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On the Algebraic and Arithmetical Structure of Generalized Polynomial Algebras.

FRANZ HALTER-KOCH (*)

ABSTRACT - We introduce a new kind of polynomial rings in infinitely many indeterminates (called large polynomial rings). The large polynomial ring over a factorial or a Krull domain is itself factorial or a Krull domain. The algebra of polynomial functions on an abelian group turns out to be essentially a large polynomial ring.

Introduction.

The classical notion of a polynomial function permits far-reaching generalizations, see [3], Ch. IV, [7], [13] and only recently [12]. In this paper we deal with polynomial functions defined on a module over a commutative ring R with values in an R-algebra. These polynomial functions form a commutative ring, whose algebraic structure is determined by means of a new kind of formal polynomial rings (called *large polynomial rings*). These large polynomial rings have nice arithmetical properties: They are factorial resp. Krull domains if the base ring is a factorial resp. a Krull domain.

1. Large polynomials and power series.

Throughout this paper, let $I \neq \emptyset$ be a set, denote by $\mathcal{S}(I)$ the set of all finite subsets of I, and let $\leqslant$ be a total order on I. For $n \in \mathbb{N}_0$, we set

$$I^n_{\leqslant} = \{(i_1, \ldots, i_n) \in I^n \mid i_1 \leqslant i_2 \leqslant \ldots \leqslant i_n\};$$

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in particular, $I^0_{\leqslant}$ is the singleton consisting of the empty sequence.

Let R be a commutative ring (always with $1 \neq 0$) and $\boldsymbol{X} = (X_i)_{i \in I}$ a family of (algebraic independent) indeterminates over R. For a subset $J \subset I$, we set $\boldsymbol{X}_J = (X_i)_{i \in J}$. Let

$$R[\![\boldsymbol{X}]\!] = R[\![(X_i)_{i \in I}]\!] = R[\![\mathcal{F}(\boldsymbol{X})]\!]$$

be the total algebra of the free abelian monoid $\mathcal{F}(\boldsymbol{X})$ with basis $\boldsymbol{X}$; see [3], ch. III, § 2, no. 11, 12. We call $R[\![\boldsymbol{X}]\!]$ the *large power series ring in $\boldsymbol{X}$ over R*; it coincides with the ring A_1 investigated in [2] and with the ring $R[\![(X_i)_{i \in I}]\!]_3$ investigated in [9].

PROPOSITION 1. *Let R be a domain.*

i) *$R[\![\boldsymbol{X}]\!]$ is a domain.*

ii) *Suppose that all power series rings $R[\![X_1, \ldots X_m]\!]$ in finitely many indeterminates over R are factorial; then $R[\![\boldsymbol{X}]\!]$ is also factorial. In particilar, if R is a regular factorial ring, then $R[\![\boldsymbol{X}]\!]$ if factorial.*

iii) *If R is a Krull domain, then $R[\![\boldsymbol{X}]\!]$ is a Krull domain.*

PROOF. i) [3], ch. IV, § 4, no. 8 or [6] or [15].

ii) [2], [6] or [15].

iii) [9].

In [15] a more general class of rings is dealt with. ■

Every $f \in R[\![\boldsymbol{X}]\!]$ has a unique representation in the form

$$f = \sum_{P \in \mathcal{F}(\boldsymbol{X})} \lambda_P P = \sum_{n \geqslant 0} \sum_{(i_1, \ldots, i_n) \in I^n_{\leqslant}} \lambda_{i_1, \ldots, i_n} X_{i_1} \cdot \ldots \cdot X_{i_n}$$

with coefficients λ_P, $\lambda_{i_1, \ldots, i_n} \in R$; addition and multiplication in $R[\![\boldsymbol{X}]\!]$ are defined in the usual way. For f as above and $J \subset I$, we set

$$f_J = \sum_{n \geqslant 0} \sum_{(i_1, \ldots, i_n) \in J^n_{\leqslant}} \lambda_{i_1, \ldots, i_n} X_{i_1} \cdot \ldots \cdot X_{i_n} \in R[\![\boldsymbol{X}_J]\!],$$

and we define $\pi_J \colon R[\![\boldsymbol{X}]\!] \to R[\![\boldsymbol{X}_J]\!]$ by $\pi_J(f) = f_J$. For $J' \subset J \subset I$ we define $\pi_{J, J'} \colon R[\![\boldsymbol{X}_J]\!] \to R[\![\boldsymbol{X}_{J'}]\!]$ by

$$\pi_{J, J'}\left(\sum_{n \geqslant 0} \sum_{(i_1, \ldots, i_n) \in J^n_{\leqslant}} \lambda_{i_1, \ldots, i_n} X_{i_1} \cdot \ldots \cdot X_{i_n} \right) =$$

$$= \sum_{n \geqslant 0} \sum_{(i_1, \ldots, i_n) \in J'^n_{\leqslant}} \lambda_{i_1, \ldots, i_n} X_{i_1} \cdot \ldots \cdot X_{i_n} .$$

π_J and $\pi_{J,J'}$ are ring epimorphisms satisfying $\pi_{J,J'} \circ \pi_J = \pi_{J'}$ and $\pi_{J',J''} \circ \pi_{J,J'} = \pi_{J,J''}$ whenever $J'' \subset J' \subset J \subset I$. With the mappings $\pi_{J,J'}$, the system $(R[\![X_J]\!])_{J \in \mathcal{E}(I)}$ becomes a projective system of R-algebras, and

$$\pi = \varprojlim_{J \in \mathcal{E}(I)} \pi_J : R[\![X]\!] \to \varprojlim_{J \in \mathcal{E}(I)} R[\![X_J]\!]$$

is an isomorphism of R-algebras; if $f \in R[\![X]\!]$, then $\pi(f) = (f_J)_{J \in \mathcal{E}(I)}$.

If $\mathfrak{J} \subset \mathcal{E}(I)$ is cofinal, we identify

$$\varprojlim_{J \in \mathcal{E}(I)} R[\![X_J]\!] = \varprojlim_{J \in \mathfrak{J}} R[\![X_J]\!],$$

and we shall in the sequel simply write $\varprojlim$ to denote the inverse limit over $\mathcal{E}(I)$ or some cofinal subset.

The constructions performed so far suggest to endow $R[\![X]\!]$ with a topology as follows; for the topological concepts used in the sequel we refer to [4].

For every $J \in \mathcal{E}(I)$, we give $R[\![X_J]\!]$ the discrete topology. We endow $\varprojlim R[\![X_J]\!]$ with the topology of the projective limit and shift this topology to $R[\![X]\!]$ by means of π. This topology on $R[\![X]\!]$ (which makes π into a homeomorphism) will be called the *limit topology*; it is obviously different from the usual topology on power series rings, and it is discrete if I is finite.

The limit topology makes $R[\![X]\!]$ into a separated complete topological R-algebra. For $f \in R[\![X]\!]$ and $J \in \mathcal{E}(I)$, we set

$$\mathcal{U}_J(f) = \{ g \in R[\![X]\!] \mid g_J = f_J \};$$

then $\{\mathcal{U}_J(f) \mid J \in \mathcal{E}(I)\}$ is a fundamental system of neighbourhoods of f, and the family $(f_J)_{J \in \mathcal{E}(I)}$ converges to f in the limit topology.

As an R-module, $R[\![X]\!]$ is of the form

$$R[\![X]\!] = \prod_{d \geqslant 0} R[\![X]\!]_d, \qquad \text{where} \qquad R[\![X]\!]_d = \prod_{(i_1, \ldots, i_d) \in I^d_{\leqslant}} RX_{i_1} \cdot \ldots \cdot X_{i_d};$$

in particular, $R[\![X]\!]_0 = R$, and the elements of $R[\![X]\!]_d$ are of the form

$$\sum_{(i_1, \ldots, i_d) \in I^d_{\leqslant}} \lambda_{i_1, \ldots, i_d} X_{i_1} \cdot \ldots \cdot X_{i_d} \qquad \text{where} \qquad \lambda_{i_1, \ldots, i_d} \in R;$$

they are called *large forms of degree d*; if $f \in R[\![X]\!]_d$ and $g \in R[\![X]\!]_e$, then $fg \in R[\![X]\!]_{d+e}$.

