RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

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Triangular stochastic differential equations with boundary conditions

Rendiconti del Seminario Matematico della Università di Padova, tome 90 (1993), p. 159-188
<http://www.numdam.org/item?id=RSMUP_1993__90__159_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 90 (1993)

Triangular Stochastic Differential Equations with Boundary Conditions.

MARCO FERRANTE (*)

ABSTRACT - In the present paper we study a particular class of stochastic differential equations with boundary conditions at the endpoints of a time interval. We present existence and uniqueness results and study the Markov property of the solution. We are able to prove, in two different situations and for every dimension $d \geq 1$, a necessary and sufficient condition that our equation has to satisfy if the solution is a Markov field process.

1. Introduction.

The stochastic calculus with anticipating integrands has been recently developed by several authors (see e.g. [5] and [6]). This theory allows to define $\int \varphi_s \, dW_d$, when the integrand φ_s is not adapted to the filtration generated by the Brownian motion $\{W_t \colon t \in [0, 1]\}$. Moreover it allows us to study different types of stochastic differential equations driven by $\{W_t \colon t \in [0, 1]\}$, where the solution turns out to be non necessarily adapted to the filtration generated by W_t.

In the present paper we are concerned with stochastic differential equations of the type

$$\frac{dX_t}{dt} + f(X_t) = \frac{dW_t}{dt}, \qquad h(X_0, X_1) = 0 \tag{1}$$

where $t \in [0, 1]$ and instead of the usual initial condition, where we fix the value of X_0, we impose a boundary condition which involves both X_0 and X_1. We assume that $\{W_t \colon t \in [0, 1]\}$ is a d-dimensional Brownian

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Partially supported by a Grant from the Fond. Ing. Aldo Gini.

motion and $\{X_t\}$ takes values in $\mathbb{R}^d$ (h being a function from $\mathbb{R}^{2d}$ into $\mathbb{R}^d$).

The goal of this paper is to study a particular class of boundary value problems of the type (1). In fact in the following we shall assume that $\forall i \in \{1, \dots, d\}$, the function $f_i(x_1, \dots, x_d)$ depends only on the first i variables and we shall consider a similar condition about the functions h_i. The main interest is the study of the Markov property of the solution of equation (1). In the paper of Nualart-Pardoux [7] a general results in this direction is given in the one-dimensional case. More precisely, in [7] it is proved in dimension one (i.e. $d = 1$), that the solution of (1) (if it exists and is unique) is a Markov field if and only if $f'' \equiv 0$ and it is proved via a counterexample that in dimension larger than one the solution can be a Markov process, even with non linear f's.

In the present paper we first provide a necessary measurability condition for the solution X_i of equation (1) to be a Markov field. Using this new condition, we can state two necessary and sufficient results, in dimension larger than one, in the present «triangular» case. In the first case, assuming that the boundary conditions are quite general, we shall prove that the solution of our problem is a Markov field process if and only if the functions $f_i(\cdot)$ are linear in the last variable. In the second, for the $f_i(\cdot)$'s sufficiently general, we will show that the process X_t is a Markov field process if and only if the boundary condition has a particular form (some of the coordinates of X_0 and X_1 are given).

The paper is organized as follows: in section 2 we recall brifely some definitions about the Anticipative Calculus (referring to [5] for a comprehensive exposition) and state one Lemma that we need in the following sections. In section 3 we prove some existence and uniqueness theorems (in the spirit of [7]) for the stochastic differential equation of type (1) in the present triangular case. In section 4 we state an extended version of the Girsanov theorem for non necessarily adapted processes which is due to Kusuoka (in [4]). Moreover we prove that we can apply it to our problem and compute a Radon-Nikodym derivative. In section 5 we study the Markov property and find out a measurability condition that the solution of a general non linear stochastic differential equation of type (1) has to satisfy if we assume that it is a Markov field. Applying the previous measurability condition to the present class we prove two necessary results about the f_i and the h_i respectively, and prove easily the sufficiency.

Acknowledgements. This paper is part of the Thesis submitted for the degree of Magister Philosophiae at the International School for Advanced Studies (SISSA) of Trieste. I wish to thank Prof.

David Nualart, supervisor of this Thesis, for the many helpful suggestions and for the critical reading of the manuscript.

2. Some remarks about the Anticipative Calculus.

In this section we shall recall the notions of derivation on Wiener space and Skorohod integral (see [5] for a complete exposition of the basic results about the anticipating stochastic calculus).

Let $\{W(t), 0 \leq t \leq 1\}$ be a d-dimensional Wiener process defined on the canonical probability space $\Omega = C_0([0, 1]; \mathbb{R}^d)$. Let us denote by H the Hilbert space $L^2([0, 1]; \mathbb{R}^d)$. For any $h \in H$, we will denote by $W(h)$ the Wiener integral

$$\int_0^1 \langle h(t), dW_t \rangle .$$

We denote by S the dense subset of $L^2(\Omega)$ consisting of those random variables of the form

$$F = f(W(h_1), \ldots, W(h_n)) \tag{2}$$

where $n \geq 1$, $h_1, \ldots, h_n \in H$ and $f \in C_b^\infty(\mathbb{R}^n)$ (that means, f and all its partial derivatives are bounded).

The random variables of the form (2) are called smooth functionals. For a smooth functional $F \in S$ of the form (2) we define its derivative DF as the d-dimensional stochastic process $\{D_t F, 0 \leq t \leq 1\}$ given by

$$D_t F = \sum_{i=1}^n \frac{\partial f}{\partial x_i} (W(h_1), \ldots, W(h_n)) \, h_i(t) .$$

Then D is a closable unbounded operator from $L^2(\Omega)$ into $L^2(\Omega \times \times [0, 1]; \mathbb{R}^d)$. We will denote by $\mathbb{D}^{1,2}$ the completion of S with respect to the norm $\|\cdot\|_{1,2}$ defined by

$$\|F\|_{1,2} = \|F\|_2 + \|DF\|_{L^2(\Omega \times [0,1]; \mathbb{R}^d)} .$$

We will denote by δ the adjoint of the derivation operator D. That means, δ is a closed and unbounded operator from $L^2(\Omega \times [0, 1]; \mathbb{R}^d)$ into $L^2(\Omega)$ defined as follows: the domain of δ, $\operatorname{Dom}(\delta)$, is the set of processes $u \in L_2(\Omega \times [0, 1]; \mathbb{R}^d)$ such that there exists a positive constant c_u verifying

$$\left| E\int_0^1 \langle D_t F, u_t\rangle\, dt \right| \leq c_u \|F\|_2$$

for all $F \in S$. If u belongs to the domain of δ then $\delta(u)$ is the square integrable random variable determined by the duality relation

$$E\left(\int_0^1 \langle D_t F, u_t\rangle\, dt \right) = E(\delta(u) F) \qquad F \in \mathbb{D}^{1,2}.$$

The operator δ is an extension of the Itô integral in the sense that the class L_a^2 of processes u in $L^2(\Omega \times [0,1]; \mathbb{R}^d)$ which are adapted to the Brownian filtration is included into Dom(δ) and $\delta(u)$ is equal to the Itô integral if $u \in L_a^2$. The operator δ is called the Skorohod stochastic integral.

Define $\mathbb{L}^{1,2} = (L^2(\Omega \times [0,1]; \mathbb{R}^{1,2}))^d$. Then the space $\mathbb{L}^{1,2}$ is included into the domain of δ. The operators D and δ are local in the following sense

(*a*) $\mathbb{1}_{\{F=0\}} DF = 0 \qquad$ for all $F \in \mathbb{D}^{1,2}$;

(*b*) $\mathbb{1}_{\left\{\int_0^1 |u_t|^2 dt = 0\right\}} \delta(u) = 0 \qquad$ for all $u \in \mathbb{L}^{1,2}$.

Using these local properties one can define the spaces $\mathbb{D}_{\text{loc}}^{1,2}$ and $\mathbb{L}_{\text{loc}}^{1,2}$ by a standard localization procedure. For instance $\mathbb{D}_{\text{loc}}^{1,2}$ is the space of random variables F such that there exists a sequence $\{(\Omega_n, F_n), n \geq 1\}$ such that $\Omega_n \in \mathcal{F}$, $\Omega_n \uparrow \Omega$ a.s., $F_n \in \mathbb{D}^{1,2}$ and $F_n = F$ on Ω_n for each $n \geq 1$. By property (a) the derivation operator D can be extended to random variables of the space $\mathbb{D}_{\text{loc}}^{1,2}$.

