^1(x) \neq u_n^2(x), \\ g'(u_n^2(x) + s\varphi_1(x)) & \text{if } u_n^1(x) = u_n^2(x), \end{cases}$$

so that h_n is bounded in $L^\infty(\Omega)$ and converges in $L^\infty(\Omega)$-weak-$*$, up to a subsequence, to a function h such that $\|h\|_{L^\infty(\Omega)} \leq \mu$; thus, we can apply Lemma 6.3 to obtain that $u_n^1 - u_n^2$ tends weakly to 0 in $W_0^{1,q}(\Omega)$, that means that the solution we obtain does not depend on the approximating sequence.

We are now going to prove (6.15). Let $u_n = u_n(s, f_n)$ and $v_n = v_n(t, f_n)$ be two sequences of solutions of (6.7). Then, if we define $w_n = u_n - v_n$, we have that w_n solves the equation

$$A(w_n) = P(g(u_n + s\varphi_1) - g(v_n + t\varphi_1)) .$$

Now we define

$$h_n = \begin{cases} \dfrac{g(u_n + s\varphi_1) - g(v_n + t\varphi_1)}{(u_n - v_n) + (s - t)\varphi_1} & \text{if } (u_n - v_n) + (s - t)\varphi_1 \neq 0\,, \\[2ex] g'(v_n + t\varphi_1) & \text{if } (u_n - v_n) + (s - t)\varphi_1 = 0\,, \end{cases}$$

(for the sake of simplicity, we have omitted the dependence of h_n from x). So, the former equation becomes

$$A(w_n) = P(h_n w_n + (s - t) h_n \varphi_1)\,. \tag{6.17}$$

Now we choose w_n as test function in (6.17), obtaining

$$\int_\Omega (\mathcal{A}(x) \nabla w_n, \nabla w_n)\, dx \leq \int_\Omega h_n |w_n|^2 dx + \tag{6.18}$$

$$+ |s - t| \int_\Omega h_n |w_n| \varphi_1 dx \leq \mu \int_\Omega |w_n|^2 dx + \mu |s - t| \, \|w_n\|_{L^2(\Omega)}\,.$$

Now we recall (6.12); so, (6.18) becomes

$$(\lambda_2 - \mu)\|w_n\|^2_{L^2(\Omega)} \leq \mu |s - t| \, \|w_n\|_{L^2(\Omega)}$$

that implies, by the uniform ellipticity of A, that

$$\|w_n\|_{H_0^1(\Omega)} \leq c_1 |s - t|\,. \tag{6.19}$$

This means that w_n is bounded in $H_0^1(\Omega)$ and so it weakly converges (up to a subsequence) to a function w; since w_n weakly converges in $W_0^{1,q}(\Omega)$ to $u(s,f) - u(t,f)$, then we have $w = u(s,f) - u\,(t,f)$. By the weak lower semi-continuity of the norm, we have

$$\|u(s, f) - u(t, f)\|_{H_0^1(\Omega)} \leq c_1 |s - t|\,,$$

that is, (6.15) is proved. ∎

Now we can prove Theorem 6.1.

Proof of Theorem 6.1. Let f_n be any sequence of $L^2(\Omega)$ functions that approximates f in $L^1(\Omega)$ and such that $\int_\Omega f_n \varphi_1 dx = 0$ for every $n \in \mathbb{N}$. Let s be a real number; let $u_n = u_n(s, f_n)$ be the solutions of

(6.7), and $\overline{u} = u(s, f)$ the solution of (6.13) given by Lemma 6.4. We define

$$G_n(s) = \lambda_1 s - \int_\Omega g(u_n + s\varphi_1)\,\varphi_1\,dx\,,$$

$$G(s) = \lambda_1 s - \int_\Omega g(\overline{u} + s\varphi_1)\,\varphi_1\,dx\,.$$

By Lemma 6.4, we have that $G_n(s)$ converges to $G(s)$ for every s in $\mathbb{R}$. Since G_n is a concave function, then G is a concave function; thanks to (6.15), G is a continuous function. Moreover, it is easily seen that we have

$$\lim_{s \to -\infty} \frac{G(s)}{s} = \lambda_1 - \lambda\,, \qquad \lim_{s \to +\infty} \frac{G(s)}{s} = \lambda_1 - \mu\,,$$

so that G diverges to $-\infty$ both at $+\infty$ and $-\infty$.