The ring $R[\![X]\!]$ contains the usual polynomial ring $R[X]$, consisting of all elements

$$\sum_{P \in \mathcal{F}(X)} \lambda_P P = \sum_{n \geq 0} \sum_{(i_1, \dots, i_n) \in I^n_{\leqslant}} \lambda_{i_1, \dots, i_n} X_{i_1} \cdot \ldots \cdot X_{i_n} \in R[\![X]\!]$$

where only finitely many of the coefficients λ_P resp. $\lambda_{i_1, \dots, i_n}$ are different from zero.

The main purpose of this paper is to investigate the subring

$$R[\langle X \rangle] = R[\langle (X_i)_{i \in I} \rangle] = \coprod_{d \geq 0} R[\![X]\!]_d \subset R[\![X]\!]$$

consisting of all $f \in R[\![X]\!]$ of the form

$$f = \sum_{n=0}^{N} \sum_{(i_1, \dots, i_n) \in I^n_{\leqslant}} \lambda_{i_1, \dots, i_n} X_{i_1} \cdot \ldots \cdot X_{i_n}$$

for some $N \in \mathbb{N}_0$; we call $R[\langle X \rangle]$ the *large polynomial ring* and its elements *large polynomials* (*in* X *over* R). Any $f \in R[\langle X \rangle]$ has a unique representation in the form

$$f = \sum_{d \geq 0} f_d \,,$$

where $f_d \in R[\![X]\!]_d$ are large forms of degree d, and $f_d = 0$ for all but finitely many $d \geq 0$. As in the classical case, we call

$$\deg(f) = \sup \{ d \geq 0 \mid f_d \neq 0 \} \in \mathbb{N}_0 \cup \{ -\infty \}$$

the *degree* of f.

Clearly, an element $f \in R[\![X]\!]$ belongs to $R[\langle X \rangle]$ if and only if $f_J \in R[\![X_J]\!]$ for all $J \in \mathcal{E}(I)$ and $\sup \{ \deg(f_J) \mid J \in \mathcal{E}(I) \} < \infty$; in this case, $\deg(f) = \max \{ \deg(f_J) \mid J \in \mathcal{E}(I) \}$.

We introduce a more general class of polynomial rings, containing $R[X]$ and $R[\langle X \rangle]$ as special cases, as follows. Let $\aleph$ be an infinite cardinal, and let $R[\langle X \rangle]_\aleph$ be the set of all large polynomials

$$f = \sum_{(i_1, \dots, i_n) \in I^n_{\leqslant}} \lambda_{i_1, \dots, i_n} X_{i_1} \cdot \ldots \cdot X_{i_n}$$

for which

$$\operatorname{card} \{ (i_1, \dots, i_n) \in I^n_{\leqslant} \mid \lambda_{i_1, \dots, i_n} \neq 0 \} < \aleph \,;$$

large polynomials with this property will be calle *large* $\aleph$*-polynomials*. Clearly, $R[\langle X \rangle]_\aleph$ is a subring of $R[\langle X \rangle]$, $R[\langle X \rangle]_{\aleph_0} = R[X]$, and $\aleph \geq \max \{ \aleph_0, \operatorname{card}(I) \}$ implies $R[\langle X \rangle]_\aleph = R[\langle X \rangle]$.

We say that an indeterminate X_j *occurs* in a large polynomial

$f \in R[\langle X \rangle]$, if $f \notin R[\langle (X_i)_{i \in I \setminus \{j\}} \rangle]$ and we set

$$I_f = \{j \in I \mid X_j \text{ occours in } f\}.$$

For any $f \in R[\langle X \rangle]$, we have $f \in R[\langle X \rangle]_{\aleph}$ if and only if $\operatorname{card}(I_f) < < \aleph$.

PROPOSITION 2. *Let R be a domain.*

i) $R[\langle X \rangle]_{\aleph}$ *is a domain, and* $R[\langle X \rangle]_{\aleph}^{\times} = R^{\times}$.

ii) *If* $f, g \in R[\langle X \rangle]$ *and* $0 \neq fg \in R[\langle X \rangle]_{\aleph}$, *then* $f \in R[\langle X \rangle]_{\aleph}$ *and* $g \in R[\langle X \rangle]_{\aleph}$.

iii) *If* $f, g \in R[\langle X \rangle]$, *then* $\deg(fg) = \deg(f) + \deg(g)$.

PROOF. Since $R[\langle X \rangle]_{\aleph} \subset R[\![X]\!]$, it is a domain by Proposition 1; i) and iii) are proved as in the classical case, see [3], Ch. IV, § 9, no. 5. Since R is a domain, we have $I_{fg} = I_f \cup I_g$ for all $f, g \in R[\![X]\!] \setminus \{0\}$, which implies ii). ■

We endow $R[\langle X \rangle]$ with the subspace topology induced from the limit topology on $R[\![X]\!]$. If I is infinite, $R[\langle X \rangle]$ is not closed in $R[\![X]\!]$ and hence it is not complete. Its closure $\overline{R[\langle X \rangle]}$ consists of all $f \in R[\![X]\!]$ such that $f_J \in R[X_J]$ for all $J \in \mathcal{E}(I)$. The ring $\overline{R[\langle X \rangle]}$ coincides with the ring A_2 investigated in [2]; it was proved there, that this ring does not even satisfy the ascending chain condition for principal ideals (if I is infinite).

The ring $R[\langle X \rangle]$ has the following universal mapping property.

PROPOSITION 3. *Let* $\varphi: R \to S$ *be a homomorphism of commutative rings,* $(x_i)_{i \in I} \in S^{(I)}$, *and give S the discrete topology. Then there exists a unique continuous ring homomorphism* $\phi: R[\langle X \rangle] \to S$ *satisfying* $\phi | R = \varphi$ *and* $\phi(X_i) = x_i$ *for all* $i \in I$.

PROOF. Clearly there exists exactly one ring homomorphism φ^*: $R[X] \to S$ satisfying $\varphi^* | R = \varphi$ and $\varphi^*(X_i) = x_i$ for all $i \in I$. For any $f \in R[X]$, we have $(\varphi^*)^{-1}(\varphi^* f) \supset \mathfrak{U}_J(f)$, where $J = \{i \in I \mid x_i \neq 0\}$; hence φ^* is continuous and has a unique extension to a continuous homomorphism ϕ as asserted. ■

2. Arithmetical properties of the large polynomial ring.

In this section we shall prove that the large polynomial ring $R[\langle X \rangle]_{\aleph}$ is a factorial domain resp. a Krull domain if R is so (Theorems 1 and 2).

First we recall from [1] the notation of a *finite factorization domain* (FFD). An integral domain R is called an FFD, if every $a \in R \backslash (R^\times \cup \cup \{0\})$ is a product of irreducible elements of R and possesses (up to associates) only finitely many divisors in R. If R is an FFD, then every polynomial ring $R[X_1, \ldots, X_m]$ is an FFD by [1], Prop. 5.3; every Krull domain is an FFD by [10], Theorem 5.

PROPOSITION 4. *Let R be an* FFD *and $f \in R[\langle \boldsymbol{X} \rangle] \backslash R$. Then f is irreducible in $R[\langle \boldsymbol{X} \rangle]$ if and only if there exists some $J_0 \in \mathcal{E}(I)$ such that f_J is irreducible in $R[\boldsymbol{X}_J]$ for all $J \in \mathcal{E}(I)$ satisfying $J \supset J_0$.*

PROOF. If f is reducible in $R[\langle \boldsymbol{X} \rangle]$, then $f = gh$, where $g, h \in \in R\langle \boldsymbol{X} \rangle] \backslash R^\times$. This implies $f_J = g_J h_J$, and if J is sufficiently large; then $g_J, h_J \notin R^\times$. Therefore f_J is irreducible in $R[\boldsymbol{X}_J]$ for all sufficiently large $J \in \mathcal{E}(I)$.

