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On the Lattice Automorphisms of Certain Simple Algebraic Groups.

MAURO COSTANTINI (*)

Introduction-notation.

Given a group G, the set $\mathcal{L}(G)$ of all subgroups of G partially ordered by inclusion is well known to be a complete algebraic lattice. A *projectivity* of a group G onto a group $\overline{G}$ is any lattice isomorphism form $\mathcal{L}(G)$ onto $\mathcal{L}(\overline{G})$, and an *autoprojectivity* of G is any projectivity of G onto itself. We shall denote by Aut $\mathcal{L}(G)$ the group of all autoprojectivities of G. Two groups G, $\overline{G}$ will be called *projective* if there exists a projectivity of G onto $\overline{G}$.

Let G, $\overline{G}$ be groups, and let α be an isomorphism of G onto $\overline{G}$. We can define in a natural way the projectivity α^* of G onto $\overline{G}$ given by $X^{\alpha^*} = X^{\alpha}$ for every $X \leqslant G$. α^* is called the projectivity induced by the isomorphism α. If $\overline{G} = G$, then we have a homomorphism $*$: Aut $G \to$ Aut $\mathcal{L}(G)$ given by $\alpha \mapsto \alpha^*$ for every α in Aut G.

An interesting problem is to know in which cases a projectivity of G onto a group $\overline{G}$ is induced by an isomorphism. In this context, a group G is said to be *strongly lattice determined* if every projectivity of G onto a group $\overline{G}$ is induced by an isomorphism. It is clear that G is strongly lattice determined if and only if the following two conditions are satisfied:

G projective to $\overline{G}$ implies G isomorphic to $\overline{G}$,

the homomorphism $*$: Aut $G \to$ Aut $\mathcal{L}(G)$ is surjective.

In literature several classes of groups whose members are strongly lattice determined are known ([10], [11], [12], [15], [16], [17], [24], [25]).

From the classification of the finite simple groups, one can see that if G is a finite non abelian simple group, and G is projective to $\overline{G}$, then G is isomorphic to $\overline{G}$. Therefore, in order to prove that a finite simple

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group is strongly lattice determined, it is enough to show that the homomorphism $*$ is surjective.

The groups studied by Metelli ([11], [12]) are a special family of finite simple groups of Lie type, and a conjecture was made that for all finite simple groups of Lie type the homomorphism $*$ was sujective. In 1985 Völklein proved this for the groups of type

$$(\text{✗}) \qquad B_l,\ C_l,\ D_{2l},{}^2D_{2l},{}^3D_4,\ E_7,\ E_8,\ F_4,\ G_2,$$

where the characteristic p of the base field is sufficintly large ([22]).

If G is a group in the list (✗) over the finite field k, then G arises as the subgroup generated by the unipotent elements of the group of k-rational points of a certain adjoint simple algebraic group G defined over k. A crucial fact in the proof given in [22] is that the groups in the list (✗) arise from simple adjoint algebraic groups G whose Weyl groups have non-trivial center. Indeed in the same paper Völklein proved that for the groups $PSL_3(q)$ and $PSU_3(q^2)$, whose absolute Weyl group is S_3, the homomorphism $*$ is *not* in general surjective. We showed in [7] that $q = 17$ is the least prime-power number such that $PSL_3(q)$ has autoprojectivities not induced by automorphism. It is in this connection that in [8] we considered the behaviour of the map $*$ for a simple algebraic G group over the algebraic closure of a finite field.

For convenience here we recall some facts there proved.

Every autoprojectivity of G is index-preserving (a projectivity φ of a group G onto a group $\overline{G}$ is said to be index-preserving if given $H \leqslant K \leqslant G$ and $[K:H] = n$, we have $[K^\varphi : H^\varphi] = n$. It is well known ([26] Corollario 3) that to prove that φ is index-preserving it is enough to prove that $H \leqslant K \leqslant G$, K cyclic and $[K:H] = n$ implies $[K^\varphi : H^\varphi] = n$) (Theorem 2.3).

If φ is an autoprojectivity of G, then the image under φ of a maximal torus, a maximal unipotent subgroup and a Borel subgroup of G are respectively a maximal torus, a maximal unipotent subgroup and a Borel subgroup of G. In particular φ induces an automorphism of the building $\Delta(G)$ canonically associated to G (Proposition 2.5, 2.7, Theorem 2.8).

For every autoprojectivity φ of G there exists a unique automorphism of G inducing φ on $\Delta(G)$ (Corollary 4.6).

It then follows that the map $*$ is surjective if and only if the group $\Gamma(G)$ of all autoprojectivities of G fixing every parabolic subgroup, coincides with the identity subgroup of $\operatorname{Aut}\mathfrak{L}(G)$ (Corollary 4.9).

The main results of the present paper is that *if G is a simple algebraic group over $\overline{\mathbb{F}}_p$ with p odd and G not of type A_2, then every autoprojectivity of G is induced by a unique automorphism* *of G (Theorem A, B, C). We also prove that, in the adjoint case, G is strongly lattice determined (Theorem D).*

We shall show in a forthcoming paper that the case A_2 is in fact an exceptional one.

For algebraic groups, we use the standard notation ([5], [20]). If G is a reductive algebraic group over an algebraically closed field K, and T is a maximal torus of G, for every root α: $T \to K^\times$, X_α is the root subgroup corresponding to α, and x_α is a fixed algebraic isomorphism $x_\alpha: (K, +) \to X_\alpha$. Also, α^v: $K^* \to T$ is the coroot corresponding to α.

Let p be any prime. We shall always denote by K the algebraic closure of the field $\mathbb{F}_p$ with p elements. Unless otherwise specified, G always denotes a simple algebraic group over K and φ an element of $\Gamma(G)$.

1. The case rank G at least 3.

In this paragraph we shall shaw that $\Gamma(G) = \{1\}$ when G has rank at least 3 and p is odd. We first prove some properties of $\Gamma(G)$ which hold in general.