We shall now state a Lemma, that we shall need in the sequel (referring for the proof to [7]). Let $\mathcal{G} = \sigma(\xi(h_s); t \leq s \leq 1)$ where $\xi(h) = \int_0^1 h(u)\, dW_u$ and $h_s(u) = \mathbb{1}_{[t,s]}(u)$. Let $K = \text{span}\{h_s; t \leq s \leq 1\}$; we can prove the following:

LEMMA 2.1. *Let* $F \in \mathbb{D}_{\text{loc}}^{1,2}$, $A \in \mathcal{G}$ *and* $\mathbb{1}_A F$ *be* $\mathcal{G}$*-measurable. Then* $DF \in K$ *a.s.* ■

3. Existence and uniqueness.

In the present section we shall study existence and uniqueness of solution to the following equation

$$\begin{cases} \dfrac{dX_t}{dt} + f(X_t) = \dfrac{dW_t}{dt}\,, \\ h(X_0, X_1) = 0\,. \end{cases}$$

We shall consider in this part W as an element of $C_0([0, 1]; \mathbb{R}^d)$ and the fact that W is a d-dimensional Wiener process will be irrelevant in this third section. Therefore a solution is regarded as an element $X \in C([0, 1]; \mathbb{R}^d)$ which is such that

$$\begin{cases} X_t + \int_0^t f(X_s)\, ds = X_0 + W_t \qquad 0 \leqslant t \leqslant 1\,, \\ h(X_0, X_1) = 0\,. \end{cases} \tag{3}$$

From now on we shall assume that the mapping $f\colon \mathbb{R}^d \to \mathbb{R}^d$ satisfies

(H1) $\forall i$, $f_i(x)$ is a function of $x_1, \ldots x_i$, continuously differentiable in x_i.

Following [7], we first associate to (3) the equation with $f \equiv 0$

$$\begin{cases} \dfrac{dY_t}{dt} = \dfrac{dW_t}{dt}\,, \\ h(Y_0, Y_1) = 0\,. \end{cases} \tag{4}$$

A solution of (4) is of the form

$$Y_t = Y_0 + W_t$$

and has to satisfy the following equation

$$h(Y_0, Y_0 + W_1) = 0\,.$$

Henceforth in the sequel we shall always assume that the following assumption is satisfied

(H2) $\forall z \in \mathbb{R}^d$, the equation $h(y, y + z) = 0$ has a unique solution $y = g(z)$ and for every $0 \leqslant i \leqslant d$ the function $g_i(z)$ depends only on $z_1, \ldots z_i$ and is of class C^1 in z_i.

Under (H2) equation (4) has the unique solution

$$Y_t = g(W_1) + W_t\,.$$

Conditions (H1) and (H2) give us the possibility to study (3) in a very simple way: in fact for every $i \in \{1, \ldots, d\}$ denoting $(Y_t^1, \ldots, Y_t^d)$ the solution of (4), it holds that

$$Y_t^i = g_i(W_1^1, \ldots, W_1^i) + W_t^i .$$

Then in order to solve (3), we can start solving the first equation

$$\begin{cases} \dfrac{dX_t^1}{dt} + f_1(X_t^1) = \dfrac{dW_t^1}{dt}, \\ X_0^1 = g_1(X_1^1 - X_0^1), \end{cases}$$

which is just a scalar equation. The second equation

$$\begin{cases} \dfrac{dX_t^2}{dt} + f_1(X_t^1, X_t^2) = \dfrac{dW_t^2}{dt}, \\ X_0^2 = g_2(X_1^1 - X_0^1, X_1^2 - X_0^2), \end{cases}$$

can be solved condidering X_t^1 as a fixed process, and so on for every equation. In this way, proving the existence of a unique solution to equation (3), can be reduced to study the generic scalar equation

$$\begin{cases} \dfrac{dX_t}{dt} + f(t, X_t) = \dfrac{dW_t}{dt}, \\ X_0 = \tau(X_1 - X_0). \end{cases} \tag{5}$$

Let us prove the following

PROPOSITION 3.1. *Let f: $[0, 1] \times \mathbb{R} \to \mathbb{R}$ be a measurable function which is of class C^1 in the second variable and τ: $\mathbb{R} \to \mathbb{R}$ be a C^1 function. Furthermore let us assume that*

(i) $\exists\, k > 0$ *and* $\lambda \in \mathbb{R}$ *such that*

$$-\lambda \leq \frac{d}{dx} f(s, x) \leq K \qquad \forall (s, x)$$

and moreover

$$\sup_{0 \leq s \leq 1} |f(s, 0)| < \infty ;$$

(ii) $\exists\, \lambda' > \lambda$ *such that*

$$\exp[\lambda'] |\tau'(x)| \leq |1 + \tau'(x)| \qquad \forall x .$$

Then equation (5) *admits a unique solution.*

PROOF. Equation (5) can be written as follows

$$\begin{cases} X_t - X_0 + \int_0^t f(s, X_s)\, ds = W_t\,, \\ X_0 = \tau(X_1 - X_0)\,. \end{cases}$$

For every X_0 fixed, assumption (i) implies that the first equation admits one and only one solution, that we shall denote by $(X_s(X_0))_{s\in[0,1]}$. Therefore we have to prove that there exists one and only one X_0 that solves the following equation

$$X_0 = \tau\left(W_1 - \int_0^1 f(s, X_s(X_0))\, ds\right).$$

Denoting by $\varphi(x)$ the real function

$$x \to x - \tau\left(W_1 - \int_0^1 f(s, X_s(x))\, ds\right)$$

we have to prove that there exist one and only one $x \in \mathbb{R}$ such that $\varphi(x) = 0$.

Since τ is of class C^1 and $f(s, y)$ is continuously differentiable in the second variable (in the sequel we shall denote by f' the derivative of f w.r.t. y), we have

$$\varphi'(x) = 1 - \tau'(X_1(x) - x)\left[- \int_0^1 f'(s, X_s(x)) \frac{\partial X_s}{\partial x}\, ds \right]$$

where $\partial X_s / \partial x$ is the solution of the following linear differential equation

$$\frac{\partial X_t}{\partial x} - 1 + \int_0^1 f'(s, X_s(x)) \frac{\partial X_s}{\partial x}\, ds = 0\,,$$

and clearly it holds that

$$\frac{\partial X_t}{\partial x} = \exp\left[- \int_0^t f'(s, X_s(x))\, ds \right].$$

At the end we obtain that

$$\varphi'(x) = 1 - \tau'(X_1(x) - x)\left[\exp\left[-\int_0^t f'(s, X_s(x))\,ds\right] - 1\right].$$

Evaluating the absolute value of $\varphi'(x)$ we obtain

$$|\varphi'(x)| = \left|1 - \tau'(X_1(x) - x)\left[\exp\left[-\int_0^t f'(s, X_s(x))\,ds\right] - 1\right]\right| =$$

$$= \left|(1 + \tau'(X_1(x) - x)) - \tau'(X_1(x) - x)\exp\left[-\int_0^t f'(s, X_s(x))\,ds\right]\right| \geqslant$$

$$\geqslant \left|\,|(1 + \tau'(X_1(x) - x))| - \left|\tau'(X_1(x) - x)\exp\left[-\int_0^t f'(s, X_s(x))\,ds\right]\right|\,\right|.$$

By condition (i), we have

$$\exp\left[-\int_0^t f'(s, X_s(x))\,ds\right] \leqslant \exp[\lambda].$$

An easy computation shows that condition (ii) is equivalent to the following one

$$\exists \varepsilon > 0 \quad \text{such that} \quad \varepsilon + \exp[\lambda]|\tau'(x)| \leqslant |1 + \tau'(x)| \qquad \forall x,$$

and therefore

$$|\varphi'(x)| \geqslant \varepsilon \qquad \forall x.$$

This implies that $\varphi(x)$ is a strictly monotone function and therefore there exist one and only one $x \in \mathbb{R}$ such that $\varphi(x) = 0$. ■

We can now give, as a Corollary of Proposition 3.1, an existence and uniqueness result to equation (3).