For the converse, suppose that, for any $J_0 \in \mathcal{E}(I)$, there exists some $J \in \mathcal{E}(I)$ such that $J \supset J_0$ and f_J is reducible in $R[\boldsymbol{X}_J]$. We set $n = \deg(f) \in \mathbb{N}$, and we shall prove that f is reducible in $R[\langle \boldsymbol{X} \rangle]$. By assumption, the set

$$\mathfrak{J} = \{J \in \mathcal{E}(I) \mid \deg(f_J) = n\,,\ f_J \text{ is reducible in } R[\boldsymbol{X}_J]\}$$

is cofinal in $\mathcal{E}(I)$, and for $J \in \mathfrak{J}$ the set

$$E_J = \{\varphi \in R[\boldsymbol{X}_J] \backslash R^\times \mid \deg(\varphi) < n,\ \varphi \mid f_J\}$$

is not empty. If $J, J' \in \mathfrak{J}$, $J' \subset J$ and $\varphi \in E_J$, then there exists some $\psi \in \in R[\boldsymbol{X}_J] \backslash R$ such that $f_J = \varphi\psi$, and consequently $f_{J'} = \pi_{J,J'}(f_J) = = \pi_{J,J'}(\varphi)\pi_{J,J'}(\psi)$. This implies $\pi_{J,J'}(\varphi) \mid f_{J'}$ and $n = \deg(f_{J'}) = = \deg(\pi_{J,J'}(\varphi)) + \deg(\pi_{J,J'}(\psi)) \leq \deg(\varphi) + \deg(\psi) = \deg(f_J) = n$, whence $\deg(\pi_{J,J'}(\varphi)) = \deg(\varphi) < n$. If $\pi_{J,J'}(\varphi) \in R$, then $0 = \deg(\pi_{J,J'}(\varphi)) = = \deg(\varphi)$, which implies $\varphi \in R$ and thus $\pi_{J,J'}(\varphi) = \varphi \notin R^\times$. In any case, we obtain $\pi_{J,J'}(\varphi) \notin R^\times$ and therefore $\pi_{J,J'}(\varphi) \in E_{J'}$.

Now we consider the projective system $((E_J)_{J \in \mathfrak{J}}, (\pi_{J,J'} | E_J : E_J \to \to E_{J'})_{J' \subset J})$, and we assert that $\varprojlim E_J \neq \emptyset$.

Since $R[\boldsymbol{X}_J]$ is an FFD, the set

$$\overline{E}_J = \{\varphi R^\times \mid \varphi \in E_J\}$$

of classes of associates in E_J is finite. For $J, J' \in \mathfrak{J}$, $J' \subset J$, we define $\overline{\pi}_{J,J'} : \overline{E}_J \to \overline{E}_{J'}$, by $\overline{\pi}_{J,J'}(\varphi R^\times) = \pi_{J,J'}(\varphi) R^\times$; then we hawe $\varprojlim \overline{E}_J \neq \emptyset$. by [5], (Ch. III, § 7, no. 4, Ex. II). Let $(\varphi_J')_{J \in \mathfrak{J}}$ be a family of polynomials $\varphi_J' \in R[\boldsymbol{X}_J]$ such that $(\varphi_J' R^\times)_{J \in \mathfrak{J}} \in \varprojlim \overline{E}_J$, and fix some $J_0 \in \mathfrak{J}$. If $J \in \in \mathfrak{J}$, $J \supset J_0$, then $\pi_{J,J_0}(\varphi_J') = u_J \varphi_{J_0}'$ for some $u_J \in R^\times$, and we set $\varphi_J =$

$= u_J^{-1}\varphi'_J \in E_J$; this implies $\pi_{J,J_0}(\varphi_J) = \varphi_{J_0} = \varphi'_{J_0}$ for all $J \supset J_0$. If $J, J' \in \mathfrak{I}$, $J \supset J' \supset J_0$, then $\pi_{J,J'}(\varphi_J) = v\varphi'_{J'}$ for some $v \in R^\times$, and $\varphi_{J_0} = \pi_{J,J_0}(\varphi_J) = \pi_{J',J_0} \circ \pi_{J,J'}(\varphi_J) = \pi_{J',J_0}(v\varphi'_{J'}) = v\varphi_{J_0}$ implies $v = 1$; therefore $(\varphi_J)_{J\in\mathfrak{I}, J\supset J_0} \in \varprojlim E_J$.

If $\boldsymbol{\varphi} = (\varphi^{(J)})_{J\in\mathfrak{I}} \in \varprojlim E_J \subset \varprojlim R[\![X_J]\!]$ and $g = \pi^{-1}(\boldsymbol{\varphi}) \in R[\![X]\!]$, then $g_J = \varphi^{(J)} \in R[X_J]$ for all $J \in \mathfrak{I}$ and $\deg(g_J) = \deg(\varphi^{(J)}) < n$, which implies $g \in R[\langle X\rangle]$ and $\deg(g) < n$. If $g \in R$, then $\varphi^{(J)} = g_J = g$ for all $J \in \mathfrak{I}$, and consequently $g \notin R^\times$.

For any $J \in \mathfrak{I}$, we have $f_J = \varphi^{(J)}\psi^{(J)}$ for some polynomial $\psi^{(J)} \in R[X_J]$; this implies $\boldsymbol{\psi} = (\psi^{(J)})_{J\in\mathfrak{S}} \in \varprojlim R[X_J]$ and (as above) $h = \pi^{-1}(\boldsymbol{\psi}) \in R[\langle X\rangle]$. Since $f_J = g_J h_J = (gh)_J$ for all $J \in \mathfrak{I}$, we obtain $f = g\,h$. Since $g \notin R^\times$ and $\deg(g) < n$, f is reducible in $R[\langle X\rangle]$. ∎

Proposition 5. *Let R be a domain, $f \in R[\langle X\rangle]$ and suppose that there exists some $J_0 \in \mathcal{E}(I)$ such that, for any $J \in \mathcal{E}(I)$ satisfying $J \supset J_0$, f_J is a prime element of $R[X_J]$. Then f is a prime element of $R[\langle X\rangle]$.*

Proof. Suppose that $f \mid gh$ for some $g, h \in R[\langle X\rangle]$. For any $J \in \mathcal{E}(I)$, this implies $f_J \mid g_J h_J$, and if $J \supset J_0$, then either $f_J \mid g_J$ or $f_J \mid h_J$. We set

$$\mathfrak{I}' = \{J \in \mathcal{E}(I) \mid f_J \mid g_J\}, \qquad \mathfrak{I}'' = \{J \in \mathcal{E}(I) \mid f_J \mid h_J\},$$

and we obtain

$$\{J \in \mathcal{E}(I) \mid J \supset J_0\} \subset \mathfrak{I}' \cup \mathfrak{I}''$$

which implies that either $\mathfrak{I}'$ or $\mathfrak{I}''$ is cofinal in $\mathcal{E}(I)$. Without restriction, let $\mathfrak{I}'$ be cofinal in $\mathcal{E}(I)$. For $J \in \mathfrak{I}'$, there exists some polynomial $\varphi^{(J)} \in R[X_J]$ such that $g_J = f_J\varphi^{(J)}$. If $J, J' \in \mathfrak{I}'$ and $J \supset J'$, then $g_{J'} = \pi_{J,J'}(g_J) = \pi_{J,J'}(f_J)\pi_{J,J'}(\varphi^{(J)}) = f_{J'}\pi_{J,J'}(\varphi^{(J)}) = f_{J'}\varphi^{(J')}$ implies $\pi_{J,J'}(\varphi^{(J)}) = \varphi^{(J')}$ and hence $\boldsymbol{\varphi} = (\varphi^{(J)})_{J\in\mathfrak{I}'} \in \varprojlim R[\![X_J]\!]$. If $q = \pi^{-1}(\boldsymbol{\varphi})$, then $q_J = \varphi^{(J)} \in R[X_J]$ and $\deg(q_J) = \deg(\varphi^{(J)}) \leqslant \deg(g_J) \leqslant \deg(g)$ for all $J \in \mathfrak{I}'$; this implies $q \in R[\langle X\rangle]$. Since $g_J = f_J q_J = (fq)_J$ for all $J \in \mathfrak{I}'$, we obtain $g = fq$, whence $f \mid g$ in $R[\langle X\rangle]$. ∎

Next we adopt Gauss' Lemma for large polynomials. An element $f \in R[\langle X\rangle]$ is called *primitive*, if $f = \lambda f^*$ where $\lambda \in R$ and $f^* \in R[\langle X\rangle]$ implies $\lambda \in R^\times$ (i.e., 1 is a g.c.d. of all coefficients of f in R). Hence an element of R is primitive if and only if it lies in $R^\times$.

