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On the Multiplicity of Holomorphic Maps and a Residue Formula.

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ABSTRACT - We obtain integral formulas for the multiplicity of a holomorphic map at an isolated zero of it. The proof is based on Stokes' theorem and a process of passing to a residue.

1. Introduction.

Let $U \subset \mathbb{C}^n$ be an open set, $0 \in U$ and $f = (f_1, \ldots, f_n)$: $U \to \mathbb{C}^n$ a holomorphic map with 0 an isolated zero of f. Then it is defined, by various equivalent ways, the multiplicity $\operatorname{mult}(f, 0)$ of f at 0; see [3, p. 667]. This multiplicity turns put to be the following integral

$$\operatorname{mult}(f, 0) = \int_{S_\varepsilon} \frac{\omega_n'(\bar{f}) \wedge \omega_n(f)}{|f|^{2n}} \tag{1}$$

where

$$\omega_n'(\bar{f}) = c(n) \det [\bar{f}_j, \overbrace{\bar{\partial}\bar{f}_j}^{n-1}] = c(n)(n-1)! \sum_{j=1}^{n} (-1)^{j-1} \bar{f}_j \bigwedge_{k \neq j} \bar{\partial}\bar{f}_k,$$

$$\omega_n(f) = \det [\overbrace{\partial f_j}^{n}] = n!\, \partial f_1 \wedge \ldots \wedge \partial f_n, \qquad |f|^2 = \sum_{j=1}^{n} |f_j|^2,$$

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$$S_\varepsilon = \{z \in \mathbb{C}^n\colon |z| = \varepsilon\} \quad \text{and} \quad c(n) = \frac{1}{n!}\frac{1}{(2\pi i)^n};$$

see [1, p. 20].

(In the above determinants j runs from $j = 1$ to $j = n$ forming the n rows; the integer above a column means that this column is to be repeated so many times as the integer indicates.)

If $n = 1$ then the above integral is reduced to $(1/2\pi i)\int_{S_\varepsilon}(f'/f)\,dz$ which by the residue theorem is equal to $\mathrm{Res}(f'/f, 0)$.

In this paper we generalize, in a sense, this situation (to the case $n > 1$) by writing integral (1) which is an integral of a $(2n-1)$-form as an integral of a differential form of lower degree, more precisely of $(2n-2p-1)$ degree for $1 \leqslant p \leqslant n-1$. In particular (by applying our formula for $p = n-1$) we express the multiplicity of f at 0 as a line integral. Let us also point out that there are certain choices that can be made in constructing these differential forms which give various formulas.

The $(2n-2p-1)$-dimensional cycles on which the integrals are taken lie on appropriately chosen analytic varieties which pass from 0 and which could be singular at 0.

As for the process of obtaining these formulas it is the classical process of passing to a residue after the use of Stokes' theorem (see [3, chap. 3]).

The arrangement of the paper is as follows: in Section 2 we state the formula, in Section 3 we give the proof of it and in Sections 4 and 5 we obtain some consequences of it in some special cases.

2. Statement of the result.

With notation as above let us consider a holomorphic map $h = (h_1, \ldots, h_p)\colon U \to \mathbb{C}^p$ so that

$$h_i = h_{i1} f_1 + \ldots + h_{in} f_n, \qquad i = 1, 2, \ldots, p$$

for some $h_{ij} \in O(U)$, $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant n$.

Let us also assume that $M =\colon \{z \in U\colon h(z) = 0\}$ is smooth near the points of S_ε and that M meets S_ε transversally so that $T_\varepsilon =\colon M \cap S_\varepsilon$ is a smooth $(2n-2p-1)$-dimensional manifold. Of course $0 \in M$ and 0 could be a singular point of M.

Let us define

$$A(z) = c(n-p)\det[h_{1j}, \ldots, h_{pj}, \bar{f}_j, \overbrace{\bar{\partial}\bar{f}_j}^{n-p-1}]$$

and

$$B(z) = \det\left[\frac{\partial \overline{h_1}}{\partial z_j}, \dots, \frac{\partial \overline{h_p}}{\partial z_j}, \overbrace{dz_j}^{n-p}\right] \Big/ |\nabla h|^2$$

where

$$|\nabla h|^2 = \sum_{1 \leq j_1 < \dots < j_p \leq n} \left| \frac{\partial(h_1, \dots, h_p)}{\partial(z_{j_1}, \dots z_{j_p})} \right|^2 .$$

With this notation we will prove the following

THEOREM 1. The multiplicity of f at 0 is given by the formula

$$\operatorname{mult}(f, 0) = \int_{z \in T_\varepsilon} \frac{\partial(f_1, \dots, f_n)}{\partial(z_1, \dots, z_n)} \frac{A(z) \wedge B(z)}{|f(z)|^{2(n-p)}} .$$

3. Proof of Theorem 1.

The idea of this proof is similar to the one in the proof of Theorem 2.1 of [4]; so we will give only the modifications which are needed to carry out the proof in this case and we will refer to [4] where more details can be found about some calculations.