COROLLARY 3.1. *Suppose that* (H1), (H2) *are satisfied and the func-*

tions f and g belong to $C^1(\mathbb{R}^d, \mathbb{R}^d)$ and verify

$$\text{(H3)} \begin{cases} \textit{For every } i \in \{1, \ldots, d\} \textit{ there exists } K_i > 0 \textit{ and } \lambda_i \in \mathbb{R} \textit{ such that} \\ \text{(i)} \quad -\lambda_i \leqslant \dfrac{\partial}{\partial x_i} f_i(x_1, \ldots x_i) \leqslant K_i\,, \qquad \forall (x_1, \ldots x_i)\,, \\ \text{(ii)} \ \exp[\lambda_i'] \left| \dfrac{\partial}{\partial x_i} g_i(x_1, \ldots x_i) \right| \leqslant \left| 1 + \dfrac{\partial}{\partial x_i} g_i(x_1, \ldots x_i) \right|, \\ \forall (x_1, \ldots x_i) \textit{ for some } \lambda_i' > \lambda_i\,. \end{cases}$$

Then equation (3) *admits a unique solution.* ■

REMARK 3.1. *In the previous results the C^1 regularity conditions can be weakned by assuming in* (i) *the customary Lipschitz condition in the last variable and in* (ii) *an equivalent monotonicity condition on g, as done in* [7]. ■

To conclude this part, let us recall a result presented in [7], that we will use in the following sections. Let us define the set

$$\Sigma = \left\{ \xi \in C([0, 1]; \mathbb{R}^d) \colon h(\xi_0, \xi_1) = 0 \right\}.$$

It is easy to prove that there exists a bijection ψ from $C_0([0, 1]; \mathbb{R}^d)$ into Σ such that

$$Y_t = \psi_t(W).$$

Defining the mapping T from $C_0([0, 1]; \mathbb{R}^d)$ into itselt by

$$T(\eta) = \eta + \int_0^{\cdot} f(\psi_s(\eta))\, ds\,, \tag{6}$$

it holds

PROPOSITION 3.2. *T is a bijection if and only if equation* (3) *has the unique solution*

$$X = \psi \circ T^{-1}(W).$$

4. Computation of a Radon-Nikodym derivative.

In this section we first state the extended Girsanov theorem of Kusuoka (Theorem 6.4 of [4]) and then apply it to our situation. We assume that $\Omega = C_0([0, 1]; \mathbb{R}^d)$ equipped with the topology of uniform convergence, $\mathscr{F}$ is the Borel field over Ω, P is standard Wiener measure and $W_t(\omega) = \omega(t)$ is the canonical process.

THEOREM 4.1. *Let T: $\Omega \to \Omega$ be a mapping of the form*

$$T(\omega) = \omega + \int_0^{\cdot} K_s(\omega)\, ds$$

where K is a measurable mapping from Ω into $H = L^2([0, 1]; \mathbb{R}^d)$, and suppose that the following conditions are satisfied

(i) *T is bijective;*

(ii) *For all $\omega \in \Omega$, there exists a Hilbert-Schmidt operator $D\,K(\omega)$ from H into itself such that*

(1) $\left\| K\left(\omega + \int_0^{\cdot} h_s ds\right) - K(\omega) - DK(\omega)h \right\|_H = o(\|h\|)$ *as* $\|h\|_H$ *tends to zero;*

(2) $h \to DK\left(\omega + \int_0^{\cdot} h_s ds\right)$ *is continuous from H into $\mathscr{L}^2(H)$, the space of Hilbert-Schmidt operators;*

(3) *$I + D\,K(\omega)$ is invertible.*

Then if Q is the measure on $(\Omega, \mathscr{F})$ s.t. $P = QT^{-1}$, Q is absolutely continuous with respect to P and

$$\frac{dQ}{dP} = |d_c(-DK)| \exp\left(-\delta(K) - \frac{1}{2}\int_0^1 |K_t|^2 dt\right),$$

where $d_c(-DK)$ denotes the Carleman-Fredholm determinant of the Hilbert-Schmidt operator $-D\,K$ (see e.g. [10]*), and $\delta(K)$ is the Skorohod integral of K.* ■

We want to apply Theorem 4.1 to the mapping T defined in section 3. Moreover we shall compute the Radon-Nikodym derivative dQ/dP in this particular case where $K_t(\omega) = f(\psi_t(\omega))$, and $\psi_t(\omega) = g(\omega_1) + \omega_t$. Let us assume that f, $g \in C^1(\mathbb{R}^d; \mathbb{R}^d)$; denoting by f' the gradient matrix of f, $(\partial f_i / \partial x_j)_{i, j \in \{1, \dots, d\}}$, and the same for g', it holds that

$$\begin{aligned} D_s K_t(\omega) &= f'(\psi_t(\omega))\, D_s \psi_t(\omega), \\ &= f'(\psi_t(\omega))\,[g'(W_1) + \mathbb{1}_{[0,t]}(s)]. \end{aligned}$$

The operator $DK(\omega) \in \mathscr{L}^2(H)$ is given by

$$(DK(\omega)(h))^i(t) = \sum_{j=1}^{d} \int_0^1 D_s^J K_t^i(\omega)\, h_s^j ds\,.$$

Conditions (ii.1) and (ii.2) are here satisfied. Before checking condition (ii.3) let us compute the Carleman-Fredholm determinant of $-DK$. We will use the following general Lemma (for the proof see Appendix A of [2]), which is a slight generalization of a similar result established in [1]. Let f be an element of $L^2([0, 1]; M_d)$ and g, $h \in L^4([0, 1]; M_d)$ (such that $g \cdot h \in L^2([0, 1]; M_d)$) where M_d is the vector space of the $d \times d$ real matrices. For every s and t belonging to $\mathbb{R}_+$, let us define the L^2-kernel

$$K(s, t) = f(t)\,(g(s) + \mathbb{1}_{[0, t]}(s))\,h(s)\,. \tag{7}$$

Let us denote by K the operator of $L^2([0, 1]; \mathbb{R}^d)$ into itself defined by

$$(K\varphi)(t) = \int_0^1 K(s, t)\,\varphi(s)\,ds$$

$\varphi \in L^2([0, 1]: \mathbb{R}^d)$. Let ψ_t be the solution of the following differential equation

$$\begin{cases} \dfrac{d}{dt}\psi(t) = -h(t)f(t)\,\psi(t)\,, \\ \psi(0) = I\,. \end{cases}$$

Then we have

LEMMA 4.1. *The Carleman-Fredholm determinant of the Hilbert-Schmidt operator* $-K$, *defined by* (7), *is given by*

$$d_c(-K) = \det\left\{I + \int_0^1 g(t)\,h(t) f(t)\,\psi(t)\,dt\right\} \cdot \exp\left(-\int_0^1 \mathrm{Tr}\,(f(t)\,g(t)\,h(t))\,dt\right). \quad \blacksquare$$

Let $\{\phi(t), 0 \leqslant t \leqslant 1\}$ denote the $d \times d$ matrix valued solution of

$$\begin{cases} \dfrac{d}{dt}\phi(t) = -f'(\psi_t)\,\phi(t)\,, \\ \phi(0) = I\,. \end{cases}$$

Notice that, under assumption (H1), it holds that

$$
(8) \qquad \begin{cases} \phi_{ij}(t) = 0 & \forall i < j\,, \\ \phi_{ii}(t) = \exp\left[-\int\limits_0^t \dfrac{\partial f_i}{\partial x_i}(Y_r)\, dr \right] & \forall i\,. \end{cases}
$$

In the present case, when $K_t = f(\psi(t))$, we obtain, from the previous Lemma, that

$$
(9) \qquad d_c(-DK) = \det(I - \phi(1) g'(W_1) + g'(W_1)) \cdot \\ \cdot \exp\left(-\int\limits_0^1 \mathrm{Tr}\,[f'(\psi(t)) g'(W_t)]\, dt \right).
$$

The main result of this section is the following.

THEOREM 4.2. *Suppose that* $f \in C^1(\mathbb{R}^d, \mathbb{R}^d)$, (H.1) *and* (H.2) *hold and* $g \in C^1(\mathbb{R}^d, \mathbb{R}^d)$. *Assume moreover that the tranformation* T *defined in* (6) *is bijective, and furthermore that*

$$
(10) \qquad \det(I - \phi(1) g'(W_1) + g'(W_1)) = \\ = \prod_{i=1}^d \left(1 - \phi_{ii}(1) \frac{\partial g_i}{\partial z_i}(W_1) + \frac{\partial g_i}{\partial z_i}(W_1) \right) \neq 0\,.
$$

Then the conditions of Theorem 4.1 *for,* $K_t = f(\psi(t))$, *are satisfied and*

$$
(11) \qquad J = |\det(I - \phi(1) g'(W_1) + g'(W_1))| \cdot \\ \cdot \exp\left[-\frac{1}{2} \int\limits_0^1 \mathrm{Tr} f(\psi(t))\, dt - \int\limits_0^1 f(\psi(t)) \circ dW_t - \frac{1}{2} \int\limits_0^1 |f(\psi(t))|^2 dt \right],
$$

where $\int\limits_0^1 f(\psi(t)) \circ dW_t$ *is the generalized Stratonovich integral.* ■

PROOF. We have to check condition (ii.3). From the computation of the Carleman-Fredholm determinant in this particular case, we obtain that condition (10) is equivalent to the fact that $d_c(-DK) \neq 0$ and, from the theory of the Hilbert-Schimdt operators (see e.g. [10]), this property is equivalent to condition (ii.3) in Theorem 4.1. Formula (11) follows from (9) and the following relation between the Skorohod and the

Stratonovich integrals (see [6], pag. 597)

$$\delta(f(\psi)) = \int_0^1 f(\psi(t))\, dW_t\,,$$

$$= \int_0^1 f(\psi(t)) \circ dW_t - \frac{1}{2}\int_0^1 \mathrm{Tr}\left[(D^+ f(\psi))(t) + (D^- f(\psi))(t)\right] dt\,,$$

$$= \int_0^1 f(\psi(t)) \circ dW_t -$$

$$- \frac{1}{2}\int_0^1 \mathrm{Tr}\, f'(\psi(t))\, dt - \int_0^1 \mathrm{Tr}\left[f'(\psi(t))\, g'(W_t)\right] dt\,. \qquad \blacksquare$$

To conclude this section, we shall provide sufficient conditions for Theorem 4.2 to hold.

