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On the Equations of Ideal Incompressible Magneto-Hydrodynamics.

PAOLO SECCHI (*)

1. Introduction.

We consider the equations of motion of an ideal incompressible plasma both homogeneous and non-homogeneous, in a bounded domain Ω of $\boldsymbol{R}^n$, $n \geqslant 2$; here «ideal» means inviscid and non resistive. The equations of motion (in non dimensional form) in the non-homogeneous case are (see [8])

$$
\text{(NH)} \begin{cases}
\rho[\dot{u} + (u\cdot\nabla)u - f] - (B\cdot\nabla)B + \nabla\left(p + \frac{1}{2}|B|^2\right) = 0 \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{in } Q_T \equiv (0, T)\times\Omega\,, \\
\dot{B} + (u\cdot\nabla)B - (B\cdot\nabla)u = 0 \quad \text{in } Q_T, \\
\dot{\rho} + u\cdot\nabla\rho = 0 \quad \text{in } Q_T, \\
\operatorname{div} u = 0,\ \operatorname{div} B = 0 \quad \text{in } Q_T, \\
u\cdot\nu = 0,\ B\cdot\nu = 0 \quad \text{on } \Sigma_T \equiv (0, T)\times\Gamma, \\
u|_{t=0} = u_0,\ B|_{t=0} = B_0,\ \rho|_{t=0} = \rho_0 \quad \text{in } \Omega\,.
\end{cases}
$$

Here $u = u(t, x) = (u_1, \dots, u_n)$ is the plasma velocity, $B = B(t, x) = (B_1, \dots, B_n)$ the magnetic field, $p = p(t, x)$ the pressure, $\rho = \rho(t, x)$ the density; $f = f(t, x) = (f_1, \dots, f_n)$ is the given external force field, $\nu = \nu(x)$ denotes the unit outward normal to $\Gamma \equiv \partial\Omega$. The initial data u_0, B_0, ρ_0 are assumed to satisfy $\operatorname{div} u_0 = 0$, $\operatorname{div} B_0 = 0$ in Ω, $u_0\cdot\nu = 0$, $B_0\cdot\nu = 0$ on Γ and $0 < m_0 \leqslant \rho_0(x) \leqslant m_1$ in $\overline{\Omega}$. In the homogeneous case ($\rho(t, x) \equiv$ const > 0, say equal to one without loss of generality) the

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equations of motion become

$$\text{(H)}\qquad \begin{cases} \dot u + (u\cdot\nabla)u - (B\cdot\nabla)B + \nabla\left(p + \dfrac{1}{2}|B|^2\right) = f & \text{in } Q_T, \\ \dot B + (u\cdot\nabla)B - (B\cdot\nabla)u = 0 \quad \text{in } Q_T, \\ \operatorname{div} u = 0,\ \operatorname{div} B = 0 \quad \text{in } Q_T, \\ u\cdot\nu = 0,\ B\cdot\nu = 0 \quad \text{on } \Sigma_T, \\ u|_{t=0} = u_0,\ B|_{t=0} = B_0 \quad \text{in } \Omega\,. \end{cases}$$

As a particular case, if $B = 0$, (NH) contains the Euler equations for non-homogeneous incompressible flow and (H) the homogeneous ones. Moreover, there is an obvious structural analogy between (NH), (H) and the corresponding Euler equations. It is therefore natural to try to extend the known results for the Euler equations to (H) and (NH). The aim of the present paper is to show the existence and uniqueness to a solution of (H) and (NH), and the persistence property, namely that the solution at each time t belongs to the same function space X as does the initial state, and describes a continuous trajectory in X (see Theorems 2.1 and 2.4); we will also show the continuous dependence of the solutions on the data (see Theorems 2.2 and 2.5).

2. Notations and results.

Throughout the paper we assume Ω to be an open bounded subset of $\boldsymbol{R}^n$, $n \geq 2$, that lies (locally) on one side of its boundary Γ; the regularity of Γ will be indicated below. We set $\nabla = (D_1, \dots, D_n)$ where $D_i = \partial/\partial x_i$; $\dot u$ stands for the time derivative $\partial u/\partial t$ of u, $(v\cdot\nabla)u = \sum_{i=1}^{n} v_i D_i u$ where $v = (v_1, \dots, v_n)$. $\int$ denotes the integral over Ω. Let $p \in (1, \infty)$, k a positive integer; we denote by W^k the Sobolev space $W^{k,p}(\Omega)$ and by $\|\cdot\|_k$ its norm. If $p = 2$ we write H^k instead of W^k. For $p \in (1, \infty]$ we denote by L^p the space $L^p(\Omega)$ and by $|\cdot|_p$ its canonical norm.

We define $\overset{\circ}{W}{}^k$, $k \geq 1$, as the closure of $C_0^\infty(\Omega)$ in W^k and set $W_l^k \equiv \equiv W^k \cap \overset{\circ}{W}{}^l$ where $0 \leq l \leq k$. Clearly $W_0^k = W^k$, $W_k^k = \overset{\circ}{W}{}^k$. If $l \geq 1$, W_l^k is the subspace of W^k consisting of functions vanishing on Γ together with their derivatives up to order $l-1$. The above notations will be also used to denote functions spaces whose elements are vector fields, and analogously for their norms. The only exception is for particular vectors U of the form $U = (u, B, \rho)$ (respectively $U = (u, B)$) where $u = = (u_1, \dots, u_n)$, $B = (B_1, \dots, B_n)$; we will use the same symbol $\|U\|_k$ to denote the norm in W^k defined by $\|U\|_k = \|u\|_k + \|B\|_k + \|\rho\|_k$ (respectively

$\|U\|_k = \|u\|_k + \|B\|_k$). Observe that here $\|u\|_k$, $\|B\|_k$ are the norms in W^k of $\boldsymbol{R}^n$-valued vectors and $\|\rho\|_k$ is the norm in W^k of a scalar function. Given $T > 0$, we set $IT = [0, T]$. We denote by $C(IT; X)$, $L^\infty(IT; X)$, $L^1(IT; X)$ the function spaces of continuous, essentially bounded, summable functions on IT with values in the Banach space X. The norms of $L^\infty(IT; W^k)$, $L^1(IT; W^k)$ will be denoted by $\|\cdot\|_{T,k}$ and $|||\cdot|||_{T,k}$ respectively.