We devide the proof into several steps.

STEP 1. Let

$$g_j = \frac{\bar{f}_j}{|f|^2} \quad \text{and} \quad s_j = \frac{1}{|h|^2} \sum_{i=1}^{p} h_{ij} \bar{h}_i , \qquad 1 \leq j \leq n .$$

Then $s_j(z)$ is defined for $z \in U - M$ and

$$\sum_{j=1}^{n} s_j f_j = 1 \quad \text{and} \quad \sum_{j=1}^{n} g_j f_j = 1 . \tag{1}$$

Let us set

$$\eta = -c(n) \sum_{l=0}^{n-2} \det[g_j, s_j, \overbrace{\bar{\partial} g_j}^{l}, \overbrace{\bar{\partial} s_j}^{n-l-2}] .$$

We claim that $\bar\partial\eta = \omega_n'(g_j) = \omega_n'(\bar f_j)/|f|^{2n}$ on $U - M$.

To prove this it suffices to work close to a point where $f_1 \neq 0$ and to write

$$\eta = -\frac{c(n)}{f_1}\sum_{l=0}^{n-2}\det \begin{array}{c} \begin{array}{cccc} & & \overbrace{}^{l} & \overbrace{}^{n-l-2} \end{array} \\ \begin{bmatrix} g_1f_1 & s_1f_1 & \bar\partial(g_1f_1) & \bar\partial(s_1f_1) \\ g_j & s_j & \bar\partial g_j & \bar\partial s_j \end{bmatrix}_{2\leq j\leq n} \end{array},$$

(in the last determinant j runs from $j = 2$ to $j = n$ forming the 2nd to n-th row of it).

Then, in view of (1), η can be written

$$\eta = -\frac{c(n)}{f_1}\sum_{l=0}^{n-2}\det \begin{array}{c} \begin{array}{cccc} & & \overbrace{}^{l} & \overbrace{}^{n-l-2} \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ g_j & s_j & \bar\partial g_j & \bar\partial s_j \end{bmatrix}_{2\leq j\leq n} \end{array},$$

which implies that $\bar\partial\eta = (1/f_1)\sum_{l=0}^{n-2}(X_{l+1} - X_l)$ where

$$X_l = c(n)\det[\,\overbrace{\bar\partial g_j}^{l},\ \overbrace{\bar\partial s_j}^{n-l-1}\,]_{2\leq j\leq n}\,.$$

Therefore $\bar\partial\eta = (1/f_1)(X_{n-1} - X_0)$.

But it is easy to calculate and find that $X_{n-1} = f_1\,\omega_n'(g_j)$ and $X_0 = 0$, which proves the claim.

Step 2. Let us start with the integral

$$\int_{S_\varepsilon} \frac{\omega_n'(\bar f)\wedge\omega_n(f)}{|f|^{2n}} = \operatorname{mult}(f, 0)$$

and write it as

$$\lim_{\delta\to 0}\int_{S_{\varepsilon,\delta}} \frac{\omega_n'(\bar f)\wedge\omega_n(f)}{|f|^{2n}}$$

where $S_{\varepsilon,\delta} = \{z \in S_\varepsilon\colon |h(z)| > \delta\}$.

But for $z \in S_{\varepsilon,\delta}$ we have (by Step 1)

$$d[\eta\wedge\omega_n(f)] = \frac{\omega_n'(\bar f)\wedge\omega_n(f)}{|f|^{2n}}\,.$$

Therefore the above limit can be written (also in view of Stokes' theorem) as

$$\lim_{\delta \to 0} \int_{T_{\varepsilon, \delta}} \eta \wedge \omega_n(f)$$

where $T_{\varepsilon, \delta} = \{z \in S_\varepsilon \colon |h(z)| = \delta\}$ (here ε and δ are small enough and so that the various sets over which we integrate are smooth).

STEP 3. We can write $\eta = \tilde{\eta} + \sum_{m=0}^{p-2} r_m$ where

$$\tilde{\eta} = -c(n) \det [g_j, s_j, \underbrace{\bar{\partial} g_j}^{n-p-1}, \underbrace{\bar{\partial} s_j}^{p-1}]$$

and

$$r_m = -c(n) \det [g_j, s_j, \underbrace{\bar{\partial} g_j}^{n-m-2}, \underbrace{\bar{\partial} s_j}^{m}]$$

provided that differential forms are restricted to $T_{\varepsilon, \delta}$.

This follows from the fact that on $T_{\varepsilon, \delta}$ we have $r_m = 0$ if $m \geqslant p$; see [4, p. 791].