COROLLARY 4.1. *Suppose that* (H1) *and* (H2) *hold and moreover that* $f, g \in C^1(\mathbb{R}^d, \mathbb{R}^d)$. *If condition* (H3) *in Corollary* 3.1 *is satisfied, then* (10) *holds and* T *is bijective, so the assumptions of Theorem* 4.2 *are satisfied.*

PROOF. From Corollary 3.1 and Proposition 3.2, we immediatly obtain that T is a bijection. To prove that (10) holds, it is clearly equivalent to prove that $\forall i \in \{1, \ldots, d\}$

$$\left| 1 - \phi_{ii}(1)\frac{\partial g_i}{\partial z_i}(W_i) + \frac{\partial g_i}{\partial z_i}(W_1) \right| > 0\,. \tag{12}$$

From (H3) and (8), we obtain that

$$\frac{\partial f_i}{\partial x_i}(x_1, \ldots x_i) \geqslant -\lambda_i\,, \qquad \forall (x_1, \ldots x_i) \in \mathbb{R}^i\,,$$

and

$$\phi_{ii}(t) = \exp\left[-\int_0^1 \frac{\partial f_i}{\partial x_i}(Y_r)\, dr\right] \leqslant \exp[\lambda]\,.$$

Proceeding as in the proof of Proposition 3.1, we obtain that for every i

$$\left| 1 - \phi_{ii}(1) \frac{\partial g_i}{\partial z_i}(W_1) + \frac{\partial g_i}{\partial z_i}(W_1) \right| \geqslant$$

$$\geqslant \left| \left| 1 + \frac{\partial g_i}{\partial z_i}(W_1) \right| - \left| \exp\left[- \int_0^1 \frac{\partial f_i}{\partial x_i}(Y_r)\, dr \right] \frac{\partial g_i}{\partial z_i}(W_1) \right| \right| \geqslant$$

$$\geqslant \left| \left| 1 + \frac{\partial g_i}{\partial z_i}(W_1) \right| - \left| \exp[\lambda] \frac{\partial g_i}{\partial z_i}(W_1) \right| \right| > 0,$$

and (12) is proved. ∎

5. The Markov property.

In this sectin we shall study the Markov property of the solution $\{X_t\}$ of equation (1). We can define the two type of Markov properties which are of interest in the present paper:

DEFINITION 5.1. *A continous process* $\{X_t, t \in [0, 1]\}$ *is said to be* Markov *if for any* $t \in [0, 1]$, $\sigma\{X_s;\, 0 \leqslant s \leqslant t\}$ *and* $\sigma\{X_s;\, t \leqslant s \leqslant 1\}$ *are conditionally independent given* $\sigma\{X_t\}$. ∎

DEFINITION 5.2. *A continuous process* $\{X_t, t \in [0, 1]\}$ *is said to be* Markov field *if for any* $0 \leqslant r < t \leqslant 1$, $\sigma\{X_s;\, 0 \leqslant s \leqslant r,\ t \leqslant s \leqslant 1\}$ *and* $\sigma\{X_s;\, r \leqslant s \leqslant t\}$ *are conditionally independent given* $\sigma\{X_r, X_t\}$, ∎

It is possible to prove (see [3]) that any Markov process is a Markov field, but the converse is not true in general. In the case of periodic boundary condition $X_0 = X_1$, we can not expect $\{X_t\}$ to be a Markov process, but it could be a Markov field.

It has been proved (see [9]) that in the Gaussian case (f and h affine) the solution is always a Markov field and it is moreover a Markov process if $h(x, y) = H_0 x + H_1 y - h_0$ is such that $\mathrm{Im}\, H_0 \cap \mathrm{Im}\, H_1 = \{0\}$. It is possible to extend the previous result (see [7]) to the case where the function h is not linear obtaining again that the solution of (1) is a Markov field.

We shall divide the present section in two subsections. In the first one we shall assume that our problem (1) admits a unique solution, that is a Markov field process, and we derive a necessary condition on $f(\cdot)$ and one on $g(\cdot)$. In the second part we shall prove the opposite results, i.e. that both necessary conditions, of f and g respectively, are sufficient for $\{X_t\}$ to be a Markov field process.

5.1. Necessary conditions.

First of all we shall prove a meaurability condition that $\{X_t, t \in \in [0, 1]\}$, solution of (1), has to satisfy if is is a Markov field process.

PROPOSITION 5.1. *Suppose that* (H1) *and* (H2) *hold equation* (1) *has a unique solution for every* $\omega \in \Omega$, f *and* g *are of class* C^2 *and* (10) *in Proposition* 4.1 *holds. Then, if* $\{X_t, t \in [0, 1]\}$ *is a Markov field, we have that*

$$\sum_{i=1}^{d} \frac{\phi_{ii}(1) \dfrac{\partial^2}{\partial x_l \partial x_i} f_i(Y_s) \dfrac{\partial}{\partial z_i} g_i(W_1)}{1 - (\phi_{ii}(1) - 1) \dfrac{\partial}{\partial z_i} g_i(W_1)} \quad \textit{is } \mathcal{F}_t^e\textit{-measurable}$$

$\forall t \in (0, 1)$, $t \leq s \leq 1$ *and* $l = 1, \ldots, i$.

PROOF. Let $t \in [0, 1]$ and define the following three σ-algebras:

$$\mathcal{F}_t := \sigma\{Y_0, Y_t\} = \sigma\{g(W_1), W_t\},$$

$$\mathcal{F}_t^i := \sigma\{Y_s;\ 0 \leq s \leq t\} = \sigma\{g(W_1), W_s;\ 0 \leq s \leq t\},$$

$$\mathcal{F}_t^e := \sigma\{Y_0, Y_s;\ t \leq s \leq 1\} = \sigma\{W_s;\ t \leq s \leq 1\},$$

(where e stands for «exterior» and i for «interior»).

Since

$$X(\omega) = \psi \circ T^{-1}(\omega),$$

$$Y(\omega) = \psi(\omega),$$

we have, for every non negative measurable function f on Ω, that:

$$\int_{\Omega} f(X(\omega))\, dP(\omega) = \int_{\Omega} f(Y(\omega))\, dQ(\omega),$$

i.e. the law of $\{X_t, t \in [0, 1]\}$ under P is the same as the law of $\{Y_t, t \in \in [0, 1]\}$ under Q. Since (1) admits a unique solution, we have that T is a bijection; then from Theorem 4.1 and Theorem 4.2 it holds that Q is absolutely continuous with respect to P (we denote again by J the Radon-Nikodym derivative dQ/dP, whose computation is done in Theorem 4.2). Therefore we can assume that $\{Y_t\}$ is a Markov field under Q, i.e. for any non negative (or equivalently Q integrable) random variable χ

which is $\mathcal{F}_t^i$ measurable,

$$\Lambda_\chi = E_Q(\chi | \mathcal{F}_t^e) = \frac{E_P(\chi J | \mathcal{F}_t^e)}{E_P(J | \mathcal{F}_t^e)}$$

is $\mathcal{F}_t$-measurable.

Define

$$Z = |\det(I - \phi(1) g'(W_1) + g'(W_1))| .$$

It is easy to prove that the previous condition implies that

$$\frac{E_P(\eta Z | \mathcal{F}_t^e)}{E_P(Z | \mathcal{F}_t^e)} \quad \text{is } \mathcal{F}_t\text{-measurable} \tag{13}$$

for any non negative $\mathcal{F}_t^i$-measurable random variable η.