Given a positive definite and bounded matrix $A_0(t, x)$, i.e. $0 < a_0 I \leq \leq A_0(t, x) \leq a_1 I$ for any $(t, x) \in \overline{Q}_T$ and some positive numbers a_0, a_1, we will consider equivalent norms $\|U\|_{k,t}$ in H^k depending on t and defined by

$$\|U\|^2_{k,t} = \sum_{|\alpha| \leq k} \int (A_0(t, \cdot) D^\alpha U, D^\alpha U),$$

where $(\,,)$ denotes the scalar product in $\boldsymbol{R}^{2n+1}$ and $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index, $|\alpha| = \alpha_1 + \dots + \alpha_n$.

In the sequel c, c' denote different positive constants. The symbol $c(\Omega, n, p, k, m_0, m_1)$ means that c depends at most on the quantities inside brackets.

Let us state now our results. We consider first the simplest problem (H) of homogeneous flow. The first result concerns the local solvability.

THEOREM 2.1. *Let $n \geq 2$, $p > 1$, $k > 1 + n/p$. Assume that $\Gamma \in \in C^{k+2}$, $u_0 \in W^k$, $B_0 \in W^k$, $\operatorname{div} u_0 = \operatorname{div} B_0 = 0$ in Ω, $u_0 \cdot \nu = B_0 \cdot \nu = 0$ on Γ, $f \in L^1(IT_0; W^k)$. Then there exists a unique solution u, $B \in \in C(IT; W^k)$ of problem* (H) *on IT where $T = c(\Omega, n, p, k)(\|u_0\|_k + + \|B_0\|_k + |||f|||_{T,k})^{-1} < T_0$. Moreover $\|u\|_{T,k} + \|B\|_{T,k} \leq c'(\Omega, n, p, k) \cdot \cdot (\|u_0\|_k + \|B_0\|_k + |||f|||_{T,k})$.*

The next theorem concerns the continuous dependence of the solutions of (H) on the data. Consider sequences $\{u_0^{(m)}\}$, $\{B_0^{(m)}\}$, $\{f_m\}$ of functions satisfying the assumptions of Theorem 2.1, namely for any m

$$\text{(2.1)} \quad \begin{cases} u_0^{(m)} \in W^k, \quad B_0^{(m)} \in W^k, \quad f_m \in L^1(IT_0; W^k), \\ \operatorname{div} u_0^{(m)} = \operatorname{div} B_0^{(m)} = 0 \text{ in } \Omega, \quad u_0^{(m)} \cdot \nu = B_0^{(m)} \cdot \nu = 0 \text{ on } \Gamma. \end{cases}$$

Assume also that

$$\text{(2.2)} \quad \begin{cases} u_0^{(m)} \to u_0, \quad B_0^{(m)} \to B_0 \text{ in } W^k, \\ f_m \to f \text{ in } L^1(IT_0; W^k). \end{cases}$$

For any m, let us denote by $(\mathrm{H})_m$ the problem (H) with data u_0, B_0, f substituted by $u_0^{(m)}, B_0^{(m)}, f_m$. Theorem 2.1 guarantees the local existence of a solution u_m, B_m on some interval IT_m. From (2.2) it follows that $\|u_0^{(m)}\|_k + \|B_0^{(m)}\|_k + |||f_m|||_{T,k} \leqslant M$ for some constant M, uniformly in m. Hence from Theorem 2.1 we see that u_m, B_m exist on some common interval IT'.

THEOREM 2.2. *Let n, p, k, Γ be as in Theorem* 2.1; *moreover* $k \geqslant 3$. *Let u_0, B_0, f be as in Theorem* 2.1 and $u_0^{(m)}, B_0^{(m)}, f_m$ satisfy (2.1) *and* (2.2). *Let $u, B \in C(IT; W^k)$ be the solution of problem (H) with data u_0, B_0, f. Then, for m large enough, there exists a solution $u_m, B_m \in C(IT; W^k)$ of problem $(H)_m$ (with data $u_0^{(m)}, B_0^{(m)}, f_m$). Moreover*

$$u_m \to u,\ B_m \to B \quad in\ C(IT;\, W^k),$$

$$\nabla p_m \to \nabla p \quad in\ L^1(IT;\, W^k).$$

If $f_m \to f$ in $C(IT; W^k)$ (respect. in $L^q(IT; W^k)$, $q \in (1, \infty)$) then $\nabla p_m \to \nabla p$ in the same topology.

REMARK 2.3. The solution u, B exists in IT if T is small enough by Theorem 2.1. However, in Theorem 2.2, the existence interval IT can be arbitrarily large.

The results contained in the two previous theorems are not completely new. Existence and uniqueness in W^k-spaces as in Theorem 2.1 have been proved by Alekseev in [1], except that the solution is shown to be bounded in time with values in W^k but not continuous. The results of both Theorem 2.1 and 2.2 have been proved by Schmidt in [16] provided that $p = 2$. Moreover the continuous dependence on the data has been proved under the additional assumption of a dominated convergence of f_m to f, namely $f_m \to f$ in $L^1(IT_0; H^k)$ and $\|f_m(t)\|_k \leqslant \alpha(t)$ a.e. in IT for some $\alpha \in L^1(IT_0; \boldsymbol{R}_+)$.

Such results are obtained by Schmidt by adapting the methods of Temam [18] and of Kato and Lai [13] for the Euler equations of ideal fluid flow. In the present work we apply the abstract Kato's theory as done by Beirão da Veiga in [2], [3], [5], again for the Euler equations. By using a similar approach we can prove analogous results for the more difficult non-homogeneous problem (NH). As far as we know such a problem has never been considered in the literature. Our existence result is as follows.