STEP 4. With differential forms restricted to $T_{\varepsilon, \delta}$, $\tilde{\eta}$ can be written (up to a sign) as follows:

$$\tilde{\eta} = A \wedge \omega_p' \left(\frac{\bar{h}}{|h|^2} \right).$$

The proof of this is similar to the proof of Lemma 3 of [4].

STEP 5. $\lim_{\delta \to 0} \int_{T_{\varepsilon, \delta}} r_m \wedge \omega_n(f) = 0$ for $0 \leqslant m \leqslant p-2$.

This follows from an analogous estimate to the one in Lemma 4 of [4].

STEP 6. By Steps 2, 3 and 5 we obtain

$$\operatorname{mult}(f, 0) = \lim_{\delta \to 0} \int_{T_{\varepsilon, \delta}} \tilde{\eta} \wedge \omega_n(f).$$

But

$$\omega_n(f) = \frac{\partial(f_1, \ldots, f_n)}{\partial(z_1, \ldots, z_n)}\omega_n(z) = \frac{\partial(f_1, \ldots, f_n)}{\partial(z_1, \ldots, z_n)}B(z) \wedge \omega_p(h)$$

(up to a sign); this follows from Lemma 1 of [4].

Therefore (also by Step 4)

$$\lim_{\delta\to 0}\int_{T_{\varepsilon,\delta}} \bar{\eta} \wedge \omega_n(f) = \lim_{\delta\to 0}\int_{T_{\varepsilon,\delta}} \frac{\partial(f_1, \ldots, f_n)}{\partial(z_1, \ldots, z_n)} \frac{A \wedge B}{|f|^{2(n-p)}} \wedge \omega_p'\left(\frac{\bar{h}}{|h|^2}\right) \wedge \omega_p(h) =$$

$$= \int_{T_\varepsilon} \frac{\partial(f_1, \ldots, f_n)}{\partial(z_1, \ldots, z_n)} \frac{A \wedge B}{|f|^{2(n-p)}}$$

which completes the proof.

4. Parametrizing T_ε.

With the notation of Theorem 1 let us consider the case $p = n - 1$ in which case the integrand θ in the integral of Theorem 1 is a differential form of type $(1, 0)$.

Now suppose that ζ is a holomorphic map from a neighborhood $V \subset \mathbb{C}$ of 0 to U so that the map $\lambda \mapsto \zeta(\lambda) = (\zeta_1(\lambda), \ldots, \zeta_n(\lambda))$ is a parametrization of $M = \{z \in U: h_1(z) = \ldots = h_{n-1}(z) = 0\}$ with $\zeta(0) = 0$.

Let $\tau_\varepsilon = \left\{\lambda \in V: \sum_{j=1}^{n} |\zeta_j(\lambda)|^2 = \varepsilon^2\right\}$. Then $\zeta: \tau_\varepsilon \to T_\varepsilon$ and let $\deg \zeta$ denote the degree of ζ.

With this notation we have the following.

THEOREM 2. The multiplicity of f at 0 is given by the formula

$$\operatorname{mult}(f, 0) = 2\pi i (\deg \zeta) \operatorname{Res}\left(\frac{\zeta^*(\theta)}{d\lambda}, \lambda = 0\right)$$

where $\zeta^*(\theta)$ in the pull-back of θ via ζ.

PROOF. First let us mention that $\bar{\partial}\theta = 0$ (as a computation shows) whence $\bar{\partial}(\zeta^*(\theta)) = 0$; from this it follows that there is a holomorphic function in $V - \{0\}$, denoted by $\zeta^*(\theta)/d\lambda$ which has a pole at 0 so that $\zeta^*(\theta) = (\zeta^*(\theta)/d\lambda)\, d\lambda$.

Now, by Theorem 1, $\operatorname{mult}(f, 0) = \int_{T_\varepsilon} \theta$. But by [2, p. 253] we have:

$$\int_{T_\varepsilon} \theta = (\deg \zeta) \int_{\tau_\varepsilon} \zeta^*(\theta).$$

Also, by the residue theorem,

$$\int_{\tau_\varepsilon} \zeta^*(\theta) = 2\pi i \operatorname{Res}\left(\frac{\zeta^*(\theta)}{d\lambda}, \lambda = 0\right)$$

and the formula of the theorem follows.

REMARK. Of course an analogue of Theorem 2 can be proved in the case $1 \leqslant p \leqslant n - 2$ too.

5. A special case.

We will prove the following.

THEOREM 3. Let $U \subset \mathbb{C}^n$ be an open neighborhood of 0 and $f = (f_1, \ldots, f_n)\colon U \to \mathbb{C}^n$ a holomorphic map with 0 isolated zero of f. Suppose that 0 is an isolated singular point of $M = \{z \in U\colon f_1 = \ldots = f_p = 0\}$.