PROPOSITION 1.1. *φ fixes every maximal torus and every maximal unipotent subgroup of G. If Φ is the set of roots relative to a maximal torus, then we have $X_\alpha^\varphi = X_\alpha$ and $(\ker \alpha)^\varphi = \ker \alpha$ for every α in Φ.*

PROOF. The first part is clear. So let now T be a maximal torus of G, and let Φ be the set of roots. For every α in Φ there exists a Borel subgroup B of G containing T such that α lies in the fundamental system Π of Φ relative to the choice of B. Then we have $X_\alpha = U \wedge P^{n_0}$, where P is the parabolic subgroup $\langle B, n_\beta \rangle$, n_β is any representative in $\mathcal{N}(T)$ of the fundamental reflection s_β and β is given by the relation $w_0 s_\alpha w_0 = s_\beta$, where w_0 is the longest element of the Weyl group $\mathcal{N}(T)/T$ ([5] page 59). So we get $X_\alpha^\varphi = (U \wedge P^{n_0})^\varphi = U \wedge P^{n_0} = X_\alpha$. We finally prove that $(\ker \alpha)^\varphi = \ker \alpha$. For every non-trivial u in X_α, we have $\ker \alpha = T \wedge \mathcal{C}(u)$. We fix a non-trivial element u in X_α. As $X_\alpha^\varphi = X_\alpha$, there exists a non-trivial $\bar{u}$ in X_α such that $\langle u \rangle^\varphi = \langle \bar{u} \rangle$. If now s is in $\ker \alpha$ and $\bar{s}$ is an element of T such that $\langle s \rangle^\varphi = \langle \bar{s} \rangle$, then $\bar{s}$ lies in $\mathcal{C}(\bar{u})$, as $\langle u, s \rangle^\varphi$ is cyclic. Hence $\bar{s}$ lies in $T \wedge \mathcal{C}(\bar{u}) = \ker \alpha$. It follows that $(\ker \alpha)^\varphi =$ $= \ker \alpha$. ■

For every α in Φ, we define L_{α^v} to be the subgroup $\alpha^v(K^\times)$ of T. L_{α^v} is therefore a 1-dimensional subtorus of T.

PROPOSITION 1.2. *$L_{\alpha^v}^\varphi = L_{\alpha^v}$ for every α be in Φ.*

PROOF. We show that $L_{\alpha^v} = T \wedge \langle X_\alpha, X_{-\alpha}\rangle$. Let n_α be any representative of the reflection w_α of $\mathcal{N}(T)/T$ in $\langle X_\alpha, X_{-\alpha}\rangle$ ([5] page 19). As L_{α^v} is a maximal torus of $\langle X_\alpha, X_{-\alpha}\rangle$, we have the Bruhat decomposition $\langle X_\alpha, X_{-\alpha}\rangle = L_{\alpha^v}X_\alpha \cup L_{\alpha^v}X_\alpha n_\alpha X_\alpha$. But $L_{\alpha^v}X_\alpha n_\alpha X_\alpha \cap T = \emptyset$, as $B \cap \cap Bn_\alpha B = \emptyset$, and so we have $T \wedge \langle X_\alpha, X_{-\alpha}\rangle \subseteq T \cap (L_{\alpha^v}X_\alpha) = L_{\alpha^v}$. Hence $L_{\alpha^v} = T \wedge \langle X_\alpha, X_{-\alpha}\rangle$, so that $L_{\alpha^v}^\varphi = T^\varphi \wedge \langle X_\alpha^\varphi, X_{-\alpha}^\varphi\rangle = T \wedge \langle X_\alpha, X_{-\alpha}\rangle = = L_{\alpha^v}$ by 1.1. ■

The crucial point in the case of groups of rank at least 3 is the following fact about tori.

PROPOSITION 1.3. Let T be a torus of dimension l. If l is at least 3, then every autoprojectivity of T is induced by an automorphism.

PROOF. Let ψ be an autoprojectivity of T. By Theorem 2.3 in [8], ψ fixes every q-component T_q of T. We have $T_q \cong C_{q^\infty} \times \ldots \times C_{q^\infty}$ (l copies) for every prime $q \neq p$. As $l \geq 3$, there exists, by a theorem by Baer ([21] Theorem 2 page 35), an automorphism f_q of T_q inducing ψ on $\mathcal{L}(T_q)$. As $T = \oplus T_q$ (sum over all primes q different from p), if we define $f = \oplus f_q$ in the obvious way, we get that f is an automorphism of T inducing ψ. ■

If H is any group, an automorphism α of H is called a power automorphism if for every subgroup K of H we have $K^\alpha = K$. Hence the group of power automorphisms of H is the kernel of the homomorphism $*$: Aut $H \to$ Aut $\mathcal{L}(H)$. If H is a periodic group, then an automorphism α is a power automorphism if and only if for every x in H, there exists m in $\mathbb{Z}$ such that $x^\alpha = x^m$.

Let T be any maximal torus of G. If the rank of G is at least 3, then, by 1.1 and 1.3, there exists an automorphism f of T inducing φ on $\mathcal{L}(T)$. We shall show that f is in fact a power-automorphism of T. We first state some facts which hold in general for semisimple groups. So assume G is a semisimple algebraic group over an algebraically closed field and let T be a maximal torus of G. We denote by X the character group of T and by Φ the set of roots relative to T. Let B be a Borel subgroup of G containing T and let $\Pi = \{\alpha_1, \ldots \alpha_l\}$ be the fundamental system of Φ relative to the choice of B. For every closed subgroup S of T we put $S^\perp = \{\chi \in X \mid \chi(s) = 1 \ \forall s \in S\}$, and for every subgroup A of X we put $A^\perp = \{t \in T \mid \alpha(t) = 1 \ \forall \alpha \in A\}$. $A^\perp$ is a closed subgroup of T. We are interested in the family of subgroups $\{S^\perp \mid S \text{ is a subtorus of } T\}$ of X. Let us denote by $\mathcal{A}$ the set of all subgroups A of X such that X/A is torsion free, and by $\mathcal{S}$ the set of all subtori of T. We have the following result:

PROPOSITION 1.4. The map given by $A \mapsto A^{\perp}$ for every A in $\mathcal{A}$, is a bijection of $\mathcal{A}$ onto $\mathcal{S}$, its inverse being the map $S \mapsto S^{\perp}$ for every S in $\mathcal{S}$. Also we have $\dim A^{\perp} = \operatorname{rank}(X/A)$ for every A in $\mathcal{A}$.