THEOREM 2.4. *Let $n \geqslant 2$, $k > 1 + n/2$. Assume that $\Gamma \in C^{k+2}$, $u_0 \in H^k$, $B_0 \in H^k$, $\rho_0 \in H^k$, $\operatorname{div} u_0 = \operatorname{div} B_0 = 0$ in Ω, $u_0 \cdot \nu = B_0 \cdot \nu = 0$ on Γ,*

$0 < m_0 \leqslant \rho_0(x) \leqslant m_1$ in $\overline{\Omega}$, $f \in L^1(IT_0; H^k)$. Then there exists a positive constant ε_0 such that if

$$\|\nabla \rho_0\|_{k-1} < \varepsilon_0 \tag{2.3}$$

then problem (NH) *admits a unique solution $u, B, \rho \in C(IT; H^k)$ on IT where $T = c(\Omega, n, k, m_0, m_1)(\|u_0\|_k + \|B_0\|_k + \|\rho_0\|_k + |||f|||_{T,k})^{-1} < T_0$. Moreover $m_0 \leqslant \rho(t, x) \leqslant m_1$ for all $(t, x) \in \overline{Q}_T$ and*

$$\|u\|_{T,k} + \|B\|_{T,k} + \|\rho\|_{T,k} \leqslant$$
$$\leqslant c'(\Omega, n, k, m_0, m_1)(\|u_0\|_k + \|B_0\|_k + \|\rho_0\|_k + |||f|||_{T,k}).$$

The previous results is unsatisfactory not only because, due to the particular structure of (NH), we are forced to consider only the case $p = 2$, but especially because of condition (2.3). In fact, in analogy with the results for the non-homogeneous Euler equations (see [6], [7]), we would expect the existence of the solution without any restriction on the size of the gradient of the initial density. On the other hand, the extension of results known for the Euler equations to the ideal Magneto-Hydrodynamics doesn't always appear a simple matter; in this concern see [14], [17], [20].

Consider now sequences $\{u_0^{(m)}\}, \{B_0^{(m)}\}, \{\rho_0^{(m)}\}, \{f_m\}$ of functions satisfying the assumptions of Theorem 2.4, namely for any m

$$\begin{cases} u_0^{(m)} \in H^k, \quad B_0^{(m)} \in H^k, \quad \rho_0^{(m)} \in H^k, \quad f_m \in L^1(IT_0; H^k), \\ \operatorname{div} u_0^{(m)} = \operatorname{div} B_0^{(m)} = 0 \text{ in } \Omega, \quad u_0^{(m)} \cdot \nu = B_0^{(m)} \cdot \nu = 0 \text{ on } \Gamma, \\ m_0 \leqslant \rho_0^{(m)}(x) \leqslant m_1 \quad \text{in } \overline{\Omega}. \end{cases} \tag{2.4}$$

Assume also that

$$\begin{cases} u_0^{(m)} \to u_0, \quad B_0^{(m)} \to B_0, \quad \rho_0^{(m)} \to \rho_0 \text{ in } H^k, \\ f_m \to f \text{ in } L^1(IT_0; H^k). \end{cases} \tag{2.5}$$

For any m, let us denote by $(\text{NH})_m$ the problem (NH) with data $u_0^{(m)}$, $B_0^{(m)}$, $\rho_0^{(m)}$, f_m instead of u_0, B_0, ρ_0, f. Let $u_m, B_m, \rho_m \in C(IT_m; H^k)$ be the solution of $(\text{NH})_m$, defined on some interval IT_m. Observe that we don't assume (2.3) for $\rho_0^{(m)}$. Obviously, if $\rho_0^{(m)}$ satisfies (2.3), Theorem 2.4 guarantees the existence of u_m, B_m, ρ_m and from (2.5) we deduce the existence on some common interval IT'.

THEOREM 2.5. *Let n, k, Γ be as in Theorem 2.4; moreover $k \geqslant 3$. Let u_0, B_0, ρ_0, f be as in Theorem 2.3 and $u_0^{(m)}$, $B_0^{(m)}$, $\rho_0^{(m)}$, f_m satisfy* (2.4), (2.5). *Let $u, B, \rho \in C(IT; H^k)$ be the solution of problem* (NH) *with data* u_0, B_0, ρ_0, f. *Then, for m large enough, there exists a solution* $u_m, B_m, \rho_m \in C(IT; H^k)$ *of problem* $(\mathrm{NH})_m$ *with data* $u_0^{(m)}, B_0^{(m)}, \rho_0^{(m)}, f_m$. *Moreover*

$$u_m \to u,\ B_m \to B,\ \rho_m \to \rho \quad in\ C(IT; H^k),$$

$$\nabla p_m \to \nabla p \quad in\ L^1(IT; H^k).$$

If $f_m \to f$ *in* $C(IT; H^k)$ *(resp. in* $L^q(IT; H^k)$, $q \in (1, \infty)$*) then* $\nabla p_m \to \nabla p$ *in the same topology.*

REMARK 2.6. In the above theorem the existence interval IT can be arbitrarily large as well as $\|\nabla \rho_0\|_{k-1}$, $\|\nabla \rho_0^{(m)}\|_{k-1}$.

The plan of the paper is the following: in the next section we consider a linearized problem associated to (H) and give results for the corresponding evolution operator. In sections 4 and 5 we prove Theorems 2.1 and 2.2 respectively. In section 6 we study a linearized problem associated to (NH) and the corresponding evolution operator. The proofs of Theorems 2.4 and 2.5 are given in section 7 and 8 respectively.

3. The linearized problem associated to (H).

Let $v = (v_1, \dots, v_n)$ and $H = (H_1, \dots, H_n)$ be two vectors field; v and H are defined over $\overline{Q}_T$. Assume further that

$$v \cdot \nu = 0, \qquad H \cdot \nu = 0 \quad \text{on } \Sigma_T. \tag{3.1}$$

Let us consider the differential operator defined on vectors $U = (u, B)$, $u = (u_1, \dots, u_n)$, $B = (B_1, \dots, B_n)$,

$$\mathcal{A}(t)\, U = ((v \cdot \nabla) u - (H \cdot \nabla) B, (v \cdot \nabla) B - (H \cdot \nabla) u).$$

The operator $\mathcal{A}(t)$ is defined on W_l^k in the domain $\{U \in W_l^k : \mathcal{A}(t)\, U \in W_l^k\}$ for any fixed $t \in IT$ and for each couple of integers k, l such that $0 \leq l \leq k$, $1 \leq k$. The lower index l means that there are $l - 1$ boundary conditions. Let us consider the initial boundary value problem

$$\begin{cases} \dot{U} + \mathcal{A}(t)\, U = F & \text{in } Q_T, \\ U = \dots = D^{l-1} U = 0 & \text{on } \Sigma_T, \\ U|_{t=0} = U_0(x) & \text{on } \Omega, \end{cases} \tag{3.2}$$

where $F = F(t, x) = (f, g)$, $f = (f_1, f_2, f_3)$, $g = (g_1, g_2, g_3)$, and $U_0 =$

$= (u_0, B_0)$ are given. Here l is a fixed nonnegative integer (if $l = 0$, equation $(3.2)_2$ has to be dropped).