Denote

$$\Theta_{s,l} = (I - \phi(1) g'(W_1) + g'(W_1))^{-1} (\phi(1) \phi^{-1}(s) \nabla_l \partial f(Y_s) \phi(s) g'(W_1)).$$

Proceeding as in the proof of Proposition 5.3 in [2], from the condition (13) we can deduce the following equality

$$\operatorname{Tr}(\Theta_{s,l}) E_P(Z | \mathcal{F}_t^e) = E_P(\operatorname{Tr}(\Theta_{s,l}) Z | \mathcal{F}_t^e) ,$$

for any $t \leq s \leq 1$ and $l = 1, \ldots, i$. This clearly implies that $\operatorname{Tr}(\Theta_{s,l})$ is $\mathcal{F}_t^e$-measurable for any $l = 1, \ldots, d$ and $t \leq s \leq 1$.

Recalling that, under assumption (H1) and (H2), it holds that

$$\phi_{ii}(t) = 0 \qquad \forall i < j ,$$

and

$$\phi_{ii}(t) = \exp\left[- \int_0^1 \frac{\partial f_i}{\partial x_i}(Y_r)\, dr \right] \qquad \forall i = 1, \ldots, d ,$$

we have that

$$[I - \phi(1) g'(W_1) + g'(W_1)]_{ii}^{-1} = \left[1 - (\phi_{ii}(1) - 1) \frac{\partial}{\partial z_i} g_i(W_1) \right]^{-1}$$

and

$$(\phi(1) \phi^{-1}(s) \nabla_l \partial f(Y_s) \phi(s) g'(W_1))_{ii} = \phi_{ii}(1) \left[\frac{\partial^2}{\partial x_l \partial x_i} f_i(Y_s) \right] \frac{\partial}{\partial z_i} g_i(W_1) ,$$

for $i = 1, \ldots, d$, $l \leq i$ and $t \leq s \leq 1$.

Consequently we obtain that

$$\mathrm{Tr}(\Theta_{s,l}) = \sum_{i=1}^{d} \frac{\phi_{ii}(1) \dfrac{\partial^2}{\partial x_l \partial x_i} f_i(Y_s) \dfrac{\partial}{\partial z_i} g_i(W_1)}{1 - (\phi_{ii}(1) - 1) \dfrac{\partial}{\partial z_i} g_1(W_1)} \quad \text{is } \mathcal{F}_t^e\text{-measurable}$$

for all $t \in (0, 1)$, $t \leqslant s \leqslant 1$ and $l = 1, \ldots, i$, which completes the proof. ■

We shall present in this subsection two results concerning necessary properties that our equation (1) has to satisfy, if its unique solution is a Markov field process. The first one, following the ideas of Theorem 4.4 in [6], is a condition on the function $f(\cdot)$ in (1), assuming some additional hypothesis on the function $g(\cdot)$. Conversely, in the second Theorem of this part we are able to prove that, if the function $f(\cdot)$ satisfies a quite general condition, the Markov field property of the solution of (1) implies a condition on $g(\cdot)$. In the following subsection we shall prove that both conditions are also sufficient for the Markov field property.

Let us start with the following

THEOREM 5.1. *Let assumptions* (H1), (H2) *and* (H3) *hold, f be of class C^3 while g is of class C^2. Let us assume furthermore that one of the two following assumptions holds:*

(i) $-1 < (\partial g_i / \partial z_i) < 0$ *for every* $i \in \{1, \ldots, d\}$;

(ii) $(\partial g_i / \partial z_i) < -1$ *or* $(\partial g_i / \partial z_i) > 0$ *for every* $i \in \{1, \ldots, d\}$.

If $\{X_t, t \in [0, 1]\}$, *solution of* (1), *is a Markov field, then* $\forall i = 1, \ldots, d$, *there exists* $u_i(x_1, \ldots, x_{i-1}) \in C^1(\mathbb{R}^{i-1})$ *and* $a_i, b_i \in \mathbb{R}$ *such that*

$$f_i(x_1, \ldots x_i) = a_i x_i + b_i + u_i(x_1, \ldots x_{i-1}). \tag{14}$$

PROOF. Under (H1), (H2) and (H3), the assumptions of previuos Proposition 5.1 are satisfied; therefore

$$\sum_{i=1}^{d} \frac{\phi_{ii}(1) \dfrac{\partial^2}{\partial x_l \partial x_i} f_i(Y_s) \dfrac{\partial}{\partial z_i} g_i(W_1)}{1 - (\phi_{ii}(1) - 1) \dfrac{\partial}{\partial z_i} g_1(W_1)} \quad \text{is } \mathcal{F}_t^e\text{-measurable}$$

$\forall\, t \in (0, 1)$, $t \leqslant s \leqslant 1$ and $l = 1, \ldots, i$.

Putting

$$\alpha_i(x) = \frac{x \dfrac{\partial}{\partial z_i} g_i(W_1)}{1 - (x-1)\dfrac{\partial}{\partial z_i}(W_1)} \tag{15}$$

we have that

$$(\alpha_1(\phi_{11}(1)), \ldots, \alpha_d(\phi_{dd}(1))) \cdot \tag{16}$$

$$\cdot \begin{bmatrix} \dfrac{\partial^2 f_1}{\partial x_1^2}(Y_s) & 0 & \ldots & 0 \\ \dfrac{\partial^2 f_1}{\partial x_1 \partial x_2}(Y_s) & \dfrac{\partial^2 f_2}{\partial x_2^2}(Y_s) & \ldots & 0 \\ \cdot & \cdot & \ldots & \cdot \\ \dfrac{\partial^2 f_d}{\partial x_1 \partial x_d}(Y_s) & \dfrac{\partial^2 f_d}{\partial x_2 \partial x_d}(Y_s) & \ldots & \dfrac{\partial^2 f_d}{\partial x_d^2}(Y_s) \end{bmatrix}$$

is $\mathcal{F}_t^e$-measurable.

From tha last column of the matrix, we obtain that

$$\alpha_d(\phi_{dd}(1)) \frac{\partial^2 f_d}{\partial x_2^2}(Y_s(\omega)) \quad \text{is } \mathcal{F}_t^e\text{-measurable for all } 0 \leqslant t \leqslant 1, \quad t \leqslant s \leqslant 1.$$

Let us suppose that there exists $x_0 \in \mathbb{R}^d$ such that

$$\frac{\partial^2 f_d}{\partial x_d^2}(x_0) \neq 0.$$

From the continuity of $\partial^2 f_d / \partial x_d^2$, we obtain that there exists an open set U in $\mathbb{R}^d$ such that

$$\frac{\partial^2 f_d}{\partial x_d^2}(x) \neq 0, \qquad \forall x \in U.$$

Define

$$G^d := \left\{ \omega \colon \frac{\partial^2 f_d}{\partial x_d^2}(Y_t(\omega)) \neq 0 \right\} \in \mathcal{F}_t^e.$$

Since in the present case the law of the random vector

$$Y_t = g(W_1) + W_t$$

has $\mathbb{R}^d$ as its support, we have

$$P(G^d) > 0 .$$

Therefore $\mathbb{1}_{G^d}\alpha_d(\phi_{dd}(1))$ is $\mathcal{F}_t^e$-meaurable and, since

$$\phi_{dd}(1) = \exp\left[- \int_0^1 \frac{\partial f_d}{\partial x_d}(Y_r)\,dr \right],$$

the random variable $\mathbb{1}_{G^d}\int_0^1 (\partial f_d/\partial x_d)(Y_r)\,dr$ is also $\mathcal{F}_t^e$-measurable. Applying Lemma 2.1 to the present case, we obtain

$$\mathbb{1}_{G^d}\left(\frac{d}{du}\left(D_u^k \int_0^1 \frac{\partial f_d}{\partial x_d}(Y_r)\,dr \right)\right) = 0 \qquad a.s.$$

for $0 \leqslant u \leqslant t$, $1 \leqslant k \leqslant d$. We have

$$\begin{aligned} D_u^k\left[\int_0^1 \frac{\partial f_d}{\partial x_d}(Y_r)\,dr \right] &= \int_0^1 D_u^k\left(\frac{\partial f_d}{\partial x_d}(Y_r)\right) dr , \\ &= \int_0^1 \sum_{l=1}^d \frac{\partial^2 f_d}{\partial x_l \partial x_d}(Y_r)\, D_u^k Y_r^l\, dr , \\ &= \int_0^1 \sum_{l=1}^d \frac{\partial^2 f_d}{\partial x_l \partial x_d}(Y_r) \frac{\partial}{\partial z_k} g_l(W_1)\, dr + \\ &\qquad + \int_u^1 \sum_{l=1}^d \frac{\partial^2 f_d}{\partial x_l \partial x_d}(Y_r)\, \delta_{lk}\, dr . \end{aligned}$$