PROPOSITION 6. *Let* R *be an* FFD *and* $f \in R[\langle X\rangle]\setminus R$. *Then the following assertions are equivalent:*

a) f *is primitive.*

b) f_J *is primitive for some* $J \in \mathcal{E}(I)$.

c) There exists some $J_0 \in \mathcal{E}(I)$ *such that* f_J *is primitive for all* $J \supset J_0$.

PROOF. Obviolusly, $c) \Rightarrow b) \Rightarrow a)$. Now set

$$f = \sum_{P \in \mathcal{F}(X)} \lambda_P P \in R[\langle X\rangle]$$

and suppose that f is primitive, i.e., 1 is a g.c.d. of $\{\lambda_P \mid P \in \mathcal{F}(X)\}$. Since R is an FFD, there exists a finite subset $\mathcal{P} \subset \mathcal{F}(X)$ such that 1 is a g.c.d. of $\{\lambda_P \mid P \in \mathcal{P}\}$. If $J_0 \in \mathcal{E}(I)$ is such that $\mathcal{P} \subset \mathcal{F}(X_{J_0})$, then $\mathcal{P} \subset \mathcal{F}(X_J)$ for all $J \in \mathcal{E}(I)$ satisfying $J \supset J_0$; therefore 1 is a g.c.d. of $\{\lambda_P \mid P \in \mathcal{F}(X_J)\}$ for any such J, which means that

$$f_J = \sum_{P \in \mathcal{F}(X)} \lambda_P P \in R[X_J]$$

is primitive. ∎

PROPOSITION 7 (Gauss' lemma). *Let* R *be a factorial domain and* K *a quotient field of* R.

i) *If* $f, g \in R[\langle X\rangle]$ *are primitive, then* fg *is also primitive.*

ii) *If* $f \in R[\langle X\rangle]$ *is primitive,* $g \in K[\langle X\rangle]$ *and* $fg \in R[\langle X\rangle]$, *then already* $g \in R[\langle X\rangle]$.

PROOF. For classical polynomials $f \in K[X]$, we use the notation of the content $c(f)$ as in [8], § 8. Then we have $c(fg) = c(f)c(g)$ for all $f, g \in K[X]$; $f \in R[X]$ if and only if $c(f)$ is integral; $f \in R[X]$ is primitive if and only if $c(f) = 1$.

i) If $f, g \in R[\langle X\rangle]$ are primitive, then f_J, $g_J \in R[X_J]$ are primitive for some $J \in \mathcal{E}(I)$ by Proposition 6. Then $(fg)_J = f_J g_J$ is also primitive, and again Proposition 6 implies that fg is primitive.

ii) By Proposition 6, there exists some $J_0 \in \mathcal{E}(I)$ such that f_J is primitive for all $J \in \mathcal{E}(I)$ satisfying $J \supset J_0$. For such J, $c(f_J g_J) = c(g_J)$ is integral, since $f_J g_J = (fg)_J \in R[X_J]$; this implies $g_J \in R[X_J]$, and consequently $g \in R[\langle X\rangle]$. ∎

PROPOSITION 8 *Let* R *be a factorial domain,* K *a quotient field of* R *and* $f \in R[\langle X\rangle]_{\aleph} \setminus R$. *Then the following assertions are equivalent:*

a) f *is a prime element of* $R[\langle X\rangle]_{\aleph}$.

b) f *is irreducible in* $R[\langle X\rangle]_{\aleph}$.

c) f *is primitive and irreducible in* $K[\langle X\rangle]$.

PROOF. For finite I, this is classical; see [14], Ch. V, § 6.

a) $\Rightarrow$ *b)* is obvious.

b) $\Rightarrow$ *c)* If f is irreducible in $R[\langle X\rangle]_{\aleph}$, then f is irreducible in $R[\langle X\rangle]$ by Proposition 2, ii). By Proposition 4, there exists some $J_0 \in \mathcal{E}(I)$ such that f_J is irreducible in $R[X_J]$ and hence in $K[X_J]$ for all $J \in \mathcal{E}(I)$ satisfying $J \supset J_0$. Again by Proposition 4 it follows that f is irreducible in $K[\langle X\rangle]$. Being irreducible in $R[\langle X\rangle]$, f is primitive by definition.

c) $\Rightarrow$ *a)* By Propositions 4 and 6, there exists some $J_0 \in \mathcal{E}(I)$ such that f_J is primitive and irreducible in $K[X_J]$ for all $J \in \mathcal{E}(I)$ satisfying $J \supset J_0$. Hence f_J is a prime element in $R[X_J]$ for all such J, and Proposition 5 implies that f is a prime element of $R[\langle X\rangle]$. By Proposition 2, ii),

$$fR[\langle X\rangle] \cap R[\langle X\rangle]_{\aleph} = fR[\langle X\rangle]_{\aleph},$$

and hence f is also a prime element of $R[\langle X\rangle]_{\aleph}$. ∎

THEOREM 1. *Let* R *be a factorial domain and* K *a quotient field of* R. *Then* $R[\langle X\rangle]_{\aleph}$ *is a factorial domain; the prime elements of* $R[\langle X\rangle]_{\aleph}$ *are the primes of* R *and the primitive polynomials* $f \in R[\langle X\rangle]_{\aleph} \setminus R$ *which are irreducible in* $K[\langle X\rangle]$.

PROOF. If $f \in R[\langle X\rangle]_{\aleph} \setminus R$ is primitive and irreducible in $K[\langle X\rangle]$, then f is a prime element of $R[\langle X\rangle]_{\aleph}$ by Proposition 8. If $p \in R$ is a prime element of R, then $R[\langle X\rangle]_{\aleph} / pR[\langle X\rangle]_{\aleph}$ is a domain, and thus p is a prime element of $R[\langle X\rangle]_{\aleph}$.

We must prove that every $f \in R[\langle X\rangle]_{\aleph} \setminus (R^{\times} \cup \{0\})$ has a factorization $f = p_1 \cdot \ldots \cdot p_r f_1 \cdot \ldots \cdot f_s$, where $p_i \in R$ are prime elements and $f_j \in R[\langle X\rangle]_{\aleph} \setminus R$ are irreducible. For $f \in R$, this is obvious. Thus we may suppose that $f \in R[\langle X\rangle]_{\aleph} \setminus R$ and that the assertation is proved for all large polynomials of smaller degree. Clearly, $f = p_1 \cdot \ldots \cdot p_r f^*$, where $p_i \in R$ are primes of R and $f^* \in R[\langle X\rangle]_{\aleph} \setminus R$ is primitive. If f^* is irreducible, we are done; otherwise $f^* = f_1^* f_2^*$ where $f_i^* \in R[\langle X\rangle]_{\aleph} \setminus R$ and hence $\deg(f_i^*) < \deg(f^*) = \deg(f)$ $(i = 1, 2)$. Applying the induction hypothesis for f_i^*, the assertion follows. ∎

For the next result, we need a Lemma.