THEOREM 3.1. *Let $n \geqslant 2$, $p > 1$, $k > 1 + n/p$ and $\Gamma \in C^{k+2}$. Assume that $v, H \in L^\infty(IT; W^k) \cap C(IT; W^{k-1})$ and that* (3.1) *holds. Then for any $U_0 \in W^k$ and $F \in L^1(IT; W^k)$ the Cauchy problem* $(3.2)_1$, $(3.2)_3$ *has a unique strong solution $U \in C(IT; W^k)$. If $0 < l \leqslant k$ and if $0 < l \leqslant k$ and if $U_0 \in W_l^k$, $F \in L^1(IT; W_l^k)$ the above solution U belongs to $C(IT; W_l^k)$. Moreover*

$$\|U\|_{T,k-1} \leqslant 2(\|U_0\|_{k-1} + |||F|||_{T,k-1}) \exp(\theta_k T), \tag{3.3}$$

$$\|U\|_{T,k} \leqslant 2(\|U_0\|_k + |||F|||_{T,k}) \exp(\theta_k T), \tag{3.4}$$

where $\theta_k = c(\Omega, n, p, k)(\|v\|_{T,k} + \|H\|_{T,k})$.

PROOF. We set $V = (w, z)$, $w = u + B$, $z = u - B$ and define the operator

$$\mathcal{B}(t)V = (((v - H)\cdot\nabla)w, ((v + H)\cdot\nabla)z).$$

Then problem (3.2) is equivalent to

$$\begin{cases} \dot{V} + \mathcal{B}(t)V = G & \text{in } Q_T, \\ V = \dots = D^{l-1}V = 0 & \text{on } \Sigma_T, \\ V|_{t=0} = V_0(x) & \text{in } \Omega, \end{cases} \tag{3.5}$$

where $G = (f + g, f - g)$, $V_0 = (u_0 + B_0, u_0 - B_0)$. We solve (3.5) by applying Theorem 5.2 of [10] as in [2]; the proof is essentially the same. The crucial a priori estimates

$$\|V\|_h \leqslant (|\lambda| - \theta_k)\|G\|_h, \qquad \text{for any } h \text{ such that } 0 \leqslant h \leqslant k, \tag{3.6}$$

for the solution V of $\lambda V + \mathcal{B}V = G$, where $|\lambda| > \theta_k$, are obtained by multiplication of the equations for w and z separately by the corresponding suitable quantities; then the two estimates are summed to give the estimate for V. Estimate (3.6) is used to prove the $(1, \theta_k)$-stability of $\{\mathcal{B}(t)\}$ in W_l^h, $h = k - 1$ and k. The other assumtions of Theorem 5.2 of [10] are easily verified by adapting the method of [2]. Thus we can construct in W_l^k the evolution operator associated to $\{\mathcal{B}(t)\}$, satisfying the properties described in Theorem 5.2 and Remarks 5.3, 5.4 of [10]. Estimates (3.3) and (3.4) follow from the representation formula plus (a) in theorem 4.1 and (e) in theorem 5.1 of [10], respectively. The multiplicative factor 2 in (3.3), (3.4) follows from

$$\|V_0\| \equiv \|u_0 + B_0\| + \|u_0 - B_0\| \leqslant 2(\|u_0\| + \|B_0\|) \equiv 2\|U_0\|$$

and analogously for $|||G||| \leqslant 2|||F|||$. ■

With Theorem 3.1 at hand we will prove the local solvability of (H). The continuous dependence on the data will be shown by means of the following perturbation result. Consider v, H and sequences $\{v_m\}$, $\{H_m\}$ satisfying the assumptions of Theorem 3.1 for any m. Assume that

$$\|v_m\|_{T,k} \quad \text{and} \quad \|H_m\|_{T,k} \text{ are bounded uniformly in } m$$

and that

$$v_m \to v\,, \qquad H_m \to H \quad \text{in } C(IT;\, W^{k-1})\,.$$

By using the coefficients v_m instead of v, H_m instead of H we define operators $\mathcal{a}^{(m)}(t)$ and the associated evolution operators $W^{(m)}(t, s)$. Let us denote by $W_l^{k,(m)}(t, s)$ the evolution operator $W^{(m)}(t, s)$ when defined on W_l^k.

THEOREM 3.2. *Let $n \geqslant 2$, $p > 1$, $k > 1 + n/p$, $k \geqslant 3$. Under the above assumptions on the coefficients of $\mathcal{a}(t)$, $\mathcal{a}^{(m)}(t)$,*

$$W_l^{2,(m)}(t, s) \to W_l^2(t, s) \qquad \text{as } m \to \infty$$

strongly in $\mathcal{L}(W_l^2)$, uniformly in $(t, s) \in IT \times IT$. Here $l = 0$ or 1. $\mathcal{L}(W_l^2)$ denotes the space of linear and continuous operators from W_l^2 onto itself.

PROOF. The result is obtained by means of Theorem VI of [11]. Since the verification of the hypotheses of such theorem is essentially the same of [3], we omit the details. ■

4. Proof of Theorem 2.1.

We solve (H) by a fixed point argument. Consider

$$K = \{(u, B) \in L^\infty(IT;\, W^k) \cap C(IT;\, W^{k-1}) \colon u|_{t=0} = u_0,\, B|_{t=0} = B_0,$$
$$\operatorname{div} u = \operatorname{div} B = 0 \text{ in } Q_T,\; u \cdot \nu = B \cdot \nu = 0 \text{ on } \Sigma_T,$$
$$\|u\|_{T,k} + \|B\|_{T,k} \leqslant \alpha,\; \|u\|_{T,k-1} + \|B\|_{T,k-1} \leqslant \beta\}\,.$$

The constants T, α, β will be chosen later on. K is a convex, closed and bounded subset of $C(IT;\, W^{k-1})$. Given $(v, H) \in K$ we solve the Neumann problem

$$(4.1)\quad \begin{cases} -\Delta q = \sum_{i,j}(D_i v_j D_j v_i - D_i H_j D_j H_i) - \operatorname{div} f \equiv \varphi & \text{in } \Omega\,, \\ \dfrac{\partial q}{\partial \nu} = \sum_{i,j} D_i \nu_j (v_i v_j - H_i H_j) + f\cdot \nu \equiv \psi & \text{on } \Gamma\,. \end{cases}$$