PROOF. This follows from the fact that for every torus S we have $\dim S = \operatorname{rank} \operatorname{Hom}(S, K^{\times})$, and from the fact that a diagonalizable group S is a torus if and only in $\operatorname{Hom}(S, K^{\times})$ is a free abelian group ([19] Proposition 2.5.8). ■

We now consider the cocharacter group Y of T. If we denote by Φ^v the set of coroots, then $\Pi^v = \{\alpha_1^v, \ldots \alpha_l^v\}$ is fundamental system of Φ^v. Let $(\chi_1, \ldots, \chi_l)$, $(\gamma_1, \ldots, \gamma_l)$ be $\mathbb{Z}$-bases resp. of X and of Y. Let T_i be the subgroup $\{t \in T \mid \chi_j(t) = 1 \text{ for every } j \neq i\}$, and L_i be the subgroup $\gamma_i(K^{\times})$ of T. Then both T_i and L_i are 1-dimensional subtori of T. For every $i = 1, \ldots, l$, we consider the algebraic homomorphisms

$$\xi_i \colon T_i \to K^{\times} \quad \text{given by } \xi_j(t) = \chi_i(t) \quad \text{for every } t \text{ in } T_i, \quad \text{and}$$

$$\zeta_i \colon K^{\times} \to L_i \quad \text{given by } \zeta_i(t) = \gamma_i(k) \quad \text{for every } k \text{ in } K^{\times}.$$

LEMMA 1.5. Let $(\chi_1, \ldots, \chi_l)$, $(\gamma_1, \ldots, \gamma_l)$ be dual $\mathbb{Z}$-bases resp. of X and Y, in the usual duality $X \times Y \to \mathbb{Z}$. Then, for every $i = 1, \ldots, l$, we have $T_i = L_i$, and ξ_i, ζ_i are one the inverse of the other (in particular they are algebraic isomorphisms).

PROOF. This is obvious. ■

PROPOSITION 1.6. Let $(\chi_1, \ldots, \chi_l)$, $(\gamma_1, \ldots, \gamma_l)$ be $\mathbb{Z}$-bases resp. of X and of Y. Then we have the following decompositions of T as algebraic group:

$$T = T_1 \times \ldots \times T_l \quad \text{and} \quad T = L_1 \times \ldots \times L_l .$$

PROOF. It is clear that $\langle T_1, \ldots, T_l \rangle = T_1 \times \ldots \times T_l$, so that $T = T_1 \times \ldots \times T_l$ as they both have dimension l. This decomposition is of algebraic groups by 1.5. From this it also follows that $T = L_1 \times \ldots \times L_l$ by cosidering the dual basis of $(\gamma_1, \ldots, \gamma_l)$ and 1.5. ■

Going back to the case when G is a simple algebraic group over K, we consider separately groups of different isogeny types. We start with G of adjoint type, i.e. $\{\alpha_1, \ldots, \alpha_l\}$ is a $\mathbb{Z}$-basis of X.

Let I be the set $\{1, \ldots, l\}$. For every $J \subseteq I$, we denote by A_J the subgroup $\langle \alpha_i \mid i \in J \rangle$ of X. Hence $A_J^{\perp}$ is a subtorus of T of dimension

$l - |J|$. For every i in I we put $T_{\alpha_i} = \{t \in T \mid \alpha_j(t) = 1 \ \forall j \in I, j \neq i\}$. Let ξ_i be the restriction to T_{α_i} of the algebraic homomorphism $\alpha_i \colon T \to K^\times$. By 1.5 ξ_i is an algebraic isomorphism from T_{α_i} onto $K^\times$. We shall denote by ν_i the inverse of ξ_i.

For every i, j in I such that $\alpha_i + \alpha_j$ lies in Φ, we define $T_{ij} = = (\bigwedge_{\substack{k \in I \\ k \neq i,j}} \ker \alpha_k) \wedge \ker(\alpha_i + \alpha_j)$.

T_{ij} is a 1-dimensional subtorus of T by 1.4.

PROPOSITION 1.7. *φ fixes all the subtori T_{α_i}, T_{ij}.*

PROOF. It is enough to obverve that all these subtori are elements of the sublattice of $\mathcal{L}(T)$ spanned by the set $\{\ker \alpha \mid \alpha \in \Phi\}$, and then use 1.1. ∎

For every i, j in I such that $\alpha_i + \alpha_j$ lies in Φ, we introduce the algebraic isomorphism $\rho_{ij} \colon K^\times \times K^\times \to T_{\alpha_i} \times T_{\alpha_j}$ given by the map $(a, b) \mapsto \nu_i(a)\nu_j(b^{-1})$ for every a, b in $K^\times$.

PROPOSITION 1.8. *Let D be the diagonal of $K^\times \times K^\times$. Then, for every i, j in I such that $\alpha_i + \alpha_j$ lies in Φ, we have $D^{\rho_{ij}} = T_{ij}$.*

PROOF. It is enough to show that $D^{\rho_{ij}} \leqslant T_{ij}$, as ρ_{ij} is injective, D and T_{ij} are 1-dimensional connected algebraic groups, and ρ_{ij} is an algebraic map. So let a be an element of $K^\times$. Let m be in I, $m \neq i, j$. We have $\alpha_m(\nu_i(a)) = \alpha_m(\nu_j(a^{-1})) = 1$ as $\alpha_m(T_{\alpha_r}) = \{1\}$ for every $r \neq m$. Also we have $(\alpha_i + \alpha_j)(\nu_i(a)\nu_j(a^{-1})) = \alpha_i\nu_i(a)\alpha_j\nu_j(a^{-1})$. But, for every s in I and for every k in $K^\times$, we have $\alpha_s\nu_s(a) = \xi_s\nu_s(a) = a$. Hence $(\alpha_i + \alpha_j)(\nu_i(a)\nu_j(a^{-1})) = \alpha_i\nu_i(a)\alpha_j\nu_j(a^{-1}) = aa^{-1} = 1$, and $(a, a)^{\rho_{ij}}$ lies in T_{ij}. ∎