LEMMA 1. *Let $(R_\alpha)_{\alpha\in\Lambda}$ be a family of Krull domains contained in a field K, and set*

$$R = \bigcap_{\alpha\in\Lambda} R_\alpha .$$

Suppose that for every $0 \neq x \in R$ the set $\{\alpha \in \Lambda \mid x \notin R_\alpha^\times\}$ is finite. Then R is Krull domain.

PROOF. [9], Lemma 1.2. ■

THEOREM 2. *If R is a Krull domain, then $R[\langle X\rangle]_\aleph$ is also a Krull domain.*

PROOF. Let K be a quotient field of R and $(V_\alpha)_{\alpha\in\Lambda}$ a family of discrete valuation rings of K such that $R = \bigcap_{\alpha\in\Lambda} V_\alpha$ and, for each $0 \neq x \in R$, the set $\{\alpha \in \Lambda \mid x \notin V_\alpha^\times\}$ is finite. For $\alpha \in \Lambda$, set

$$N_\alpha = \{f \in V_\alpha[\langle X\rangle] \mid f \text{ is primitive}\}.$$

By Proposition 7, N_α is a multiplicatively closed subset of $V_\alpha[\langle X\rangle]$. By Theorem 1, the domains $K[\langle X\rangle]_\aleph$ and $V_\alpha[\langle X\rangle]$ are factorial and hence the localisations $V_\alpha[\langle X\rangle]_{N_\alpha}$ are also factorial. If $0 \neq f \in R[\langle X\rangle]$ then the set $\{\alpha \in \Lambda \mid x \notin N_\alpha^\times\}$ is finite; this implies $f \in (V_\alpha[\langle X\rangle]_{N_\alpha})^\times$ for all but finitely many $\alpha \in \Lambda$. By Lemma 1 it is sufficient to prove that

$$R[\langle X\rangle]_\aleph = K[\langle X\rangle]_\aleph \cap \bigcap_{\alpha\in\Lambda} V_\alpha[\langle X\rangle]_{N_\alpha} .$$

Obviously $R[\langle X\rangle]_\aleph$ is contained in $K[\langle X\rangle]_\aleph$ and in each $V_\alpha[\langle X\rangle]_{N_\alpha}$. If $\alpha \in \Lambda$ and $f \in K[\langle X\rangle] \cap V_\alpha[\langle X\rangle]_{N_\alpha}$, then there exists some $g_\alpha \in N_\alpha$ such that $fg_\alpha \in V_\alpha[\langle X\rangle]$. Since $g_\alpha \in V_\alpha[\langle X\rangle]$ is primitive, Proposition 7, ii) implies $f \in V_\alpha[\langle X\rangle]$. Thus we obtain

$$K[\langle X\rangle]_\aleph \cap \bigcap_{\alpha\in\Lambda} V_\alpha[\langle X\rangle]_{N_\alpha} \subset K[\langle X\rangle]_\aleph \cap \bigcap_{\alpha\in\Lambda} V_\alpha[\langle X\rangle] = R[\langle X\rangle]_\aleph . \quad ■$$

3. Polynomial functions on modules.

Throughout this section, let F be a commutative ring, R a commutative F-algebra and V an F-module. A mapping $p:V \to R$ is called a *homogeneous F-polynomial function of degree $d \in \mathbb{N}$*, if there exists an F-multilinear mapping $p^*: V^d \to R$ such that $p(x) = p^*(x, \ldots, x)$ for all $x \in V$. We denote by $\mathcal{P}_F(V, R)_d$ the set of all homogeneous F-polynomial functions $p: V \to R$ of degree $d \in \mathbb{N}$; $\mathcal{P}_F(V, R)_0$ denotes the set of all constant functions $p: V \to R$ which we call *homogeneous* **F-polynomial functions of degree* 0. For any $d \in \mathbb{N}_0$, $\mathcal{P}_F(V, R)_d$ is an R-module under pointwise addition and scalar multiplication. If $p \in \mathcal{P}_F(V, R)_d$, $x \in V$ and $t \in F$, then $p(tx) = t^d p(x)$.

It is usual to define polynomial functions with values in F-modules, see e.g. [3; Ch. IV, § 5, no. 9]. In this paper however, we are mainly interested in the polynomial algebra (with a pointwise multiplication), and therefore we restrict ourselves to polynomial functions taking values in an F-algebra.

PROPOSITION 9. *Let* $d, e \in \mathbb{N}_0$, $p \in \mathcal{P}_F(V, R)_d$ *and* $q \in \mathcal{P}_F(V, R)_e$ *be given. If* $pq: V \to R$ *is defined pointwise, i.e.* $(pq)(x) = p(x)q(x)$, *then* $pq \in \mathcal{P}_F(V, R)_{d+e}$.

PROOF. Let $p^*: V^d \to R$ and $q^*: V^e \to R$ be F-multilinear mappings such that $p(x) = p^*(x, \dots, x)$ and $q(x) = q^*(x, \dots, x)$ for all $x \in V$. If $r: V^{d+e} = V^d \times V^e \to R$ is defined by $r(x_1, \dots, x_d, y_1, \dots y_e) = p(x_1, \dots, x_d) q(y_1, \dots, y_e)$, then r is F-multilinear, and $(pq)(x) = r(x, \dots, x)$ for all $x \in V$. ■

A mapping $p: V \to R$ is called an *F-polynomial function*, if there exists some $d \in \mathbb{N}_0$ and homogeneous F-polynomial functions $p_0, \dots, p_d: V \to R$ such that $p(x) = p_0(x) + \dots + p_d(x)$ for all $x \in V$; p is called a *local F-polynomial function*, if $p|M: M \to R$ is an F-polynomial function for every finitely generated R-submodule M of V. We denote by $\mathcal{P}_F(V, R)$ the set of all F-polynomial functions and by $\bar{\mathcal{P}}_F(V, R)$ the set of all local F-polynomial functions $p: V \to R$. Obviously,

$$\mathcal{P}_F(V, R) \subset \bar{\mathcal{P}}_F(V, R) \subset R^V$$

are R-subalgebras if R^V is viewed as the R-algebra of all functions $f: V \to R$ under pointwise addition, multiplication and scalar multiplication.

On the algebra R^V we introduce a topology as follows. Denote by $\mathfrak{E}(V)$ the set of all finitely generated F-submodules of V. For $M \in \mathfrak{E}(V)$, define $\pi_M: R^V \to R^M$ by $\pi_M(f) = f|M$, and for $M, M' \in \mathfrak{E}(V)$, $M \supset M'$, define $\pi_{M,M'}: R^M \to R^{M'}$ by $\pi_{M,M'}(g) = g|M'$. With the mappings $\pi_{M,M'}$, the system $(R^M)_{M \in \mathfrak{E}(V)}$ becomes a projective system of R-algebras, and

$$\pi = \varprojlim_{M \in \mathfrak{E}(V)} \pi_M : R^V \to \varprojlim_{M \in \mathfrak{E}(V)} R^M$$

is an R-algebra isomorphism. If $f \in R^V$, then $\pi(f) = (f|M)_{M \in \mathfrak{E}(V)}$. For every $M \in \mathfrak{E}(V)$, we give R^M the discrete topology, and we shift the topology of the projective limit to R^V be means of the isomorphism π. This topology on R^V (obviously different from the product topology) will be called the *limit topology*.

With the limit topology, R^V is a separated complete topological R-algebra. For $f \in R^V$ and $M \in \mathfrak{E}(V)$, we set

$$\mathcal{U}_M(f) = \{g \in R^V \mid g|M = f|M\}.$$

Then $\{\mathcal{U}_M(f) | M \in \mathfrak{E}(V)\}$ is a fundamental system of neighbourhoods of f, and therefore the limit topology on R^V coincides with the topology of $\mathfrak{E}(V)$-convergence; see [4], Ch. X, § 1.

For the next result, let $\mathfrak{E}^+(V)$ be the set of all finitely generated F-submodules of V which are F-direct summands.