Choosing $k = d$, we obtain

$$\begin{aligned} \frac{d}{du}\left[D_u^d\left[\int_0^1 \frac{\partial f_d}{\partial x_d}(Y_r)\,dr \right]\right] &= \frac{d}{du}\left(\int_u^1 \frac{\partial^2 f_d}{\partial x_d^2}(Y_r)\,dr \right), \\ &= - \frac{\partial^2 f_d}{\partial x_d^2}(Y_u) = 0 , \qquad 0 \leqslant u \leqslant t , \end{aligned}$$

almost surely on G^d. But this is possible only in $P(G^d) = 0$ and this leads to a contradiction. So we have that

$$\frac{\partial^2 f_d}{\partial x_d^2}(x) = 0\,, \qquad \forall x \in \mathbb{R}^d\,,$$

and, therefore, there exist $\gamma(x_1, \ldots, x_{d-1})$ and $\delta(x_1, \ldots, x_{d-1})$ such that

$$f_d(x) = \phi(x_1, \ldots, x_{d-1})\, x_d + \delta(x_1, \ldots, x_{d-1})\,. \tag{17}$$

Consequently, from (16) and (17), we have that

$$(\alpha_1(\phi_{11}(1)), \ldots, \alpha_d(\phi_{dd}(1))) \begin{bmatrix} \frac{\partial^2 f_1}{\partial x_1^2}(Y_s) & \cdots & 0 & 0 \\ \cdot & \cdots & \cdot & \cdot \\ \frac{\partial^2 f_{d-2}}{\partial x_1 \partial x_{d-2}}(Y_s) & \cdots & 0 & 0 \\ \frac{\partial^2 f_{d-1}}{\partial x_1 \partial x_{d-1}}(Y_s) & \cdots & \frac{\partial^2 f_{d-1}}{\partial x_{d-1}^2}(Y_s) & 0 \\ \frac{\partial^2 f_d}{\partial x_1 \partial x_d}(Y_s) & \cdots & \frac{\partial^2 f_d}{\partial x_{d-1} \partial x_d}(Y_s) & 0 \end{bmatrix}$$

is $\mathcal{F}_t^e$-measurable. Consider now the $d-1$ component of the above product which is equal to

$$\alpha_{d-1}(\phi_{d-1,\,d-1}(1))\frac{\partial^2 f_{d-1}}{\partial x_{d-1}^2}(Y_s) + \alpha_d(\phi_{dd}(1))\frac{\partial^2 f_d}{\partial x_{d-1}\partial x_d}(Y_s)\,.$$

From the previous condition (17), we have that $(\partial^2 f_d / \partial x_{d-1}\partial x_d)(x)$ depends only on the first $d-1$ variables. Let us suppose that there exist $x_0 \in \mathbb{R}^{d-1}$ such that

$$\frac{\partial^2 f_{d-1}}{\partial x_{d-1}^2}(x_0) \neq 0 \quad \textit{or} \quad \frac{\partial^2 f_d}{\partial x_{d-1}\partial x_d}(x_0) \neq 0\,.$$

Define

$$G_{d-1} := \left\{ \omega : \left(\frac{\partial^2 f_{d-1}}{\partial x_{d-1}^2} (Y_t(\omega)) \right)^2 + \left(\frac{\partial^2 f_d}{\partial x_{d-1} \partial x_d} (Y_t(\omega)) \right)^2 \neq 0 \right\}.$$

Using again that the law of Y_t has $\mathbb{R}^d$ as its support, we obtain

$$P(G^{d-1}) > 0 .$$

We have that

$$\mathbb{1}_{G^d} \left\{ \alpha_{d-1}(\phi_{d-1,d-1}(1)) \frac{\partial^2 f_{d-1}}{\partial x_{d-1}^2} (Y_t) + \alpha_d(\phi_{dd}(1)) \frac{\partial^2 f_d}{\partial x_{d-1} \partial x_d} (Y_t) \right\}$$

is $\mathscr{F}_t^e$-measurable.

Proceeding as before we obtain

$$\text{(18)} \qquad \mathbb{1}_{G^d} \left(\frac{d}{du} D_u^k \left[\alpha_{d-1}(\phi_{d-1,d-1}(1)) \frac{\partial^2 f_{d-1}}{\partial x_{d-1}^2} (Y_t) + \right.\right.$$

$$\left.\left. + \alpha_d(\phi_{dd}(1)) \frac{\partial^2 f_d}{\partial x_{d-1} \partial x_d} (Y_t) \right] \right) = 0 ,$$

for $0 \leqslant u \leqslant t$ and $1 \leqslant k \leqslant d-1$.

The first term, when $k = d-1$, is

$$\frac{d}{du} D_u^{d-1} \left[\alpha_{d-1}(\phi_{d-1,d-1}(1)) \frac{\partial^2 f_{d-1}}{\partial x_{d-1}^2} (Y_t) \right] =$$

$$= \frac{d}{du} \left[\alpha'_{d-1}(\phi_{d-1,d-1}(1)) D_u^{d-1}(\phi_{d-1,d-1}(1)) \frac{\partial^2 f_{d-1}}{\partial x_{d-1}^2} (Y_t) + \right.$$

$$\left. + \alpha_{d-1}(\phi_{d-1,d-1}(1)) \sum_{l=1}^{d-1} \frac{\partial^3 f_{d-1}}{\partial x_l \partial x_{d-1}^2} (Y_t) D_u^{d-1}(Y_t^l) \right] =$$

$$= \alpha'_{d-1}(\phi_{d-1,d-1}(1)) \phi_{d-1,d-1}(1) \left(\frac{\partial^2 f_{d-1}}{\partial x_{d-1}^2} (Y_u) \right) \left(\frac{\partial^2 f_{d-1}}{\partial x_{d-1}^2} (Y_t) \right),$$

where

$$\alpha'_{d-1}(x) = \frac{\dfrac{\partial}{\partial z_{d-1}} g_{d-1}(W_1) + \left(\dfrac{\partial}{\partial z_{d-1}} g_{d-1}(W_1)\right)^2}{\left[1 - [x-1]\dfrac{\partial}{\partial z_{d-1}} g_{d-1}(W_1)\right]^2} .$$

A similar computation can be done for the second summand of (18). Then putting $u = t$ in (18) we get

$$\alpha'_{d-1}(\phi_{d-1,d-1}(1))\,\phi_{d-1,d-1}(1)\left[\frac{\partial^2 f_{d-1}}{\partial x_{d-1}^2}(Y_u)\right]^2 + \tag{19}$$
$$+\alpha'_d(\phi_{dd}(1))\,\phi_{dd}(1)\left[\frac{\partial^2 f_d}{\partial x_{d-1}\partial x_d}(Y_t)\right]^2 = 0$$

almost surely on G^{d-1}.

It is easy to prove that under assumption (i) it holds that

$$\alpha'_i(x) < 0 \qquad \forall 0 < x < 1 \text{ and } \forall i \in \{1, \dots, d\}$$

and that under assumption (ii)

$$\alpha'_i(x) > 0 \qquad \forall 0 < x < 1 \text{ and } \forall i \in \{1, \dots, d\}.$$

Therefore, since $\phi_{dd}(1)(\omega) > 0$ for all i, from (19) we have that

$$\left(\frac{\partial^2 f_{d-1}}{\partial x_{d-1}^2}(Y_t(\omega))\right)^2 + \left(\frac{\partial^2 f_d}{\partial x_{d-1}\partial x_d}(Y_t(\omega))\right)^2 = 0$$

almost surely on G^{d-1}, and this is possible if and only if $P(G^{d-1}) = 0$. Therefore both $(\partial^2 f_{d-1}/\partial x_{d-1}^2)(x)$ and $(\partial^2 f_d/\partial x_{d-1}\partial x_d)(x)$ are indentically zero.