LEMMA 1.9. *Let F be an automorphism of $K^\times \times K^\times$. Let D be the diagonal of $K^\times \times K^\times$. If F fixes the subgroups $K^\times \times \{1\}, \{1\} \times K^\times$ and D, then, for every α in $\mathbb{N}$, there exists n in $\{1, \dots, p^\alpha - 1\}$ such that $s^F = s^n$ for every s in $\mathbb{F}_{p^\alpha}^\times \times \mathbb{F}_{p^\alpha}^\times$.*

PROOF. Let α be in $\mathbb{N}$. Let k be an element of $K^\times$ such that $\langle k \rangle = = \mathbb{F}_{p^\alpha}^\times$. There exist a unique n_1 and a unique n_2 in $\{1, \dots, p^\alpha - 1\}$ such that $(k, 1)^F = (k^{n_1}, 1)$ and $(1, k)^F = (1, k^{n_2})$. As F fixed also D, we get $n_1 = n_2$. Call this common value n. Now let s be in $\mathbb{F}_{p^\alpha}^\times \times \mathbb{F}_{p^\alpha}^\times$. There exist r, t in $\mathbb{Z}$ such that $s = (k^r, k^t)$. Hence $s^F = s^n$, and we are done. ∎

We now fix a maximal torus T of G.

PROPOSITION 1.10. Let G be adjoint of rank at least 3. Then any automorphism of T inducing φ on $\mathcal{L}(T)$ is a power-automorphism.

PROOF. Let f be an automorphism of T inducing φ on $\mathcal{L}(T)$. Let t be in T. By 1.6 for every i in I there exists a unique t_i in T_{α_i} such that $t = t_1 \dots t_l$. Let us denote by k_i the element $\xi_i(t_i)$ of $K^\times$, and let $\mathbb{F}_{p^\alpha}$ be the subfield of K generated by $k_1, \dots, k_l$. Let also k be a generator of $\mathbb{F}_{p^\alpha}^\times$. By 1.7 there exists a unique n_i such that $\{1, \dots, p^\alpha - 1\}$ such that $\nu_i(k)^f = \nu_i(k^{n_i})$ for every i in I. Let now i, j be such that $\alpha_i + \alpha_j$ lies in Φ. If we denote by F_{ij} the automorphism $\rho_{ij}\, f_{ij}\, \rho_{ij}^{-1}$ of $K^\times \times K^\times$, where f_{ij} is automorphism of $T_{\alpha_i} \times T_{\alpha_j}$ induced by f, we have $(K^\times \times \{1\})^{F_{ij}} = K^\times \times \{1\}, (\{1\} \times K^\times)^{F_{ij}} = \{1\} \times K^\times$ and $D^{F_{ij}} = D$, by 1.7, 1.8. By 1.9 there exists n in $\mathbb{N}$ such that $(a, b)^{F_{ij}} = (a, b)^n$ for every a, b in $F_{p^\alpha}^\times$. Hence we have $n_i = n_j$ for every i, j in I such that $\alpha_i + \alpha_j$ lies in Φ. As the Dynkin diagram of G is connected, we conclude that $n_i = n_j$ for every i, j in I. Call this common value m, so that we have $\nu_i(k)^f = \nu_i(k^m)$ for every i in I. From $t = \nu_1(k_1) \dots \nu_l(k_l)$, it then follows that $t^f = t^m$. Therefore f is a power-automorphism of T. ■

We now consider the case when G is simply-connected. Then $\{\alpha_1^\vee, \dots, \alpha_l^\vee\}$ is a $\mathbb{Z}$-basis of Y. For every i in I, we denote by ζ_i the algebraic isomorphism $K^\times \to L_{\alpha_i^\vee}$ given by $\zeta_i(k) = \alpha_i^\vee(k)$ for every k in $K^\times$. Also, for every i, j in I such that $\alpha_i + \alpha_j$ lies in Φ, we define the algebraic isomorphism μ_{ij}: $K^\times \times K^\times \to L_{\alpha_i^\vee} \times L_{\alpha_j^\vee}$ by $(a, b) \mapsto \zeta_i(a)\zeta_j(b)$ for every a, b in $K^\times$. From the fact that $\alpha_i + \alpha_j$ lies in Φ it follows that $\alpha_i^\vee + \alpha_j^\vee$ lies in $\Phi^\vee$, so that $L_{\alpha_i^\vee + \alpha_j^\vee}$ is also well defined. But then $L_{\alpha_i^\vee + \alpha_j^\vee}$ coincides with $D^{\mu_{ij}}$ where D be the diagonal of $K^\times \times K^\times$.

PROPOSITION 1.11. Let G be simply-connected of rank at least 3. Then any automorphism of T inducing φ on $\mathcal{L}(T)$ is a power-automorphism.

PROOF. The proof is similar to the proof of 1.10. The only difference is that here, for every i, j in I such that $\alpha_i + \alpha_j$ lies in Φ, we consider the automorphism V_{ij} of $K^\times \times K^\times$ given by $\mu_{ij}\, f_{ij}\, \mu_{ij}^{-1}$ (where f_{ij} is the automorphism of $L_{\alpha_i^\vee} \times L_{\alpha_j^\vee}$ induced by f). By 1.2 we have $L_{\alpha_i^\vee}^f = L_{\alpha_i^\vee}$, $L_{\alpha_j^\vee}^f = L_{\alpha_j^\vee}$ and $L_{\alpha_i^\vee + \alpha_j^\vee} = L_{\alpha_i^\vee + \alpha_j^\vee}$, so that $K^\times \times \{1\})^{V_{ij}} = K^\times \times \{1\}, (\{1\} \times K^\times)^{V_{ij}} = \{1\} \times K^\times$ and $D^{V_{ij}} = D$. Then we can proceed as in the proof of 1.10. ■

We shall finally deal with the case left out so far. So let G be neither adjoint nor simply-connected. Such a group is therefore forced to have rank at least 3, and to be of type A_l or D_l. In particular, the Dynkin diagram of G has only single bonds. For every subgroup A of Y we denote by $A^{\frown}$ the subtorus $\langle \gamma(K^\times) \mid \gamma \in A \rangle$ of T. We have $A^{\frown} = a_1(K^\times) \dots a_r(K^\times)$ whenever $\{a_1, \dots, a_r\}$ is a set of generators of A.