The compatibility condition $\int \varphi = -\int_\Gamma \psi\, d\Gamma$ holds because $\varphi = \operatorname{div}[(v\cdot\nabla)v - (H\cdot\nabla)H - f]$ and $\psi = -[(v\cdot\nabla)v - (H\cdot\nabla)H - f]\cdot\nu$; observe that $v\cdot\nu = H\cdot\nu = 0$ on Γ implies $0 = (v\cdot\nabla)(v\cdot\nu) = (v\cdot\nabla)v\cdot\nu + (v\cdot\nabla)\nu\cdot v$ and similarly for the term in H. Standard estimates give

$$(4.2)\qquad |||\nabla q|||_{T,k} \leq c\alpha^2 T + c|||f|||_{T,k}\,.$$

We then consider the Cauchy problem

$$(4.3)\quad \begin{cases} \dot{u} + (v\cdot\nabla)u - (H\cdot\nabla)B = f - \nabla q & \text{in } Q_T\,, \\ \dot{B} + (v\cdot\nabla)B - (H\cdot\nabla)u = 0 & \text{in } Q_T\,, \\ u|_{t=0} = u_0\,,\ B|_{t=0} = B_0 & \text{in } \Omega\,. \end{cases}$$

Theorem 3.1 guarantees the existence and uniqueness of the solution $u, B \in C(IT; W^k)$. We introduce the Helmholtz decomposition of L^q, see ref. [9]. Denote by P the projection on the subspace of solenoidal and tangential on Γ vectors and set $Q = I - P$. The restrictions of P and Q are continuous from W^h into W^h, $h = k-1$ and k. Thus, by using (3.4) and (4.2), we have

$$\|Pu\|_{T,k} + \|PB\|_{T,k} \leq c_1(\|u_0\|_k + \|B_0\|_k + |||f|||_{T,k} + \alpha^2 T)\exp(c\alpha T)\,.$$

We choose $\alpha \equiv 4c_1(\|u_0\|_k + \|B_0\|_k)$. Then, for T sufficiently small, we prove that Pu, PB satisfy the first estimate required in K. Directly from (4.3) we obtain an estimate for $\dot{u}$, $\dot{B}$ in $L^1(IT; W^{k-1})$, by using (3.4), and this gives an estimate for u, B in $C(IT; W^{k-1})$ from which we derive the second estimate required in K, provided that T is sufficiently small, if β is a suitable constant multiplied by α. Hence the map Λ: $(v, H) \to (Pu, PB)$ satisfies $\Lambda(K) \subset K$. Similar calculations and the use of (3.3) yield that Λ is a contraction with respect to $C(IT; W^{k-1})$ provided that T is sufficiently small. Thus we obtain a fixed point $v = Pu$, $H = PB$. Finally we have to show that $u = Pu$, $B = PB$, namely $Qu = 0$, $QB = 0$. We observe that (4.1) implies $Q[(v\cdot\nabla)v - (H\cdot\nabla)H + \nabla q - f] = 0$. On the other hand, we have also $Q[(v\cdot\nabla)H - (H\cdot\nabla)v] = 0$ because $\operatorname{div} v = \operatorname{div} H = 0$ in Ω gives $\operatorname{div}[(v\cdot\nabla)H - (H\cdot\nabla)] = 0$ in Ω, and $v\cdot\nu = H\cdot\nu = 0$ on Γ gives $[(v\cdot\nabla)H - ((H\cdot\nabla)v]\cdot\nu = -\sum_{i,j} v_i H_j (D_i\nu_j - D_j\nu_i) = 0$ on Γ since $\nu = \pm\nabla\phi$, if ϕ is a map defined in a neighbourhood of Γ such that $\phi = 0$ describes Γ. We apply Q to $(4.3)_{1,2}$. Since $u = v + Qu$, $B = H + QB$ we obtain

$$Q\dot{u} + Q[(v\cdot\nabla)Qu - (H\cdot\nabla)QB] = 0\,,$$

$$Q\dot{B} + Q[(v\cdot\nabla)QB - (H\cdot\nabla)Qu] = 0\,.$$

Multiplying the two equations by Qu and QB respectively, integrating over Ω and adding the two equations easily give

$$\frac{1}{2}\,\frac{d}{dt}(\|Qu\|^2 + \|QB\|^2) = 0\,.$$

Since $Qu\,|_{t=0} = Qu_0 = 0$, $QB\,|_{t=0} = QB_0 = 0$ it follows $Qu(t) = QB(t) = 0$ for any $t\in[0, T]$. Then u, B solve problem (H) together with $p = q - (1/2)|B|^2$.

5. Proof of Theorem 2.2.

First of all we observe that, since $u_0^{(m)}$ and $B_0^{(m)}$ are uniformly bounded in W^k and f_m are uniformly bounded in $L^1(IT; W^k)$, from the results of theorem 2.1 it follows that the solutions u_m, B_m of $(\mathrm{H})_m$ exist on a common interval $IT'\subset IT$ and are uniformly bounded in $C(IT'; W^k)$. It readily follows from problems $(4.1)_m$ that ∇p_m are uniformly bounded in $L^1(IT'; W^k)$ and from $(\mathrm{H})_m$ that $\dot{u}_m$, $\dot{B}_m$ are uniformly bounded in $L^1(IT'; W^{k-1})$. By using these uniform bounds we can prove the equicontinuity of the u_m, B_m on IT' with values in W^{k-1}. Since the embedding of W^k into W^{k-1} is compact, by the Ascoli-Arzelà theorem we deduce that $\{u_m\}$, $\{B_m\}$ are relatively compact in $C(IT'; W^{k-1})$. Since limits of convergent sunsequences are solutions of (H) and since the solution of (H) is unique we obtain that $u_m\to u$, $B_m\to B$ in $C(IT'; W^{k-1})$. Consider now any space derivative D^α with $0\leq|\alpha|\leq k-2$. From (H)

$$\begin{cases} D^\alpha\dot{u} + (u\cdot\nabla)D^\alpha u - (B\cdot\nabla)D^\alpha B = \\ \qquad\qquad = F^\alpha(u, B) - D^\alpha(\nabla q) + D^\alpha f \equiv G_1^\alpha\,, \\ D^\alpha\dot{B} + (u\cdot\nabla)D^\alpha B - (B\cdot\nabla)D^\alpha u = G_2^\alpha\,, \end{cases} \tag{5.1}$$

where $q = p - (1/2)|B|^2$, $F^\alpha(u, B)\equiv(u\cdot\nabla)D^\alpha u - (B\cdot\nabla)D^\alpha B - D^\alpha(u\cdot\nabla)u + D^\alpha(B\cdot\nabla)B$, $G_2^\alpha\equiv(u\cdot\nabla)D^\alpha B - (B\cdot\nabla)D^\alpha u - D^\alpha(u\cdot\nabla)B + D^\alpha(B\cdot\nabla)u$. Observe that F^α and G_2^α contain derivative of u and B of order $k-2$

at most. A similar calculation is carried out for $(H)_m$. We easily prove that

$$\|F^\alpha(u, B)(t) - F^\alpha(u_m, B_m)(t)\|_2 \leq c(\|u(t) - u_m(t)\|_k + \|B(t) - B_m(t)\|_k),$$

and similarly for G_2^α. From the difference of problems (4.1), $(4.1)_m$ we obtain

$$\|D^\alpha \nabla(p - p_m)(t)\|_2 \leq c(\|u(t) - u_m(t)\|_k + \|B(t) - B_m(t)\|_k + \|f(t) - f_m(t)\|_k).$$