It is clear that the same computation can be done for the $(d-2)$ -th column of the previous matrix (16). At the end we shall have

$$\frac{\partial^2 f_i}{\partial x_l \partial x_i}(x) \equiv 0\,, \qquad \forall i = 1, \dots, d\,,\ \ l \leq i\,,\ \ x \in \mathbb{R}^i\,,$$

and this implies that there exist $a_i, b_i \in \mathbb{R}$ and $u_i(x_1, \dots, x_{i-1}) \in C^1(\mathbb{R}^{i-1})$ such that

$$f_i(x) = a_i x_i + b_i + u_i(x_1, \dots x_{i-1}), \qquad \forall i = 1, \dots, d\,,\ \ x \in \mathbb{R}^d\,. \qquad \blacksquare$$

REMARK 5.1. *It the one-dimensional case* (*i.e.*, $d = 1$), *Theorem* 5.1 *reduces to Theorem* 4.4 *in* [7], *i.e. that* $\{X_t, t \in [0, 1]\}$ *is a Markov field process if and only if f is linear, but the proof given here is different from that of* [7]. ■

In the second part of this section we shall assume that the function f is infinetely differentiable, satisfies (H1) and the following condition holds:

$$\text{(H4) Span}\left\langle\left(\frac{\partial}{\partial x_{i_1}}\cdots\frac{\partial}{\partial x_{i_m}}\frac{\partial}{\partial x_1}f_1(x_0), \ldots, \frac{\partial}{\partial x_{i_1}}\cdots\frac{\partial}{\partial x_{i_m}}\frac{\partial}{\partial x_d}f_d(x_0)\right); \right.$$

$$\left. i_1, \ldots i_m \in \{1, \ldots d\},\quad m \geq 1\right\rangle = \mathbb{R}^d\,, \quad \forall x_0 \in \mathbb{R}^d\,.$$

We are able to prove the following

THEOREM 5.2. *Let us assume that f is of class* C^∞, *hypotheses* (H1), (H2) *and* (H4) *hold g is of class* C^2, *and equation* (1) *admits an unique solution* $\{X_t, t \in [0, 1]\}$. *Moreover let us assume that* (10) *in Theorem* 4.1 *holds. Then, if* $\{X_t, t \in [0, 1]\}$ *is a Markov field process, g has to satisfy the following condition:*

$$\text{(H5)}\begin{cases}\forall i \in \{1, \ldots, d\}, \quad \textit{one the two conditions holds}: \\ (a)\ \dfrac{\partial}{\partial z_i} g_i(z_1, \ldots, z_i) = 0\,, \qquad \forall (z_1, \ldots, z_i) \in \mathbb{R}^i\,. \\ (b)\ \dfrac{\partial}{\partial z_i} g_i(z_1, \ldots, z_i) = -1\,, \qquad \forall (z_1, \ldots, z_i) \in \mathbb{R}^i\,.\end{cases}$$

PROOF. Under conditions (H1), (H2) and (13), by Proposition 5.1 we again obtain that

$$(20)\qquad \sum_{i=1}^{d}\frac{\phi_{ii}(1)\dfrac{\partial^2}{\partial x_l\,\partial x_i}f_i(Y_s)\dfrac{\partial}{\partial z_i}g_i(W_1)}{1-(\phi_{ii}(1)-1)\dfrac{\partial}{\partial z_i}g_1(W_1)}\quad \text{is } \mathcal{F}_t^e\text{-measurable}$$

for all $t \in (0, 1)$, $t \leq s \leq 1$ and $l = 1, \ldots, i$.

In terms of the function $\alpha_i(x)$ introduced in (15), condition (20) means that

$$\sum_{i=1}^{d} \alpha_i(\phi_{ii}(1)) \frac{\partial^2}{\partial x_l \partial x_i} f_i(Y_s) \quad \text{is } \mathcal{F}_t^e\text{-measurable} \tag{21}$$

for all $t \in (0, 1)$, $t \leqslant s \leqslant 1$ and $l = 1, \ldots, i$.

We shall use the following Lemma, whose proof will be given at the end of this proof:

LEMMA 5.1. *Under the assumption of Theorem* 5.2, *for all* $t \in (0, 1)$ *we have*

$$P\left[\omega\colon \mathrm{Spain}\left\langle\left(\frac{\partial^2}{\partial x_l \partial x_1} f_1(Y_s(\omega)), \ldots, \frac{\partial^2}{\partial x_l \partial x_d} f_d(Y_s(\omega))\right); \; l \in \{1, \ldots d\}, \;\; s \in [t, 1]\right\rangle = \mathbb{R}^d\right] = 1\,, \tag{22}$$

$$P\left[\omega\colon \mathrm{Spain}\left\langle\left(\frac{\partial^2}{\partial x_l \partial x_1} f_1(Y_s(\omega)), \ldots, \frac{\partial^2}{\partial x_l \partial x_d} f_d(Y_s(\omega))\right); \; l \in \{1, \ldots d\}, \;\; s \in [0, t]\right\rangle = \mathbb{R}^d\right] = 1 \quad \blacksquare. \tag{23}$$

From (22) the measurability condition (21) implies that

$$\alpha_i(\phi_{ii}(1))) = 0 \quad \text{is } \mathcal{F}_t^e\text{-measurable} \quad \text{for all } i = 1, \ldots, d \text{ and } 0 \leqslant t \leqslant 1\,.$$

Applying Lemma 2.1 to the random variable $\alpha_i(\phi_{ii}(1))$, we have

$$\frac{d}{d\theta} D_\theta^k(\alpha_i(\phi_{ii}(1))) = 0 \quad \text{for all } i, k = 1, \ldots, d \text{ and } 0 \leqslant \theta \leqslant t \leqslant 1\,.$$

We obtain

$$\alpha_i'(\phi_{ii}(1)) \frac{d}{d\theta} D_\theta^k(\phi_{ii}(1)) = 0\,,$$

and from (8)

$$\alpha_i'(\phi_{ii}(1))\,\phi_{ii}(1)\left(\frac{d}{d\theta}D_\theta^k\left[\int_0^1 \frac{\partial f_i}{\partial x_i}(Y_r)\,dr\right]\right)=0\,.$$

We have

$$\frac{d}{d\theta}D_\theta^k\left[\int_0^1 \frac{\partial f_i}{\partial x_i}(Y_r)\,dr\right]=\frac{d}{d\theta}\left[\int_0^1\sum_{l=1}^d \frac{\partial^2}{\partial x_l\,\partial x_i}f_i(Y_r)\,D_\theta^k Y_r^l\,dr\right]=$$

$$=\frac{d}{d\theta}\left[\int_0^1\sum_{l=1}^d \frac{\partial^2}{\partial x_l\,\partial x_i}f_i(Y_r)\left(\frac{\partial g_l}{\partial z_k}(W_1)+\mathbb{1}_{[0,r]}(\theta)\,\delta_{lk}\right)dr\right]=$$

$$=\frac{d}{d\theta}\left[\int_0^1\sum_{l=1}^d \frac{\partial^2}{\partial x_l\,\partial x_i}f_i(Y_r)\frac{\partial g_l}{\partial z_k}(W_1)\,dr\right]+\frac{d}{d\theta}\left[\int_\theta^1 \frac{\partial^2}{\partial x_k\,\partial x_i}f_i(Y_r)\,dr\right]=$$

$$=-\frac{\partial^2}{\partial x_k\,\partial x_i}f_i(Y_\theta)\,.$$

Therefore

$$\alpha_i'(\phi_{ii}(1))\,\phi_{ii}(1)\frac{\partial^2}{\partial x_k\,\partial x_i}f_i(Y_\theta)=0\,,$$

for all $i, k=1, \ldots, d$ and $0\leqslant\theta\leqslant t\leqslant 1$.
Recall that

$$\alpha_i'(x)=\frac{\dfrac{\partial}{\partial z_i}g_i(W_1)+\left(\dfrac{\partial}{\partial z_i}g_i(W_1)\right)^2}{\left[1-[x-1]\dfrac{\partial}{\partial z_i}g_i(W_1)\right]^2}\,.$$

Applynig now property (23) we get

$$\alpha_i'(\phi_{ii}(1))\,\phi_{ii}(1)=0\quad\text{a.e. for all } i=1,\ldots,d\,,$$

which implies that

$$\frac{\partial}{\partial z_i}g_i(W_1)+\left(\frac{\partial}{\partial z_i}g_i(W_1)\right)^2=0\quad\text{a.e. for all } i=1,\ldots,d\,.$$

Since $\partial/\partial z_i\, g_i(\cdot)$ is a continuous function $\forall i = 1, \ldots, d$, the only possibilities are

$$\frac{\partial}{\partial z_i} g_i(z_1, \ldots z_i) \equiv 0, \qquad \text{for all } (z_1, \ldots z_i) \in \mathbb{R}^i,$$

or

$$\frac{\partial}{\partial z_i} g_i(z_1, \ldots z_i) \equiv -1, \qquad \text{for all } (z_1, \ldots z_i) \in \mathbb{R}^i,$$

and therefore condition (H5) is satisfied. ■

PROOF OF LEMMA 5.1. We shall only prove (22) and the proof of (23) would follow the same lines. Suppose that (22) does not hold. Fix $t \in (0, 1)$, denote by $(v_1, \ldots v_d)$ a base of $\mathbb{R}^d$ and let $\xi_i^l(s) := := (\partial^2/\partial x_l \partial x_i) f_i(Y_s)$. We have

$$P\left\{\omega: \inf_{r \in \{1, \ldots, d\}} \sum_{l=1}^{d} \int_t^1 \langle v_r, \xi^l(s)\rangle^2 ds = 0\right\} > 0.$$