LEMMA 1.12. Let A be a subgroup of Y of rank r. Then the dimension of $A^{\frown}$ is r. In particular we have $T = \alpha_1^\vee(K^\times) \dots \alpha_l^\vee(K^\times)$.

PROOF. There exists a $\mathbb{Z}$-basis $(\gamma_1, \dots, \gamma_l)$ of Y and positive integers $n_1, \dots, n_r$, such that $(n_1\gamma_1, \dots, n_r\gamma_r)$ is a $\mathbb{Z}$-basis of A. Then we have $T = \gamma_1(K^\times) \times \dots \times \gamma_l(K^\times)$, by 1.6, and so $A^{\frown} = \gamma_1(K^\times) \times \dots \times \gamma_r(K^\times)$ as $K^\times$ is divisible. Hence the dimension of $A^{\frown}$ is r. In particular we have $\alpha_1^\vee(K^\times) \dots \alpha_l^\vee(K^\times) = T$.

The decomposition $T = \alpha_1^\vee(K^\times) \dots \alpha_l^\vee(K^\times)$ is in general not a direct decomposition of T, as one can see for instance when G has type A_l. (It is enough to take a non-adjoint, non-simply-connected algebraic group of type A_3 and p odd). In the following we shall use the same notation we previously introduced. For every α in Φ we consider the 1-dimensional subtorus $L_{\alpha^\vee}$ of T, and for every i in I, the surjective algebraic homomorphism $\zeta_i \colon K^\times \to L_{\alpha_i^\vee}$. In the simply-connected case we also had the isomorphism $\mu_{ij} \colon K^\times \times K^\times \to L_{\alpha_i^\vee} \times L_{\alpha_j^\vee}$ for every i, j in I such that $\alpha_i + \alpha_j$ lies in Φ. In our case we do not know a priori if the product $L_{\alpha_i^\vee} L_{\alpha_j^\vee}$ is direct and if ζ_i is bijective. Anyway we can still define $\mu_{ij} \colon K^\times \times K^\times \to L_{\alpha_i^\vee} L_{\alpha_j^\vee}$ by $(a, b) \mapsto \zeta_i(a)\zeta_j(b)$ for every a, b in $K^\times$, so that μ_{ij} is a surjective algebraic homomorphism. We show that in fact the map μ_{ij} is injective (hence, in particular, $L_{\alpha_i^\vee} L_{\alpha_j^\vee}$ is a direct product and ζ_i is injective).

PROPOSITION 1.13. Let i, j be elements of I such that $\alpha_i + \alpha_j$ lies in Φ. Then the map μ_{ij} is a bijective algebraic homomorphism, ζ_i is injective for every i in I.

PROOF. We already know that μ_{ij} is a surjective algebraic homomorphism. We prove that it is injective. Let (a, b) be in $\ker \mu_{ij}$. Hence $\alpha_i^\vee(a)\alpha_j^\vee(b) = 1$. From the fact that the rank of G is at least 3, and that the Dynkin diagram of G has only single bonds, there exists k in I, $k \neq i, j$, such that we have one of the following subgraphs.

o — — o — — o   o — — o — — o
k   i   j   k   j   i

Without loss of generality, we may suppose we are in the first situation. Therefore we have $\langle \alpha_k, \alpha_i^\vee \rangle = -1$, $\langle \alpha_k, \alpha_j^\vee \rangle = 0$ and $\langle \alpha_i, \alpha_j^\vee \rangle = -1$. We have $1 = \alpha_k(1) = \alpha_k(\alpha_i^\vee(a)\alpha_j^\vee(b)) = a^{-1}$, so that $a = 1$, and consequently $\alpha_j^\vee(b) = 1$. But then we have $1 = \alpha_i(1) = \alpha_i(\alpha_j^\vee(b)) = b^{-1}$, which gives $b = 1$. Hence μ_{ij} is injective. Now let i be in I. As the Dynkin diagram of G is connected, there exists j in I such that $\alpha_i + \alpha_j$ lies in Φ. But then we proved that the map μ_{ij} is injective, and so ζ_i must be injective. ■

PROPOSITION 1.14. Let G be neither adjoint nor simply-connected. Then any automorphism of T inducing φ on $\mathcal{L}(T)$ is a power-automorphism.

PROOF. Let f be an automorphism of T inducing φ on $\mathcal{L}(T)$. Let t be in T. By 1.12, for every i in I there exists an element t_i in $L_{\alpha_i^\vee}$ such that $t = t_1 \ldots t_l$. By 1.13, the homomorphism ζ_i is injective, and so, for every i in I, there exists a unique k_i in $K^\times$ such that $\zeta_i(k_i) = t_i$. From the fact that for every i, j in I such that $\alpha_i + \alpha_j$ lies in Φ, μ_{ij} is a bijective, we can now use the same procedure we used in the simply-connected case to find m in $\mathbb{Z}$ such that $t^f = t^m$. Therefore f is a power-automorphism of T. ■

COROLLARY 1.15. Let G be of rank at least 3. Then we have $\langle s\rangle^\varphi = \langle s\rangle$ for every semisimple element s of G.

PROOF. Let s be a semisimple element of G. There exists a maximal torus T of G such that s lies in T. By 1.10, 1.11 and 1.14, there exists a power-automorphism f of T inducing φ on $\mathcal{L}(T)$. Hence we have $\langle s\rangle^\varphi = \langle s\rangle^f = \langle s\rangle$, and we are done. ■

We are now interested in the behaviour of subgroups generated by a unipotent element under the action of $\Gamma(G)$. We shall make use of the classification of unipotent classes of G. We prove the next lemma for groups over any algebraically closed field.

LEMMA 1.16. Let G be a simple algebraic group over an algebraically closed field. Then every unipotent element u of G is conjugate to its inverse.