It then follows that a similar estimate holds for G_1^α. We set $U = (u, B)$ and denote by $W(t, s)$ the evolution operator generated by the family of operators $\{\mathcal{A}(t)\}$ in the space W^2. Similarly $W^{(m)}(t, s)$ is the evolution operator generated by $\{\mathcal{A}^{(m)}(t)\}$ in W^2. From Theorem 3.2 $W^{(m)}(t, s) \to$ $\to W(t, s)$ strongly in $\mathcal{L}(W^2)$, uniformly in $(t, s) \in IT' \times IT'$. From (5.1) we have

$$D^\alpha U(t) = W(t, 0) D^\alpha U_0 + \int_0^t W(t, s) G^\alpha(s)\, ds$$

and an analogous formula for $U_m = (u_m, B_m)$. By subtracting the two equations for $D^\alpha U$ and $D^\alpha U_m$, using the estimates for G_1^α, G_2^α and adding over α, $0 \leq |\alpha| \leq k - 2$, we obtain

$$\|U - U_m\|_{\tau, k} \leq \sum_\alpha \sup_{t \in [0, \tau]} \|(W(t, 0) - W^{(m)}(t, 0)) D^\alpha U_0\|_2 +$$

$$+ c\|U_0 - U_0^{(m)}\|_k + c \sum_\alpha \int_0^\tau \sup_{t \in [0, \tau]} \|(W(t, s) - W^{(m)}(t, s)) G^\alpha(s)\|_2\, ds +$$

$$+ c \int_0^\tau \|f(s) - f_m(s)\|_k\, ds,$$

for a sufficiently small positive value of τ depending only on Ω, n, p, k, M, T' (for the definition of M see after (2.2)). By using the result of theorem 3.2 and the dominated convergence theorem it readily follows that $U_m \to U$, namely $u_m \to u$ and $B_m \to B$ in $C(I\tau; W^k)$. By applying successively this result to the intervals $[i\tau, (i+1)\tau] \cap IT'$ we prove the convergence in all IT'. Theorem 3.1 and a standard continuation argument yield that u_m, B_m exist on IT if m is large enough. The repetition in IT of the above argument gives the convergence of u_m to u, B_m to B over all IT. The last assertion in Theorem 2.2 easily follows from (4.1), $(4.1)_m$.

6. The linearized problem associated to (NH).

Given two vectors fields v and H satisfying (3.1) and a scalar function σ, all defined over $\overline{Q}_T$, we introduce the differential operator $A(t)$ acting on vectors $U = (u, B, \rho)$, where $u = (u_1, \dots, u_n)$, $B = (B_1, \dots, B_n)$ and where ρ is a scalar function. $A(t)$ is defined for any $t \in IT$ by

$$A(t)\,U = \left((v\cdot\nabla)\,u - \left(\frac{H}{\sigma}\cdot\nabla\right)B, (v\cdot\nabla)\,B - (H\cdot\nabla)\,u,\; v\cdot\nabla\rho\right)$$

on the space H_l^k for any pair of integers k, l such that $0 \leq l \leq k$, $1 \leq k$. Let us consider the initial-boundary value problem

$$\text{(6.1)} \qquad \begin{cases} \dot{U} + A(t)\,U = F & \text{in } Q_T, \\ U = \dots = D^{l-1}\,U = 0 & \text{on } \Sigma_T, \\ U|_{t=0} = U_0 & \text{on } \Omega\,, \end{cases}$$

where $F = (F_1, \dots, F_7)$ and $U_0 = (u_0, B_0, \rho_0)$ are given. Here l is a fixed non-negative integer (if $l = 0$, equation $(6.1)_2$ has to be dropped). Moreover, let us define the $(2n + 1) \times (2n + 1)$ matrix

$$A_0 = A_0(t, x) \equiv \begin{pmatrix} \sigma I_n & 0 \\ 0 & I_{n+1} \end{pmatrix}$$

where I_m is the $m \times m$ identity matrix. Observe that we have $0 < a_0 I_{2n+1} \leq A_0 \leq a_1 I_{2n+1}$ where $a_0 = \min\{1, m_0\}$, $a_1 = \max\{1, m_1\}$. Correspondingly to A_0 we consider norms $\|U\|_{k,t}$ in H^k, $t \in IT$, defined in section 2. Norms $\|\cdot\|_{k,t}$ are equivalent to the usual $\|\cdot\|_k$ for $t \in IT$.

THEOREM 6.1. *Let $n \geq 2$, $k > 1 + n/2$ and $\Gamma \in C^{k+2}$. Assume that $v, H, \sigma \in L^\infty(IT; H^k) \cap C(IT; H^{k-1})$ and that v, H satisfy* (3.1). *Assume also that $0 < m_0 \leq \sigma(t, x) \leq m_1$ holds in $\overline{Q}_T$ and that $\dot{\sigma} \in L^\infty(IT; H^{k-1})$. Then for any $U_0 \in H^k$ and $F \in L^1(IT; H^k)$ the Cauchy problem* $(6.1)_1$, $(6.1)_3$ *has a unique strong solution $U \in C(IT; H^k)$. If $0 < l \leq k$ and if $U_0 \in H_l^k$, $F \in L^1(IT; H_l^k)$, the above solution U belongs to $C(IT; H_l^k)$. Moreover*

$$\text{(6.2)} \qquad \|U\|_{T,k-1} \leq k_0(\|U_0\|_{k-1} + |||F|||_{T,k-1})\exp(\theta_k T)\,,$$

$$\text{(6.3)} \qquad \|U\|_{T,k} \leq k_0(\|U_0\|_k + |||F|||_{T,k})\exp(\theta_k T)\,,$$