PROPOSITION 10. i) *$\overline{\mathcal{P}}_F(V, R)$ is closed in R^V, and*

$$\pi(\overline{\mathcal{P}}_F(V, R)) = \varprojlim_{M \in \mathfrak{E}(V)} \mathcal{P}_F(M, R) \subset \varprojlim_{M \in \mathfrak{E}(V)} R^M.$$

ii) *Let $M \in \mathcal{E}(V)$ be given and suppose that either $M \in \mathfrak{E}^+(V)$ or R is an injective F-module. Then the restriction map*

$$\rho: \begin{cases} \mathcal{P}_F(V, R) & \to & \mathcal{P}_F(M, R), \\ f & \mapsto & f|M, \end{cases}$$

is surjective.

iii) *Suppose that either $\mathfrak{E}^+(V)$ is cofinal in $\mathfrak{E}(V)$ or R is an injective F-module. Then*

$$\overline{\mathcal{P}}_F(V, R) = \overline{\mathcal{P}_F(V, R)} \subset R^V.$$

PROOF. i) A function $f \in R^V$ lies in $\overline{\mathcal{P}}_F(V, R)$ if and only if $f|M \in \mathcal{P}_F(M, R)$ for all $M \in \mathfrak{E}(V)$, i.e.,

$$\pi(f) = (f|M)_{M \in \mathfrak{E}(V)} \in \varprojlim_{M \in \mathfrak{E}(V)} \mathcal{P}_F(M, R).$$

This implies $\pi(\overline{\mathcal{P}}_F(V, R)) = \varprojlim \mathcal{P}_F(M, R)$. If $f \in \overline{\mathcal{P}_F(V, R)}$, then

$$\pi(f) \in \overline{\pi\mathcal{P}_F(V, R)} = \varprojlim_{M \in \mathfrak{E}(V)} \pi_M \mathcal{P}_F(V, R) \subset \varprojlim_{M \in \mathfrak{E}(V)} \mathcal{P}_F(M, R),$$

and consequently $f \in \overline{\mathcal{P}}_F(V, R)$. Hence $\overline{\mathcal{P}}_F(V, R)$ is closed in R^V.

ii) It is sufficient to prove that every homogeneous F-polynomial function $q{:}M \to R$ of degree $d \geq 1$ can be extended to an F-polynomial function $\tilde{q}\colon V \to R$. Let $q^*\colon M^d \to R$ be F-multilinear such that $q(x) = q^*(x, \ldots, x)$ for all $x \in M$. If either $M \in \mathfrak{S}^+(V)$ or R is F-injective, then there exists an F-multilinear mapping $\tilde{q}^*\colon V^d \to R$ such that $\tilde{q}^* \,|M^d = q^*$, and $\tilde{q}\colon V \to R$, defined by $\tilde{q}(x) = \tilde{q}^*(x, \ldots, x)$, is an F-polynomial function extending q.

iii) If $M \in \mathcal{S}^+(V)$ or R is F-injective, ii) implies

$$\mathcal{P}_F(M, R) = \pi_M \mathcal{P}_F(V, R) \subset \pi_M \overline{\mathcal{P}}_F(V, R) \subset \mathcal{P}_F(M, R),$$

whence equality holds. This implies

$$\overline{\pi\mathcal{P}_F(V, R)} = \varprojlim_{M \in \mathfrak{S}(V)} \mathcal{P}_F(M, R) = \pi\overline{\mathcal{P}}_F(V, R),$$

and consequently $\overline{\mathcal{P}_F(V, R)} = \overline{\mathcal{P}}_F(V, R)$. ■

Next we investigate the connection between F-polynomial functions and large polynomials; we start with the case of polynomials in a finite number of indeterminates.

We say that F *has no zero divisors on* R if

$$t \in F,\ x \in R,\ tx = 0 \ \text{implies}\ t = 0 \ \text{or}\ x = 0;$$

notice that this condition implies that F itself is a domain.

A polynomial $f \in R[X_1, \ldots, X_n]$ (in $n \in \mathbb{N}$ indeterminates) is called *q-reduced* (for some $q \in \mathbb{N}$), if $\deg_{X_j}(f) < q$ for all $j \in \{1, \ldots, n\}$.

LEMMA 2. *Suppose that F has no zero divisors on R, and let $f \in R[X_1, \ldots, X_n]$ be a polynomial.*

i) *If F is infinite and $f(x_1, \ldots, x_n) = 0$ for all $(x_1, \ldots, x_n) \in F^n$, then $f = 0$.*

ii) *If $\# F = q \in \mathbb{N}$, then there exists a unique q-reduced polynomial $f_0 \in R[X_1, \ldots, X_n]$ such that $f(x_1, \ldots, x_n) = f_0(x_1, \ldots, x_n)$ for all $(x_1, \ldots, x_n) \in F^n$.*

PROOF. Exactly as in the classical case; cf. [14], Ch. V, § 4. ■

Now let again $X = (X_i)_{i \in I}$ be a family of indeterminates, and adopt all notations of section 1.

THEOREM 3. *For*

$$f=\sum_{n\geqslant 0}\sum_{(i_1,\ldots,i_n)\in I^n_{\leqslant}}\lambda_{i_1,\ldots,i_n}X_{i_1}\cdot\ldots\cdot X_{i_n}\in\overline{R[\langle X\rangle]}$$

we define $f^F\colon F^{(I)}\to R$ *by*

$$f^F((x_i)_{i\in I})=\sum_{n\geqslant 0}\sum_{(i_1,\ldots,i_n)\in I^n_{\leqslant}}\lambda_{i_1,\ldots,i_n}x_{i_1}\cdot\ldots\cdot x_{i_n}\,.$$

i) $f\in\overline{R[\langle X\rangle]}$ *implies* $f^F\in\overline{\mathcal{P}}_F(F^{(I)},R)$, *and* $f\in R[\langle X\rangle]$ *implies* $f^F\in\mathcal{P}_F(F^{(I)},R)$.

ii) *The mapping*

$$\phi^F:\begin{cases}\overline{R[\langle X\rangle]} & \to\quad \overline{\mathcal{P}}_F(F^{(I)},R),\\ \quad f & \mapsto\quad f^F,\end{cases}$$

is a homomorphism of R-algebras satisfying

$$\phi^F(R[\langle X\rangle])=\mathcal{P}_F(F^{(I)},R)\,.$$

iii) *Suppose that F has no zero divisors on R; then*

$$\phi^F(\overline{R[\langle X\rangle]})=\overline{\mathcal{P}}_F(F^{(I)},R)\,.$$

If moreover F is infinite, then ϕ^F *is a topological isomorphism.*

PROOF. We set $V=F^{(I)}$, and we denote by $(\boldsymbol{e}_i)_{i\in I}$ the cannonical basis of V, i.e.,

$$\boldsymbol{x}=(x_i)_{i\in I}=\sum_{i\in I}x_i\boldsymbol{e}_i\qquad\text{for all}\qquad \boldsymbol{x}\in V.$$

For $J\subset I$, we set

$$V_J=\bigoplus_{i\in J}R\boldsymbol{e}_i\subset V.$$

If $J\in\mathcal{E}(I)$, then $V_J\in\mathfrak{E}^+(V)$, and the system $\{V_J\,|\,J\in\mathcal{E}(I)\}$ is cofinal in $\mathfrak{E}(V)$. Identifying V_J with $F^{(J)}$ we obtain, for any $f\in\overline{R[\langle X\rangle]}$,

$$f^F\,|\,V_J=(f_J)^F\,.$$

i) We show first that $f\in R[\langle X\rangle]$ implies $f^F\in\mathcal{P}_F(V,R)$ and it suffices to do this for large forms $f\in R[\![X]\!]_n$, where $n\in\mathbb{N}$. Suppose that

$$f=\sum_{(i_1,\ldots,i_n)\in I^n_{\leqslant}}\lambda_{i_1,\ldots,i_n}X_{i_1}\cdot\ldots\cdot X_{i_n}\in R[\![X]\!]_n\,,$$

and let $p^*: V^n \to R$ be the unique F-multilinear mapping satisfying

$$p^*(e_{i_1}, \dots, e_{i_n}) = \begin{cases} \lambda_{i_1, \dots, i_n} & \text{if } (i_1, \dots, i_n) \in I^n_{\leqslant}, \\ 0 & \text{otherwise}. \end{cases}$$

If $p \in \mathcal{P}_F(V, R)$ is defined by $p(x) = p^*(x, \dots, x)$, then

$$p((x_i)_{i\in I}) = p^*\left(\sum_{i\in I} x_i e_i, \dots, \sum_{i\in I} x_i e_i\right) =$$

$$= \sum_{(i_1, \dots, i_n)\in I^n_{\leqslant}} x_{i_1} \cdot \ldots \cdot x_{i_n} \lambda_{i_1, \dots, i_n} = f^F((x_i)_{i\in I}),$$

whence $f^F = p \in \mathcal{P}_F(V, R)$.