We define

$$\bar{t} = \max\{G(s),\ s \in \mathbb{R}\}\,. \tag{6.20}$$

We are going to show that Theorem 6.1 holds with this choice of $\bar{t}$. As a first step, we prove that equation (6.1) has at least a solution when $t = \bar{t}$. To do this, we define

$$\bar{t}_n = \max\{G_n(s),\ s \in \mathbb{R}\} = G_n(s_n)$$

(we recall that, since (6.1) has a unique solution when $t = \bar{t}_n$, then s_n is uniquely determined). Then we have that s_n is bounded in $\mathbb{R}$. Arguing by contradiction, we suppose that there exists a diverging subsequence of s_n (that we will call again s_n); we suppose that this s_n diverges to $+\infty$ (the case of s_n diverging to $-\infty$ can be treated in the same way).

Let s be a real number such that $G(s) = \bar{t}$, and fix a t strictly less than $\bar{t}$. Since G is concave, there exist exactly two real numbers s^1 and s^2, such that

$$G(s^1) = G(s^2) = t\,, \qquad s^1 < s < s^2\,.$$

If s_n diverges to $+\infty$, then we have that $s_n > s^2$ for every n large enough. This means that G_n is strictly increasing in the interval $[s^1, s^2]$, and so

$$G_n(s^2) > G_n(s) > G_n(s^1)\,.$$

Letting n tend to ∞, we obtain that

$$t = G(s^2) \geqslant G(s) \geqslant G(s^1) = t\,,$$

and so $G(s) = t$, that is a contradiction. Thus, s_n is bounded and so we can extract from it a subsequence that we will still call s_n, tending to a limit $\bar{s}$. Since $\{G_n\}$ is a sequence of Lipschitz functions with the same Lipschitz constant, we have that $G_n(s_n)$ tends to $G(\bar{s})$. Then $G(\bar{s}) = \bar{t}$; to show this, suppose by contradiction that $G(\bar{s}) < \bar{t} = G(\tilde{s})$; then we have $G_n(s_n) > G_n(\tilde{s})$, and, passing to the limit, $G(\bar{s}) \geqslant G(\tilde{s})$, thus contradicting our hypothesis. So we have proved that

$$\bar{t}_n \to \bar{t}\,.$$

If we define $u_n = u_n(s_n, f_n) + s_n \varphi_1$, then u_n solves the equation

$$A(u_n) = g(u_n) + f_n + \bar{t}_n \varphi_1\,,$$

and is bounded in $W_0^{1,q}(\Omega)$ by a constant (depending on the norm of f_n in $L^1(\Omega)$, on s_n and on t_n). Thus, we can extract from it a subsequence that converges to a solution u of the equation

$$A(u) = g(u) + f + \bar{t}\varphi_1\,,$$

and this concludes the part of the proof about the case $t = \bar{t}$.

Let now t be a real number greater than $\bar{t}$, and suppose that there exists a solution u of (6.1) obtained by approximation. This means that there exists a sequence $u_n \in H_0^1(\Omega)$ of solutions of

$$A(u_n) = g(u_n) + f_n + t_n \varphi_1\,,$$

with f_n a sequence of $L^2(\Omega)$ functions that approximates f in $L^1(\Omega)$ and t_n a sequence of real numbers with limit t. Since u_n belongs to $H_0^1(\Omega)$ we can write it as

$$u_n = u_n(s_n, f_n) + s_n \varphi_1\,, \qquad s_n = \int_\Omega u_n \varphi_1 \, dx\,.$$

But this means that $G_n(s_n) = t_n$ and that s_n tends, as n goes to infinity, to a limit s that is equal to $\int_\Omega u\varphi_1 \, dx$. Hence, $G(s) = t$, and this is a contradiction since $t > \bar{t}$.

Now we fix a real number $t < \bar{t}$; then we can uniquely find two real numbers $s^1 < s^2$ such that $G(s^1) = G(s^2) = t$. If n is large enough, since