PROOF. We give a proof from the classification of unipotent conjugacy classes if the characteristic of the field is 0 or a good prime. Let u be a unipotent element of G. From the Bala-Carter theorem([3], [4]) and the results of Pommerening ([13], [14]) (we recall that the classification of unipotent classes is independent of the isogeny class of G), there exists a Levi subgroup L of G and a parabolic subgroup P of the derived subgroup L' of L, such that u lies in the unique conjugacy class C of L' such that $C \cap R_u(P)$ is open and dense in $R_u(P)$ ([5] Note on page 132). In the following we shall denote by U the unipotent radical

$R_u(P)$ of P. We consider the map ε: $L' \to L'$ given by $x \mapsto x^{-1}$ for every x in L'. This map is an automorphism of affine varieties. Let $\{C_1, \dots, C_k\}$ be the (finite) set of unipotent conjugacy classes of L'. Then ε permutes this set. $C \cap U$ is open and dense in U, and so $(C \cap \cap U)^{-1}$ must be open and dense in U, as ε is a homeomorphism of topological spaces, and $U^{-1} = U$. But $(C \cap U)^{-1} = C^{-1} \cap U^{-1} = C^{-1} \cap U$, hence we must have $C^{-1} = C$, by uniqueness. Therefore u^{-1} lies in C. In particular u and u^{-1} are conjugate in G. If p is a bad prime the classification of unipotent conjugacy classes may be different. However J. N. Spaltenstein suggested the following argument. u^{-1} is certainly cojugate to u if the conjugacy class C of u contains a dense open subset of $U \wedge {}^wU$ for some w in the Weyl group, where U is a maximal unipotent subgroup of G. It is known from the classification of unipotent classes that this actually covers all cases. Hence u is always conjugate to u^{-1}. ■

REMARK. From the structure of reductive groups, it follows that 1.16 holds also when G is reductive.

From the previous result we get.

PROPOSITION 1.17. Let u be a unipotent element of G. If ψ is an autoprojectivity of G fixing every cyclic subgroup of G of order a power of 2, then we have $\langle u \rangle^\psi = \langle u \rangle$.

PROOF. If the characteristic of the field is 2, then there is nothing to prove. So assume $p \neq 2$. By 1.16, there exists an element h of G such that $huh^{-1} = u^{-1}$. Let $2^a m$ be the order of h, with $2 \nmid m$. If we take $g = = h^m$, we have $gug^{-1} = u^{-1}$, and the order of g is 2^a. We have $(gu)^2 = g^2$ so that gu has order a power of 2 as well. Hence $\langle g, u \rangle^\psi = \langle g, gu \rangle^\psi = = \langle g, gu \rangle = \langle g, u \rangle$. But then we must have $\langle u \rangle^\psi = \langle u \rangle$ as $\langle u \rangle$ is the unique p-Sylow subgroup of the group $\langle g, u \rangle$, and ψ is index-preserving. ■

We are now able to prove the announced.

THEOREM A. Let G be a simple algebraic group over the field $K = = \overline{\mathbb{F}}_p$. If the rank of G is at least 3, and p is odd, then every autoprojectivity of G is induced by a unique automorphism of G.

PROOF. We have to show that $\Gamma(G)$ coincides with the identity subgroup of Aut $\mathcal{L}(G)$. It is enough to show that $\langle s \rangle^\varphi = \langle s \rangle$ and $\langle u \rangle^\varphi = = \langle u \rangle$ for every φ in $\Gamma(G)$, every semisimple element s of G and every unipotent element u of G. But this follows from 1.15 and

1.17, as every element of order a power of 2 in our case is semisimple. ■

In the next paragraph we shall deal with the case when the rank of G is less than 3.

2. The case rank G less than 3 (A_2 excluded).

In the previous paragraph we proved that for simple algebraic groups of rank at least 3, in odd characteristic, every autoprojectivity is induced by an automorphism. We made a crucial use of the hypothesis on the rank to use a theorem by Baer. Here we consider groups of rank 1 or 2. In this case the above mentioned theorem by Baer drastically fails. Nevertheless we are able to prove the result we proved for rank $G \geqslant 3$, for groups of rank 1 and 2, if we exclude the case when G has type A_2.

OBSERVATION. Our aim is to study the group Aut $\mathcal{L}(G)$. But clearly our aim will be achieved if we determine the group Aut $\mathcal{L}(\overline{G})$ where $\overline{G}$ is any abstract group isomorphic to our given algebraic group G. If G is any simple algebraic group, there exists an isogeny π: $G \to G_{ad}$, where G_{ad} is the adjoint simple algebraic group of the same type of G. We have $\ker \pi = Z(G)$, and so $G/Z(G)$ is isomorphic to G_{ad} as an abstract group. We introduce the abstract groups $PSL_2(K) = SL_2(K)/Z(SL_2(K))$ and $PSp_4(K) = Sp_4(K)/Z(Sp_4(K))$. We have $PGL_2(K) = (A_1)_{ad} \cong PSL_2(K)$ and $PCSp_4(K) = (B_2)_{ad} \cong PSp_4(K)$.

We shall first study the group Aut $\mathcal{L}(PSL_2(K))$. We recall the following result by C. Metelli ([11]). Let $q = p^f$, where p is a prime and f is any natural number. If q is at least 4, for every projectivity τ of the simple group $PSL_2(q)$ onto a group H, there exists a unique isomorphism α: $PSL_2(q) \to H$ inducing τ.

We can now prove.

THEOREM 2.1. Let τ be a projectivity of $PSL_2(K)$ onto a group H, where $K = \overline{\mathbb{F}}_p$. Then there exists a unique isomorphism of $PSL_2(K)$ onto H inducing τ.

PROOF. Let us denote by Z the centre of $SL_2(K)$. For every n in $\mathbb{N}$, we denote by G_n the subgroup $SL_2(p^{n!})/Z$ of $PSL_2(K)$, and we consider the restrictions τ_n: $G_n \to G^{\tau_n}$ induced by τ. By Metelli's result, for every $n = 2, 3, \ldots$, there exists a unique isomorphism α_n: $G_n \to G_n^{\tau}$ inducing τ_n. By the uniqueness of α_n, we have $g^{\alpha_n} = g^{\alpha_{n+1}}$ for every g in

G_n and every $n \geqslant 2$. This enables us to define the map α: $PSL_2(K) \rightarrow H$ by $g^\alpha = g^{\alpha_n}$ for g in $PSL_2(K)$, where n is such that g lies in G_n. α is an isomorphism, and it is clear that α induces τ. Uniqueness of α follows from the fact that the group of power-automorphisms of a perfect group is the identify group ([6] 2.2.2.). ■

In particular if G is adjoint of rank 1, every autoprojectivity of G is induced by a unique automorphism of G. We shall now consider the other possibility for groups of rank 1, i.e. the case when $G = SL_2(K)$. We prove a lemma which we shall use also when G has type B_2.