If $f \in \overline{R[\langle X \rangle]}$, then $f_J \in R[X_J]$ for all $J \in \mathcal{E}(I)$ and consequently $(f_J)^F = f^F | V_J \in \mathcal{P}_F(V_J, R)$, which implies $f^F \in \overline{\mathcal{P}_F(V, R)}$.

ii) Clearly, ϕ^F is a homomorphism of R-algebras. In order to prove the equality $\phi^F(R[\langle X \rangle]) = \mathcal{P}_F(V, R)$, it is sufficient to show that every homogeneous F-polynomial function $p: V \to R$ of degree $n \geqslant 1$ is of the form $p = f^F$ for some $f \in R[\![X]\!]_n$.

Let $p: V \to R$ be a homogeneous F-polynomial function, and let $p^*: V^n \to R$ be F-multilinear such that $p(x) = p^*(x, \dots, x)$ for all $x \in V$. For $(i_1, \dots, i_n) \in I^n$, we set

$$[i_1, \dots, i_n] = \{(i_{\sigma(1)}, \dots, i_{\sigma(n)}) | \sigma \in \mathcal{S}_n\},$$

and we define $f \in R[\![X]\!]_n$ by

$$f = \sum_{(i_1, \dots, i_n)\in I^n_{\leqslant}} \left(\sum_{(j_1, \dots, j_n)\in [i_1, \dots, i_n]} p^*(e_{j_1}, \dots, e_{j_n})\right) X_{i_1} \cdot \ldots \cdot X_{i_n}.$$

Then we obtain

$$f^F((x_i)_{i\in I}) = \sum_{(i_1, \dots, i_n)\in I^n} p^*(e_{i_1}, \dots, e_{i_n}) x_{i_1} \cdot \ldots \cdot x_{i_n} =$$

$$= p^*\left(\sum_{i=1}^{n} x_i e_i, \dots, \sum_{i=1}^{n} x_i e_i\right) = p((x_i)_{i\in I}),$$

whence $p = f^F$.

iii) For a large polynomial $f \in R[\langle X \rangle]$, we have $f^F = 0$ if an only if $0 = f^F | V_J = (f_J)^F$ for all $J \in \mathcal{E}(I)$. If F is infinite, this implies $f_J = 0$ for all $J \in \mathcal{E}(I)$ (by Lemma 2) and hence $f = 0$; therefore

$$\phi^F | R[\langle X \rangle]: R[\langle X \rangle] \xrightarrow{\sim} \mathcal{P}_F(V, R)$$

is an isomorphism and

$$(\phi^F \mid R[X_J]\colon R[X_J] \xrightarrow{\sim} \mathcal{P}_F(V_J, R))_{J \in \mathcal{E}(I)}$$

is a family of isomorphisms, compatible with the mappings $\pi_{J,J'}$ of the projective systems on either side. Taking projective limits and observing the commutative diagram

$$\begin{array}{ccc} \overline{R[\langle X \rangle]} & \xrightarrow{\phi^F} & \overline{\mathcal{P}}_F(V, R) \\ \pi \downarrow & & \downarrow \pi \\ \varprojlim R[X_J] & \xrightarrow{\sim} & \varprojlim \mathcal{P}_F(V_J, R), \end{array}$$

it follows from Proposition 9, iii) that ϕ^F is an isomorphism if F is infinite.

Now consider the case $\# F = q \in \mathbb{N}$. If $g \in \mathcal{P}_F(V, R)$ and $J \in \mathcal{E}(I)$, then $g \mid V_J \in \mathcal{P}_F(V_J, R)$, and by ii), there exists a polynomial $f^{(J)} \in R[X_J]$ such that $(f^{(J)})^F = g \mid V_J$. By Lemma 2, there exists a unique q-reduced polynomial $f_0^{(J)} \in R[X_J]$ such that $(f_0^{(J)}) = (f^{(J)})^F = g \mid V_J$. If $J, J' \in \mathcal{E}(I)$ and $J \supset J'$, then $\pi_{J,J'}(f_0^{(J)})$ is q-reduced, and $\pi_{J,J'}(f_0^{(J)})^F = (f_0^{(J)})^F \mid V_{J'} = g \mid V_{J'}$, whence $\pi_{J,J'}(f_0^{(J)}) = f_0^{(J')}$. This implies $(f_0^{(J)})_{J \in \mathcal{E}(I)} \in \varprojlim R[X_J]$, $f = \pi^{-1}((f_0^{(J)})_{J \in \mathcal{E}(I)}) \in \overline{R[\langle X \rangle]}$ and $f^F \mid V_J = (f_J)^F = (f_0^{(J)})^F = g \mid V_J$, whence $f^F = g$. ■

4. Polynomial functions on groups.

In this section we study ($\mathbb{Z}$-) polynomial functions and local ($\mathbb{Z}$-) polynomial functions $q\colon G \to R$, where G is an abelian group and R is a commutative ring containing a prime field F.

Let G be an (additively written) abelian group, F a prime field (i.e. $F = \mathbb{Q}$ or $F = \mathbb{F}_p$ for some prime number Hp) and R a commutative F-algebra. We shall be concerned with the R-algebras $\mathcal{P}(G, R) = \mathcal{P}_{\mathbb{Z}}(G, R)$ and $\overline{\mathcal{P}}(G, R) = \overline{\mathcal{P}}_{\mathbb{Z}}(G, R)$; we always write $\otimes$ instead of $\otimes_{\mathbb{Z}}$.

$F \otimes G$ is a vector space over F, and $F \otimes G = \{\lambda \otimes g \mid \lambda \in F,\ g \in G\}$. Let $\omega\colon G \to F \otimes G$ be the group homomorphism defined by

$$\omega(g) = 1 \otimes g\,.$$

Let $\mathcal{L}^n(G, R)$ resp. $\mathcal{L}_F^n(F \otimes G, R)$ be the R-module of all multiadditive functions $G^n \to R$ resp. F-multilinear functions $(F \otimes G)^n \to R$. For $p^* \in$

$\in \mathcal{L}_F^n(F \otimes G, R)$ we define $\omega^n(p^*) \in \mathcal{L}^n(G, R)$ by

$$\omega^n(p^*)(g_1, \ldots, g_n) = p^*(1 \otimes g_1, \ldots, 1 \otimes g_n).$$

Then we obtain the following Lemma.