where $k_0 \equiv |A_{0|t=0}|_{\infty}^{1/2} \exp(c_0 \|\dot{\sigma}\|_{T,k-1} T)$, c_0 *depends only on* Ω, n, k *and*

$$\theta_k = c(\Omega, n, k)(\|v\|_{T,k} + \|H\|_{T,k} + \|\sigma\|_{T,k}).$$

PROOF. We solve (6.1) by applying Theorem I of [11]. We follow the method of [2] and prove that $A(t) \in G(W_l^h, 1, \theta_k)$ for $0 \leq l \leq h$, $1 \leq h$, where W_l^h has norm $\|\cdot\|_{h,t}$, for any fixed $t \in IT$. To illustrate how, by means of such norms, we can use the special structure of the operator $A(t)$, let us show for example how we prove the analogous of Lemma 3.5 of [2]: namely that any solution $U \in H^1$ of $\lambda U + A(t) U = F$ satisfies $(|\lambda| - \theta)\|U\|_{0,t} \leq \|F\|_{0,t}$ provided that $|\lambda| > \theta$, for a suitable $\theta > 0$. The equation for U is $(F = (f, g, h))$

$$\lambda u + (v \cdot \nabla) u - \left(\frac{H}{\sigma} \cdot \nabla\right) B = f,$$

$$\lambda B + (v \cdot \nabla) B - (H \cdot \nabla) u = g,$$

$$\lambda \rho + v \cdot \nabla \rho = h.$$

We multiply the equations by σu, B, ρ respectively and integrate over Ω. We then have, by integration by parts,

$$\lambda \int \sigma |u|^2 - \frac{1}{2} \int \operatorname{div}(\sigma v) |u|^2 - \int (H \cdot \nabla) B \cdot u = \int \sigma f \cdot u,$$

$$\lambda \int |B|^2 - \frac{1}{2} \int (\operatorname{div} v) |B|^2 - \int (H \cdot \nabla) u \cdot B = \int g \cdot B,$$

$$\lambda \int \rho^2 - \frac{1}{2} \int (\operatorname{div} v) \rho^2 = \int h \rho.$$

We add the three equations; the third term in the first and in the second equation cancel. We easily obtain

$$|\lambda| \|U\|_{0,t}^2 \leq \frac{1}{2}(|\operatorname{div}(\sigma v)|_\infty + |\operatorname{div} v|_\infty)\|U\|_{0,t}^2 + \|F\|_{0,t}\|U\|_{0,t}$$

which implies the required estimate. Then we show that the assumption $\dot{\sigma} \in L^\infty(IT; H^{k-1})$ implies the (k_0, θ_k)-stability by arguing as in Lemma 3.3 of [12] and Proposition 3.4 of [10].The rest of the proof is as in [2]. ■

Consider now v, H, σ and sequences $\{v_m\}, \{H_m\}, \{\sigma_m\}$, all functions satisfying the hypotheses of Theorem 6.1. Assume further that

$$\|v_m\|_{T,k}, \ \|H_m\|_{T,k}, \ \|\sigma_m\|_{T,k}, \ \|\dot{\sigma}_m\|_{T,k-1} \text{ are bounded uniformly in } m$$

and that

$$v_m \to v, \ H_m \to H, \ \sigma_m \to \sigma \quad \text{in } C(IT; H^{k-1}).$$

By using the coefficients v_m, H_m, σ_m instead of v, H, σ we define operators $A^{(m)}(t)$ and the associated evolution operators $W^{(m)}(t, s)$. Let us denote by $W_l^{k,(m)}(t, s)$ the evolution operator $W^{(m)}(t, s)$ when defined on H_l^k. By applying Theorem VI of [11] we prove as above the following perturbation result.

THEOREM 6.2. *Let $n \geq 2$, $k > 1 + n/2$. Under the above assumptions*

$$W_l^{2,(m)}(t, s) \to W_l^2(t, s) \quad \text{as } m \to \infty$$

strongly in $\mathcal{L}(H_l^2)$, uniformly in $(t, s) \in IT \times IT$. Here $l = 0$ or 1.

7. Proof of Theorem 2.4.

We solve (NH) by a fixed point argument. Set

$$\begin{aligned}
K' = \{&(u, B, \rho) \in L^\infty(IT; H^k) \cap C(IT; H^{k-1}) \colon \dot{u} \in L^1(IT; H^{k-1}),\\
&\dot{\rho} \in L^\infty(IT; H^{k-1}),\ u|_{t=0} = u_0,\ B|_{t=0} = B_0,\ \rho|_{t=0} = \rho_0,\\
&\operatorname{div} u = \operatorname{div} B = 0 \text{ in } Q_T,\ u \cdot \nu = B \cdot \nu = 0 \text{ on } \Sigma_T,\\
&m_0 \leq \rho(t, x) \leq m_1 \text{ in } \overline{Q}_T,\ \|u\|_{T,k} + \|B\|_{T,k} + \|\rho\|_{T,k} \leq \alpha,\\
&\|u\|_{T,k-1} + \|B\|_{T,k-1} \leq \beta,\ |||\dot{u}|||_{T,k-1} \leq \gamma,\ \|\dot{\rho}\|_{T,k-1} \leq \delta,\ \|\nabla\rho\|_{T,k-1} \leq \varepsilon\}.
\end{aligned}$$

K' is a convex, closed and bounded subset of $C(IT; H^{k-1})$. Given $(v, H, \sigma) \in K'$ we solve the Neumann problem

$$\text{(7.1)} \quad \begin{cases} -\Delta q = \nabla\sigma \cdot \dot{v} + \sum_{i,j} (D_i(\sigma v_j) D_j v_i - D_i H_j D_j H_i) - \operatorname{div}(\sigma f) \equiv \varphi & \text{in } \Omega, \\ \dfrac{\partial q}{\partial \nu} = \sum_{i,j} (\sigma v_i v_j - H_i H_j) D_i \nu_j + \sigma f \cdot \nu \equiv \psi & \text{on } \Gamma. \end{cases}$$

It is easily verified that the necessary compatibility condition $\int \varphi = -$

$-\int_\Gamma \psi \, d\Gamma$ holds. The solution $\nabla q \in L^1(IT; H^k)$ verifies

$$\|\nabla q\|_{T,k} \leqslant c(\varepsilon\gamma + \alpha^3 T + \alpha^2 T + \alpha\|f\|_{T,k}). \tag{7.2}$$