$\bar{t}_n \to \bar{t}$, we can find two sequences of real numbers s_n^1 and s_n^2 such that

$$G_n(s_n^1) = G_n(s_n^2) = t, \qquad s_n^1 < s_n < s_n^2.$$

We prove now that both s_n^1 and s_n^2 are bounded. Suppose they are not; then (we work on s_n^1, but the proof can be repeated for s_n^2) there exists a subsequence (still called s_n^1) that diverges to $-\infty$ (since s_n is bounded, s_n^1 cannot diverge at $+\infty$). Then, there exists a real number s such that $s_n^1 < s < s_n$ for every n large enough, and such that $G(s) < t - 1$. So we have

$$t = G_n(s_n^1) < G_n(s) < t - \frac{1}{2}$$

if n is large, since $G_n(s) \to G(s)$. So we have a contradiction. Thus, we can extract from s_n^1 and s_n^2 two subsequences that converges to $\tilde{s}^1$ and $\tilde{s}^2$. Then $\tilde{s}^i = s^i$, since $t = G(\tilde{s}^i) = G(s^i)$. So we can consider the two subsequences $u_n^i = u_n(s_n^i, f_n) + s_n^i \varphi_1$ that solve the equations

$$A(u_n^i) = g(u_n^i) + f_n + t\varphi_1, \qquad i = 1, 2$$

and that are bounded in $W_0^{1,q}(\Omega)$; thus, we can extract two converging subsequences. Their limits satisfy the equations

$$A(u^i) = g(u^i) + f + t\varphi_1, \qquad i = 1, 2$$

so that it remains to show that $u^1 \neq u^2$ to achieve the proof. But this fact is true since $u(s^1, f) + s^1\varphi_1 = u(s^2, f) + s^2\varphi_1$ would imply

$$u(s^1, f) - u(s^2, f) = (s^2 - s^1)\varphi_1;$$

since the first term belongs to $E_1^\perp$ (we recall that the difference of two solutions is in $H_0^1(\Omega)$) while the second is in E_1, this means that both the terms are zero and this is a contradiction because $s^1 \neq s^2$.

So, we have shown that equation (6.1) has at least one or two solutions depending on the choice of t. The last thing we have to show is that the solutions we have found are uniquely obtained by means of approximation.

If $t < \bar{t}$, we have that the solutions u^1 and u^2 do not depend on the approximating sequence; this follows from the properties of $u_n(s_n^i, f_n)$ stated in Lemma 6.4, and from the fact that s_n^1 and s_n^2 cannot converge to limits different from s^1 and s^2.

If we study the case $t = \bar{t}$, we may have some problems if $G(s)$ has more than a maximum; i.e., if there exist s^1 and s^2 such that $G(s^1) = G(s^2) = \bar{t}$; since G is concave, this means that $G(s) = \bar{t}$ for every s between s^1 and s^2; in this case, equation (6.1) will have infinitely many solutions. We are going to show that this is not possible because equation (6.1) can have no more than two solutions obtained by means of approximation, and so G has a unique maximum. We repeat the argument of [10, Théoreme 1].

Actually, if there are two such solutions u and v, their difference w satisfies

$$A(w) = h(x)w, \qquad w \in H_0^1(\Omega),$$

where we have defined, as usual,

$$h(x) = \begin{cases} \dfrac{g(u(x)) - g(v(x))}{u(x) - v(x)} & \text{if } u(x) \neq v(x), \\ g'(v(x)) & \text{if } u(x) = v(x), \end{cases}$$

(we have that w belongs to $H_0^1(\Omega)$ because $w = u(s^1, f) - u(s^2, f) + (s^2 - s^1)\varphi_1$ and both terms belong to $H_0^1(\Omega)$ (see Lemma 6.4)).

So, w can be seen as an eigenfunction of a weighted eigenvalue problem with weight h. This means that there exists and index i such that $\lambda_i(h) = 1$. Since $h < \lambda_2$ almost everywhere, then, thanks to (5.8),

$$\frac{\lambda_i}{\lambda_2} = \lambda_i(\lambda_2) < \lambda_i(h) = 1$$

and so $\lambda_i < \lambda_2$, that means $i = 1$. So we have proved that w is an eigenfunction corresponding to the first eigenvalue, and so it can be chosen positive almost everywhere.

Thus, we have proved that if equation (6.1) has two solutions, then they are ordered. Suppose now that there exist three solutions u, v and w; then we have (for example) $u < v < w$ almost everywhere. If we define $w_1 = v - u$ and $w_2 = w - v$, we have

$$A(w_1) = h_1(x)w_1, \qquad A(w_2) = h_2(x)w_2,$$

where h_1 and h_2 are defined in an obvious way. This means that there exists an index i (that is again equal to 1), such that

$$\lambda_i(h_1) = \lambda_i(h_2) = 1.$$

But this is impossible, since the strict convexity of g implies that $h_1 < h_2$ almost everywhere and so $\lambda_i(h_1) > \lambda_i(h_2)$. ■

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