LEMMA 3. *The mapping* $\omega^n\colon \mathcal{L}_F^n(F \otimes G, R) \to \mathcal{L}^n(G, R)$ *is an isomorphism of R-modules.*

PROOF. ω^n is R-linear by definition. Now we consider the canonical isomorphism

$$\mathcal{L}^n(G, R) \xrightarrow{\sim} \mathrm{Hom}\,(G \otimes \ldots \otimes G, R),$$

$$\mathcal{L}_F^n(F \otimes G, R) \xrightarrow{\sim} \mathrm{Hom}_F((F \otimes G) \bigotimes_F \ldots \bigotimes_F (F \otimes G), R)$$

and

$$(F \otimes G) \bigotimes_F \ldots \bigotimes_F (F \otimes G) \xrightarrow{\sim} F \otimes (G \otimes \ldots \otimes G);$$

they induce a commutative diagram

$$\begin{array}{ccc} \mathcal{L}_F^n(F \otimes G, R) & \xrightarrow{\sim} & \mathrm{Hom}_F(F \otimes G^{\otimes n}, R) \\ {\scriptstyle \omega^n}\downarrow & & \downarrow{\scriptstyle \omega^*} \\ \mathcal{L}^n(G, R) & \xrightarrow{\sim} & \mathrm{Hom}\,(G^{\otimes n}, R) \end{array}$$

where $\omega^*(\phi)(g_1 \otimes \ldots \otimes g_n) = \phi(1 \otimes g_1 \otimes \ldots \otimes g_n)$. By [11], Lemma 2, ω^* is an isomorphism (there R is assumed to be a field, but this is immaterial). Hence ω^* is also an isomorphism. ∎

THEOREM 4. *The mapping*

$$\omega^*\colon \begin{cases} \overline{\mathcal{P}}_F(F \otimes G, R) & \to \quad \overline{\mathcal{P}}(G, R), \\ \quad p & \mapsto \quad p \circ \omega, \end{cases}$$

is an isomorphism of K-algebras satisfying

$$\omega^*(\mathcal{P}_F(F \otimes G, R)) = \mathcal{P}(G, R).$$

PROOF. We prove first that $p \in \mathcal{P}_F(F \otimes G, R)$ implies $\omega^*(p) \in \mathcal{P}(G, R)$, and it is sufficient to do this in the case where $p\colon F \otimes G \to R$ is a homogeneous F-polynomial function of degree $n \geq 1$. In this case, let

$p^* \in \mathcal{L}_F^n(F \otimes G, R)$ be such that $p(z) = p^*(z, \ldots, z)$ for all $z \in F \otimes G$. Then we obtain, for $g \in G$,

$$\omega^*(p)(g) = p(1 \otimes g) = p^*(1 \otimes g, \ldots, 1 \otimes g) = (\omega^n p)(g, \ldots, g),$$

which implies $\omega^*(p) \in \mathcal{P}(G, R)$. Now we set

$$\tilde{\omega} = \omega^* \mid \mathcal{P}_F(F \otimes G, R)\colon \mathcal{P}_F(F \otimes G, R) \to \mathcal{P}(G, R),$$

and we prove that $\tilde{\omega}$ is an isomorphism of R-algebras.

In order to prove that $\tilde{\omega}$ is surjective it suffices to show that every homogeneous polynomial function $q\colon G \to R$ of degree $n \geqslant 1$ lies in the image of $\tilde{\omega}$. Let $q\colon G \to R$ be a homogeneous polynomial function of degree $n \geqslant 1$, and let $q^* \in \mathcal{L}^n(G, R)$ be such that $q(g) = q^*(g, \ldots, g)$ for all $g \in G$. By Lemma 3, $q^* = \omega^n(p^*)$ for some $p^* \in \mathcal{L}_F^n(F \otimes G, R)$. If $p\colon F \otimes G \to R$ is defined by $p(z) = p^*(z, \ldots, z)$ then

$$\omega^*(p)(g) = p^*(1 \otimes g, \ldots 1 \otimes g) = q^*(g, \ldots, g) = q(g)$$

for all $g \in G$, whence $\omega^*(p) = q$.

In order to prove that $\tilde{\omega}$ is injective, let $p \in \mathcal{P}_F(F \otimes G, R)$ be in the kernel of $\tilde{\omega}$, i.e., $p(1 \otimes g) = 0$ for all $g \in G$.

CASE 1. char $(R) = p > 0$, $F = \mathbb{F}_p$. In this case, all elements of $F \otimes G$ are of the form $z = \overline{m} \otimes g = 1 \otimes mg$ for some $m \in \mathbb{Z}$, which implies $p = 0$.

CASE 2. char $(R) = 0$, $F = \mathbb{Q}$. We write p in the form $p = p_1 + \ldots + p_d$, where $p_i\colon F \otimes G \to R$ is a homogeneous F-polynomial function of degree i. For $t \in \mathbb{Q}$ and $g \in G$, we obtain

$$p(t \otimes g) = \sum_{i=0}^{d} t^i p_i(1 \otimes g) \in R,$$

and if $t \in \mathbb{Z}$, then $p(t \otimes g) = p(1 \otimes tg) = 0$. Hence the polynomial

$$\sum_{i=0}^{d} p_i(1 \otimes g) T^i \in R[T]$$

vanishes on $\mathbb{Z}$ which, by Lemma 2, implies $p_i(1 \otimes g) = 0$ for all $i \in \{0, \ldots, d\}$ and $g \in G$. Therefore we obtain $p(t \otimes g) = 0$ for all $t \in \mathbb{Q}$ and $g \in G$, i.e., $p = 0$.

Now we consider local polynomial functions. Let $\mathfrak{E}(G)$ be the set of all finitely generated subgroups of G and $\mathfrak{E}(F \otimes G)$ the set of all finitely generated F-submodules of $F \otimes G$. Obviously, the set

$$\mathfrak{E}_0(F \otimes G) = \{F \otimes C \mid C \in \mathfrak{E}(G)\}$$

is cofinal in $\mathfrak{E}(F \otimes G)$ and therefore a function $p\colon F \otimes G \to R$ lies in $\overline{\mathcal{P}}_F(F \otimes G, R)$ if and only if $p|F \otimes C \in \mathcal{P}_F(F \otimes G, R)$ for all $C \in \mathfrak{E}(G)$. If $p \in \overline{\mathcal{P}}_F(F \otimes G, R)$, then $(p \circ \omega)|C = (p|F \otimes C) \circ (\omega|C) \in \mathcal{P}(C, R)$ for all $C \in \mathfrak{E}(G)$ which implies $\omega^*(p) = p \circ \omega \in \overline{\mathcal{P}}(G, R)$. For $C \in \mathfrak{E}(G)$, we have established an isomorphism

$$\tilde{\omega}_C\colon \mathcal{P}_F(F \otimes C, R) \xrightarrow{\sim} \mathcal{P}(C, R)$$

satisfying $\tilde{\omega}_C(p) = p \circ \omega$; the family $(\tilde{\omega}_C)_{C \in \mathfrak{E}(G)}$ is compatible with the morphisms of the projective system, and therefore we get a commutative diagram.

$$\begin{array}{ccc}
\overline{\mathcal{P}}_F(F \otimes G, R) & \xrightarrow{\omega^*} & \overline{\mathcal{P}}(G, R) \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\
\varprojlim\limits_{C \in \mathfrak{E}(G)} \mathcal{P}_F(F \otimes C, R) & \xrightarrow[\varprojlim \tilde{\omega}_C]{\sim} & \varprojlim\limits_{C \in \mathfrak{E}(G)} \mathcal{P}(C, R).
\end{array}$$

The left vertical arrow is an isomorphism by Proposition 9. Hence the right vertical arrow is surjective, and since it clearly is injective it is also an isomorphism. Therefore ω^* is an isomorphism. ■

COROLLARY. *Let K be a field of characteristic zero. Then $\overline{\mathcal{P}}(G, K)$ is a factorial domain.*

PROOF. By Theorem 4, $\mathcal{P}(G, K) \simeq \mathcal{P}_F(F \otimes G, K)$, where $F \simeq \mathbb{Q}$ is the prime field of K. If $F \otimes G = \{0\}$ then $\mathcal{P}_F(F \otimes G, K) \simeq K$; thus we suppose that $F \otimes G \simeq F^{(I)}$ for some set $I \neq \emptyset$. Then we obtain $\mathcal{P}_F(F \otimes G, K) \simeq \mathcal{P}_F(F^{(I)}, K) \simeq K[\langle X \rangle]$ by Theorem 3, and the latter ring is a factorial domain by Theorem 1. ■

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