LEMMA 2.2. Let G be one of the groups $SL_2(K)$, $Sp_4(K)$. Suppose ψ is an autoprojectivity of G fixing every subgroup of G containing the centre of G. Then ψ is the identity.

PROOF. Let us denote by Z the centre of G. If the characteristic p of K is 2, we have $Z = \{1\}$ and there is nothing to prove. So let $p \neq 2$. Then Z is cyclic of order 2. Let x be an element of G of order r^α, where r is a prime, and $\alpha \geqslant 1$. If r is odd, then $\langle x \rangle$ is the unique r-Sylow subgroup of $\langle x, Z \rangle$. Hence we have $\langle x \rangle^\psi = \langle x \rangle$ as $\langle x, Z \rangle^\psi = \langle x, Z \rangle$, and ψ is index-preserving. Now let $r = 2$. We have two cases. If $< x > \geqslant Z$, then we have $\langle x \rangle^\psi = \langle x \rangle$ by hypothesis (this is always the case if $G = SL_2(K)$). Otherwise let T be a maximal torus of G containing x. As T is a divisible group, there exists y in T such that $y^2 = x$. We have $\langle y, Z \rangle = \langle y \rangle \times Z$ and $\mathrm{Frat}(\langle y, Z \rangle) = \langle y^2 \rangle = \langle x \rangle$. Hence $\langle x \rangle^\psi = \langle x \rangle$, as the Frattini subgroup is clearly an invariant under projectivities.

We can now prove.

PROPOSITION 2.3. Let φ be an autoprojectivity of $SL_2(K)$. Then there exists a unique automorphism of $SL_2(K)$ inducing φ.

PROOF. Let us denote by Z the centre of $SL_2(K)$. If p is 2, we have $Z = \{1\}$, and so the result follows from 2.1. So let us assume $p \neq 2$. By Corollary 2.10 in [8], we have $Z^\varphi = Z$, and so we can define the autoprojectivity $\bar{\varphi}$ of $PSL_2(K)$ by $(X/Z)^{\bar{\varphi}} = X^\varphi / Z$, for every subgroup X of $SL_2(K)$ containing Z. By 2.1, there exists an automorphism $\bar{\alpha}$ of $PSL_2(K)$ inducing $\bar{\varphi}$. Also, from the structure of the group Aut $PSL_2(K)$ ([9]), there exists an automorphism α of $SL_2(K)$ inducing $\bar{\alpha}$ on $PSL_2(K)$. We therefore have $X^\varphi = X^\alpha$ for every subgroup X of $SL_2(K)$ containing Z. By 2.2, we get $\varphi = \alpha^*$. Uniqueness follows again from the fact that $SL_2(K)$ is a perfect group. ■

We summarize the previous results in

THEOREM B. Let G be a simple algebraic group of rank 1 over K. Then every autoprojectivity of G induced by a unique automorphism of G. ■

(This gives an alternative proof of Corollary 4.6 in [8] for groups of rank 1).

We now consider the case when G has rank 2. We deal only with groups of type B_2 or G_2.

Using an argument similar to the one used by Völklein in [22] for the corresponding finite simple Chevalley groups (here the proof is even easier as in our case every unipotent element is conjugate to its inverse) it is possible to prove the following theorem:

THEOREM 2.4. Let G be an adjoint simple algebraic group of type B_2 or G_2 over the field K of odd characheristic. Then every autoprojectivity of G is induced by a unique automorphism of G. ■

The crucial point in the proof is that if the centre of the Weyl group of an adjoint simple algebraic group G is non- trivial (as in the case B_2 or G_2), then, for every maximal torus T of G, there exists an involution σ of G such that $s^{\sigma} = s^{-1}$ for every s in T. Then one first proves that if φ lies $\Gamma(G)$, φ fixes every subgroup of order 2 of G using the decomposition $T = T_{\alpha_1} \times T_{\alpha_2}$ we introduced in § 1. From the previous observation it follows that φ fixes every subgroup of G generated by a semisimple element, and finally φ fixes also every subgroup generated by a unipotent element by 1.17.

Let us now consider the simply-connected case. As the simply-connected simple algebraic group of type G_2, is also adjoint, we only have to deal with the simply-connected group of type B_2, i.e. with the symplectic group $Sp_4(K)$.

PROPOSITION 2.5. Let G be the group $Sp_4(K)$ over the field K of odd characteristic. Then every autoprojectivity of G is induced by a unique automorphism of G.

PROOF. Let φ be in $\Gamma(G)$. We consider the isogeny $\pi\colon G \to G_{ad}$. π induces an abstract isomorphism μ between $G/Z(G)$ and G_{ad}, and a bijection between the set of parabolic subgroups of G and the set of parabolic subgroups of G_{ad}. Let $\bar{\varphi}$ be the autoprojectivity of $G/Z(G)$ induced by φ. We denote by γ the projectivity of $G/Z(G)$ onto G_{ad} induced by μ, and we put $\psi = \gamma^{-1}\bar{\varphi}\gamma$. From the fact that φ lies in $\Gamma(G)$, it follows that ψ lies in $\Gamma(G_{ad})$. But then we must have $\psi = 1$, by 2.4, and so we are left with $\bar{\varphi} = 1$. Hence we have $X^{\varphi} = X$ for every subgroup X of G containing $Z(G)$, and so $\varphi = 1$, by 2.2. ■

We summarize the previous results in

THEOREM C. Let G be of type B_2 or G_2 over the field K of odd characteristic. Then every autoprojectivity of G is induced by a unique automorphism of G. ■

3. Final remarks.

The methods used so far cannot be carried over to the case A_2. We shall show in a forthcoming paper that in fact A_2 represents an exceptional case among the simple algebraic groups G over $\overline{\mathbb{F}}_p$, from the subgroup lattice point of view.