Consider now the Cauchy problem

$$\begin{cases} \dot{u} + (v\cdot\nabla)u - \left(\dfrac{H}{\sigma}\cdot\nabla\right)B = f - \dfrac{\nabla q}{\sigma} & \text{in } Q_T, \\ \dot{B} + (v\cdot\nabla)B - (H\cdot\nabla)u = 0 \quad \text{in } Q_T, \\ \dot{\rho} + v\cdot\nabla\rho = 0 \quad \text{in } Q_T, \\ u|_{t=0} = u_0,\ B|_{t=0} = B_0,\ \rho|_{t=0} = \rho_0 \quad \text{in } \Omega. \end{cases} \tag{7.3}$$

Theorem 6.1 guarantees the existence and uniqueness of the solution u, B, $\rho \in C(IT; H^k)$. We introduce the projection P for u, B. From (6.3) we then obtain

$$\begin{aligned} \|Pu\|_{T,k} + \|PB\|_{T,k} + \|\rho\|_{T,k},\ \|u\|_{T,k} + \|B\|_{T,k} + \|\rho\|_{T,k} \leqslant \\ \leqslant c_1[\|u_0\|_k + \|B_0\|_k + \|\rho_0\|_k + \|f\|_{T,k} + \\ + c_2(\alpha)(\varepsilon\gamma + \alpha^2 T + \alpha^3 T + \alpha\|f\|_{T,k})]\exp(c_3(\alpha+\delta)T), \end{aligned} \tag{7.4}$$

where $c_2(\alpha)$ is an increasing function of α depending also on m_0. From $(7.3)_{1,2}$ we have

$$\begin{aligned} \|P\dot{u}\|_{T,k-1} + \|P\dot{B}\|_{T,k-1},\ \|\dot{u}\|_{T,k-1} + \|\dot{B}\|_{T,k-1} \leqslant \\ \leqslant c_4(\alpha)(\|u\|_{T,k} + \|B\|_{T,k})T + \|f\|_{T,k} + c_4(\alpha)(\varepsilon\gamma + \alpha^3 T + \alpha^2 T + \alpha\|f\|_{T,k}). \end{aligned} \tag{7.5}$$

From $(7.3)_3$ we have

$$\|\dot{\rho}\|_{T,k-1} \leqslant c_5\alpha\|\rho\|_{T,k}. \tag{7.6}$$

Moreover, the first order space derivatives of ρ satisfy the system

$$D_i\dot{\rho} + v\cdot\nabla D_i\rho + D_i v\cdot\nabla\rho = 0 \quad \text{in } Q_T,$$

$$D_i\rho|_{t=0} = D_i\rho_0 \quad \text{in } \Omega,$$

where $i = 1, \dots, n$. Such kind of system has been studied in [2], [4]. The solution $\nabla\rho \in C(IT; H^{k-1})$ satisfies

$$\|\nabla\rho\|_{T,k-1} \leqslant \|\nabla\rho_0\|_{k-1}\exp(c_6\alpha T). \tag{7.7}$$

Fix now $\alpha \equiv 8c_1(\|u_0\|_k + \|B_0\|_k + m_1|\Omega|^{1/2} + 1)$, $\gamma = 4c_4(\alpha)$, $\delta = c_5\alpha^2$. Let $0 < \varepsilon < 1$ be such that $8c_1c_2(\alpha)\varepsilon\gamma \leqslant \alpha$, $c_4(\alpha)\varepsilon \leqslant 1/4$ and set $\varepsilon_0 = \varepsilon/2$. If

$\|\nabla\rho_0\|_{k-1} \leqslant \varepsilon_0$ and T is such that $c_6\alpha T \leqslant \log 2$ we have from (7.7) $\|\nabla\rho\|_{T,k-1} \leqslant \varepsilon$. Let now T be such that

$$\alpha T \leqslant 1, \qquad c_3(\alpha + \delta)T \leqslant \log 2,$$

$$4c_1[(1 + \alpha c_2(\alpha))|||f|||_{T,k} + c_2(\alpha)(1+\alpha)\alpha^2 T] \leqslant \alpha,$$

$$2(1 + \alpha c_4(\alpha))|||f|||_{T,k} + 2c_4(\alpha)(1+\alpha)\alpha^2 T \leqslant \gamma.$$

From (7.4) we have $\|Pu\|_{T,k} + \|PB\|_{T,k} + \|\rho\|_{T,k} \leqslant \alpha$ and from (7.5) $|||P\dot{u}|||_{T,k-1} + |||P\dot{B}|||_{T,k-1} \leqslant \gamma$; from (7.6) it follows $\|\dot{\rho}\|_{T,k-1} \leqslant \delta$. Finally, we have $\|Pu\|_{T,k-1} + \|PB\|_{T,k-1} \leqslant \beta \equiv c(\|u_0\|_k + \|B_0\|_k + \gamma)$ for a suitable c. Thus the map $\Lambda' : (v, H, \sigma) \to (Pu, PB, \rho)$ satisfies $\Lambda'(K') \subseteq K'$. Similar calculations and (6.2) yield that Λ' is a contraction with respect to $C(IT; H^{k-1})$ if T is sufficiently small. Thus there exists a fixed point $v = Pu$, $H = PB$, $\sigma = \rho$ of Λ'. Finally we have to show that $Qu = QB = 0$. This is easily obtained as in the proof of Theorem 2.1 by observing that (7.1) gives $Q[\rho\dot{v} + \rho(v\cdot\nabla)v - (H\cdot\nabla)H + \nabla q - \sigma f] = 0$. Thus u, B, ρ solves (NH) together with $p = q - (1/2)|B|^2$.

8. Proof of Theorem 2.5.

The proof is similar to the one of Theorem 2.2, where instead of Theorem 3.2 we use Theorem 6.2. We only remark that the difference of the gradients of the pressure is obtained from system

$$-\operatorname{div}\frac{\nabla q}{\rho} = \sum_{i,j}\left(D_i u_j D_j u_i - D_i\left(\frac{B_j}{\rho}\right)D_i B_j\right) - \operatorname{div} f \quad \text{in } \Omega,$$

$$\frac{\partial q}{\partial \nu} = \sum_{i,j}(\rho u_i u_j - B_i B_j)D_i \nu_j + \rho f\cdot\nu \quad \text{on } \Gamma,$$

and similar one for q_m in terms of u_m, B_m, ρ_m.

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