Then for $\varepsilon > 0$ sufficiently small we have

$$\operatorname{mult}(f, 0) = \int_{M \cap S_\varepsilon} \frac{\omega_m'(\bar{\phi}) \wedge \omega_m(\phi)}{|\phi|^{2m}}$$

where $m = n - p$ and $\phi = (f_{p+1}, \ldots, f_n)$.

PROOF. We will apply Theorem 1 with $h_1 = f_1, \ldots, h_p = f_p$. Then we may choose $h_{ij} = 1$ if $i = j$ and $h_{ij} = 0$ if $i \neq j$.

Then

$$
(1)\qquad A(z) = c(m)\det \begin{array}{c} \begin{array}{cc} \overbrace{\qquad\qquad\qquad}^{p} & \qquad \overbrace{\qquad\qquad\qquad}^{m-1} \end{array} \\ \begin{bmatrix} 1 & 0 & \cdots & 0 & \bar{f}_1 & \bar{\partial}\bar{f}_1 & \cdots & \bar{\partial}\bar{f}_1 \\ 0 & 1 & \cdots & 0 & \bar{f}_2 & \bar{\partial}\bar{f}_2 & \cdots & \bar{\partial}\bar{f}_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & \bar{f}_p & \bar{\partial}\bar{f}_p & \cdots & \bar{\partial}\bar{f}_p \\ 0 & 0 & \cdots & 0 & \bar{\phi}_1 & \bar{\partial}\bar{\phi}_1 & \cdots & \bar{\partial}\bar{\phi}_1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \bar{\phi}_m & \bar{\partial}\bar{\phi}_m & \cdots & \bar{\partial}\bar{\phi}_m \end{bmatrix} \end{array} =
$$

$$
= (-1)^p c(m)\det \begin{array}{c} \qquad\quad \overbrace{\qquad\qquad\qquad}^{n-1} \\ \begin{bmatrix} \phi_1 & \bar{\partial}\bar{\phi}_1 & \cdots & \bar{\partial}\bar{\phi}_1 \\ \vdots & \vdots & & \vdots \\ \phi_m & \bar{\partial}\bar{\phi}_m & \cdots & \bar{\partial}\bar{\phi}_m \end{bmatrix} \end{array} = (-1)^p \omega'_m(\bar{\phi}).
$$

On the other hand

$$
(2)\qquad \frac{\partial(f_1, \ldots, f_n)}{\partial(z_1, \ldots, z_n)} B(z) = (-1)^p \omega_m(\phi)
$$

provided that differential forms are restricted to M. This follows from [5, p. 483] and the representation of $B(z)$ in terms of local coordinates as is given by [4, Lemma 2]. Now (1) and (2) and Theorem 1 complete the proof.

EXAMPLES. 1) Let $f\colon \mathbb{C}^3 \to \mathbb{C}^3$ be the map

$$
f(z_1, z_2, z_3) = (z_1^2 - z_2^3,\ z_2^5 - z_3^7,\ z_1^2 z_2 z_3 - z_1^3 - z_2 z_3^2).
$$

Consider the map $\zeta\colon \mathbb{C} \to \mathbb{C}^3$ defined by $\zeta(\lambda) = (\lambda^{21}, \lambda^{14}, \lambda^{10})$. If $T_\varepsilon = \{z \in S_\varepsilon\colon z_1^2 = z_2^3,\ z_2^5 = z_3^7\}$ and $\tau_\varepsilon = \{\lambda \in \mathbb{C}\colon |\lambda|^{42} + |\lambda|^{28} + |\lambda|^{20} = \varepsilon^2\}$ then $\zeta\colon \tau_\varepsilon \to T_\varepsilon$ and $\deg \zeta = 1$.

Thus combining Theorems 1 and 2 we obtain that

$$
\operatorname{mult}(f, 0) = \operatorname{Res}\left[\frac{1}{d\lambda}\zeta^*\left(\frac{df_3}{f_3}\right), \lambda = 0\right] = 34.
$$

2) Let $g\colon \mathbb{C}^2 \to \mathbb{C}$ be the function

$$g(z_1, z_2) = \frac{1}{3} z_1^3 - z_1 z_2^3 + z_2^5$$

which defines the hypersurfaces $S = \{(z_1, z_2) \in \mathbb{C}^2 : g(z_1, z_2) = 0\}$ whose 0 is an isolated singular point.

Then the Milnor number of S at 0 is the multiplicity of the map $f = (\partial g/\partial z_1, \partial g/\partial z_2)$ at 0 (see [6]).

Now $f(z_1, z_2) = (z_1^2 - z_2^3, \; -3 z_1 z_2^2 + 5 z_2^4)$. Hence in the setting of Theorem 2, if we set $z_1 = \lambda^3$, $z_2 = \lambda^2$ we compute (using also Theorem 3) that the Milnor number of S at 0 is 7.

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