We underline that we dit not consider the case when $p = 2$, with the exception of groups of rank 1. The difficulty arises from the classification of unipotent conjugacy classes in characteristic 2, which is usually more complicated than in the case of odd characteristic. We have some results also in this direction. For instance, it is possible to extend the results obtained by Völklein ([23]) for the groups $SL_n(2^a)/D$ (where D is any central subgroup of $SL_n(2^a)$) to the groups $SL_n(\overline{\mathbb{F}}_2)/D$. In particular, if G is of type A_l, then every autoprojectivity of G is induced by an automorphism if l is not 2. In general, if G is any simple algebraic group over $\overline{\mathbb{F}}_2$, and φ lies $\Gamma(G)$, then, using the classification of involutions ([1], [2]), it is possible to prove that φ fixes all the subgroups of order 2 of G. Hence, from the results we proved if G is not of type A_2, it follows that φ, in this case, fixes every subgroup generated by simisimple elements of G.

Finally I would like to mention the problem whether the projective image of a simple algebraic group G over $\overline{\mathbb{F}}_p$ is isomorphic (as an abstract group) to G. We prove that this is true if G is of adjoint type.

LEMMA 3.1 Let G be a finite simple Chavelley group over $\mathbb{F}_q$ (q a power of a prime p with $q > 3$) and suppose τ is an autoprojectivity of G fixing every p-Sylow subgroup of G. Then for every root α and every k in $\mathbb{F}_q$, τ fixes the subgroup $\langle x_\alpha(k)\rangle$ of G.

PROOF. Let α be root of G. We observe that τ fixes every parabolic subgroup of G, as τ, being index-preserving, fixes the normalizers of the p-Sylow subgroups of G. Hence we have $X_\beta^\tau = X_\beta$ for every root β, as

X_β can be expressed as the intersection of a parabolic subgroup and a p-Sylow subgroup of G. In particular τ fixes $\langle X_\alpha, X_{-\alpha}\rangle$. Let P be a-Sylow subgroup of $\langle X_\alpha, X_{-\alpha}\rangle$ and R a p-Sylow subgroup of G containing P. Then $P^\tau = P$, as $P = \langle X_\alpha, X_{-\alpha}\rangle \wedge R$. Therefore τ fixes every p-Sylow subgroup of $\langle X_\alpha, X_{-\alpha}\rangle$. By Metelli's result, if γ is an autoprojectivity of $PSL_2(F)$ (with $|F| > 3$), fixing every p-Sylow subgroup, then γ is the identity. If $\langle X_\alpha, X_{-\alpha}\rangle$ is isomorphic to $PSL_2(\mathbb{F}_q)$, then we are done. Otherwise $\langle X_\alpha, X_{-\alpha}\rangle$ is isomorphic to $SL_2(\mathbb{F}_q)$, and p is odd. But τ fixes the center Z of $\langle X_\alpha, X_{-\alpha}\rangle$ and therefore it induces an autoprojectivity δ of $\langle X_\alpha, X_{-\alpha}\rangle/Z$ fixing every p-Sylow subgroup, so that δ must be the identify. If u is any unipotent element of $\langle X_\alpha, X_{-\alpha}\rangle$, we get $(\langle u, Z\rangle/Z)^\delta =$ $= \langle u, Z\rangle$, so that $= \langle u, Z\rangle^\tau = \langle u, Z\rangle$. Hence $\langle u\rangle^\tau = \langle u\rangle$ as τ is index-preserving. In particular we get $\langle x_\alpha(k)\rangle^\tau = \langle x_\alpha(k)\rangle$. ■

THEOREM 3.2. If G is a simple algebraic group of adjoint type over $\overline{\mathbb{F}}_p$, and H is a group projective to G, then G and H are isomorphic.

PROOF. If G has rank 1, then the result follows from 2.1. So assume rank $G \geqslant 2$. We consider the family $(G_n)_{n \in \mathbb{N}}$ of subgroups of G we defined in § 1 of [8], so that $G_n \leqslant G_{n+1}$ for every n in $\mathbb{N}$ and G is the union of all G_n's. Without loss of generality we may suppose each G_n to be a finite simple Chevalley group over a field F_n with more than 3 elements. Let φ be a projectivity of G onto H. We put $H_n = G_n^\varphi$ for every n in $\mathbb{N}$. It is well known that H_n is simple and it follows from [18] that H_n and G_n are isomorphic. We choose for every n in $\mathbb{N}$ an isomorphism β_n of G_n onto H_n. Let γ_n be the autoprojectivity $\varphi\beta_n^{-1}$ of G_n. By proposition 2 in [22], there exists a unique automorphism δ_n of G_n such that γ_n and δ_n act in the same way on every p-Sylow subgroup of G_n. Therefore if we put $\alpha_n = \delta_n\beta_n$, α_n is the unique isomorphism of G_n onto H_n such that $\psi_n = \varphi\alpha_n^{-1}$ fixes every p-Sylow subgroup of G_n. Let n be in $\mathbb{N}$. By 3.1, for every root α and every k in F_{n+1}, ψ_{n+1} fixes the subgroup $\langle x_\alpha(k)\rangle$ of G_{n+1}. As $G_n = \langle x_\alpha(k) \mid \alpha$ is a root and k lies in $F_n\rangle$, ψ_{n+1} fixes G_n. By uniqueness it follows that the restriction of ψ_{n+1} to G_n coincides with ψ_n, so that the restriction of α_{n+1} to G_n coincides with α_n. We can then define a map α: $G \to H$ as follows. For x in G we put $x^\alpha = x^{\alpha_n}$ where n is any natural number such that x lies in G_n. α is an isomorphism. Therefore G and H are isomorphic. ■

THEOREM D. Let G in a simple algebraic group of adjoint type over $\overline{\mathbb{F}}_p$. It p is odd and G is not of type A_2, then G is strongly lattice determined.

PROOF. To prove that G is strongly lattice determined is equivalent to prove the following two things. Every autoprojectivity of G is induced by an automorphism, and if H is projective to G then G and H are isomorphic. This follows from A, B, C and 3.2. ∎

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