This implies that there exists $\rho \in \{1, \ldots, d\}$ such that the set

$$A = \left\{\omega: \sum_{l=1}^{d} \int_t^1 \langle v_\rho, \xi^l(s)\rangle^2 ds = 0\right\}$$

has positive probability. Consider the function $\Phi: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, defined by

$$\Phi(y, z) := \sum_{i=1}^{d} (v_\rho)_i \frac{\partial^2}{\partial x_l \partial x_i} f_i(g(y) + z).$$

We can apply the multidimensional anticipative Itô-formula (see [6], pag. 565) on $[s, 1]$ to the function Φ, obtaining

$$\Phi(W_1, W_1) - \Phi(W_1, W_s) = \sum_{k=1}^{d} \int_s^1 \Phi'_{z_k}(W_1, W_r)\, dW_r^k +$$

$$+ \frac{1}{2} \sum_{k,j=1}^{d} \int_s^1 \Phi''_{z_k z_j}(W_1, W_r)\, dr + \sum_{k,j=1}^{d} \int_s^1 \Phi''_{z_k y_j}(W_1, W_r)\, dr.$$

The quadratic variation of the right hand side can be computed following the results of [6] and is equal to

$$\sum_{k=1}^{d} \int_{s}^{1} [\Phi'_{z_k}(W_1, W_r)]^2 \, dr .$$

Therefore

$$\sum_{k=1}^{d} \int_{s}^{1} [\Phi'_{z_k}(W_1, W_r)]^2 \, dr = 0 \qquad \text{on } A, \text{ a.s., for all } s \in [t, 1].$$

Since

$$\Phi'_{z_k}(W_1, W_s) = \sum_{k=1}^{d} (v_\rho)_i \frac{\partial^3}{\partial x_k \partial x_l \partial x_i} f_i(g(W_1) + W_s)$$

we obtain

$$\langle v_\rho, \xi^{l,k}(s) \rangle = 0 , \qquad \forall s \in [t, 1], \ \forall l, k \in \{1, \ldots, d\}, \ \forall \omega \in A , \text{ a.s.} ,$$

where $\xi_i^{l,k}(s) := (\partial^3 / \partial x_k \partial x_l \partial x_i) f_i(Y_s)$. Clearly we can apply again the previous computation to the function

$$\Psi(y, z) := \sum_{i=1}^{d} (v_\rho)_i \frac{\partial^3}{\partial x_k \partial x_l \partial x_i} f_i(g(y) + z) ,$$

obtaining that

$$\langle v_\rho, \xi^{l,k,n}(s) \rangle = 0 , \qquad \forall s \in [t, 1], \ \forall l, k, n \in \{1, \ldots, d\}, \ \forall \omega \in A , \text{ a.s.} ,$$

where $\xi_i^{l,k,n}(s) := (\partial^4 / \partial x_n \partial x_k \partial x_l \partial x_i) f_i(Y_s)$. It is clear that we can iterate the previous computation as many times we want, obtaining for all $m \geq 1$ that

$$\langle v_\rho, \xi^{i_1, \ldots, i_m}(s) \rangle = 0 , \quad \forall s \in [t, 1], \ \forall i_1, \ldots, i_m \in \{1, \ldots, d\}, \ \forall \omega \in A , \text{ a.s.} ,$$

where $\xi^{i_1, \ldots i_m}(s) := (\partial^{m+1} / \partial x_{i_1} \ldots \partial x_{i_m} \partial x_j) f_j(Y_s)$.

Since $P(A) > 0$, there shall exists $x_0 \in \mathbb{R}^d$ such that

$$\dim \left[\operatorname{Span} \left\langle \left(\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_m}} \frac{\partial}{\partial x_1} f_1(x_0), \ldots, \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_m}} \frac{\partial}{\partial x_d} f_d(x_0) \right) ; \right.\right.$$
$$\left.\left. i_1, \ldots i_m \in \{1, \ldots d\} \right\rangle \right] < d , \quad \forall m \geq 1 ,$$

and this leads to a contradiction with the hypothesis (H4). ■

REMARK 5.2. *In Theorem* 5.2 *we have to assume directly that* (1) *admits an unique solution and that* (10) *holds, instead of considering the sufficient condition* (H3) *of Corollary* 3.1. *In fact condition* (ii) *of* (H3) *implies that* $(\partial/\partial z_i) g_i(z_1, \ldots, z_i) \neq -1$ *for every* $(z_1, \ldots z_i) \in \mathbb{R}^i$. ■

5.2. Sufficient conditions.

In this second part, we shall investigate the converse implications of the previuos Theorems 5.1 and 5.2.

Let us start with some general remarks. As we have seen in the proof of Proposition 5.1, to prove that $\{X_t, t \in [0, 1]\}$, solution of (1), is a Markov field under P, it is sufficient to show that $\{Y_t, t \in [0, 1]\}$, solution of (4), is a Markov field under Q (where $P = QT^{-1}$ and T is defined by (6)). From the expression of dQ/dP it follows that the Markov field property holds if

$$Z = |\det(I - \phi(1) g'(W_1) + g'(W_1))| \tag{24}$$

can be written as $Z = Z^i \cdot Z^e$, with

$$Z^i \text{ is } \mathcal{F}_t^i\text{-measurable and } Z^e \textit{ is } \mathcal{F}_t^e\text{-measurable}.$$

Let us prove the following

THEOREM 5.3. *Under the assumption of Theorem* 5.2, *if* g *satisfies* (H5), *then the solution of* (1) *is a Markov field process.*

PROOF. It will be sufficient to prove that, Z given by (24) has the above factorization property. We have

$$Z = \prod_{i=1}^{d} \left| \left(1 - \phi_{ii}(1) \frac{\partial}{z_i} g_i(W_1) + \frac{\partial}{z_i} g_i(W_1)\right) \right| .$$

Since $(\partial/z_i) g_i$ is equal to 0 or to -1, we have, letting $R := \{j \in \{1, \ldots, d\} : (\partial/\partial z_i) g_i = -1\}$

$$Z = \prod_{i \in R} |\phi_{ii}(1)| = \prod_{i \in R} \left[\exp\left(- \int_0^1 \frac{\partial f_i}{\partial x_i}(Y_r)\, dr \right) \right] =$$

$$= \prod_{i \in R} \left[\exp\left(- \int_0^t \frac{\partial f_i}{\partial x_i}(Y_r)\, dr \right) \right] \prod_{i \in R} \left[\exp\left(- \int_t^1 \frac{\partial f_i}{\partial x_i}(Y_r)\, dr \right) \right],$$

which provides the desired factorization. ■

REMARK 5.3. *Theorem* 5.3 *is related to the result contained in* [7], *section* 5, *and in chapter* 7 *of* [2], *where it is proved, with a different technique, that if:*

$$\frac{\partial g_i}{\partial z_j} \equiv 0 \quad \forall i \neq 0 \qquad \textit{and} \ \frac{\partial g_i}{\partial z_j} = 0 \ \textit{or} \ -1\,,$$

and (1) *admits an unique solution, then this solution is a Markov process.* ∎

To conclude this subsection we can prove the converse of Theorem 5.1, i.e.:

THEOREM 5.4 *Under the assumptions of Theorem* 5.1, *if* $f\colon \mathbb{R}^d \to \mathbb{R}^d$ *satisfies for all* i:

$$f_i(x_1, \ldots, x_i) = a_i x_i + b_i + u_i(x_1, \ldots, x_{i-1}) \tag{25}$$

where $a_i, b_i \in \mathbb{R}$ *and* $u_i\colon \mathbb{R}^{i-1} \to \mathbb{R}$, *then the unique solution of* (1) *is a Markov field process.*

PROOF. Again it holds that

$$Z = \prod_{i=1}^{d} \left| \left(1 - \phi_{ii}(1) \frac{\partial}{\partial z_i} g_i(W_1) + \frac{\partial}{\partial z_i} g_i(W_1)\right) \right| .$$

Let us recall that

$$\phi_{ii}(t) = \exp\left(-\int_0^t \frac{\partial f_i}{\partial x_i}(Y_r)\,dr\right)$$

and from (25) we get $\phi_{ii}(t) = \exp(a_i t)$ and, therefore

$$Z = \prod_{i=1}^{d} \left| \left(1 - \exp(a_i) \frac{\partial}{\partial z_i} g_i(W_1) + \frac{\partial}{\partial z_i} g_i(W_1)\right) \right| ,$$

which is trivially $\mathcal{F}_t^i$-measurable. ∎

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Manoscritto pervenuto in redazione il 19 maggio 1992.