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M. Sabatini

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Bifurcation of Periodic Solutions from Inversion of Stability of Periodic O.D.E.'S.

M. SABATINI (*)

ABSTRACT - Bifurcation of periodic solutions as a consequence of a sudden inversion of stability of a given one is proved via Lefschetz fixed point theorem. The results hold in odd-dimensional spaces for periodic systems, and in even-dimensional spaces for autonomous ones.

1. Introduction.

In this paper we consider one-parameter families of periodic ordinary differential systems

$$(E_\lambda) \qquad x' = f(\lambda, t, x) \equiv f(\lambda, t + T, x),$$

where $\lambda \in [0, \lambda^\#)$, T is a positive constant, $f \in \mathcal{C}^1([0, \lambda^\#) \times R \times U, R^n)$, U is an open subset of R^n. We assume that for any $\lambda \in [0, \lambda^\#)$, E_λ has a periodic solution $u(t)$. We are interested in determining sufficient conditions for the existence of periodic solutions of E_λ bifurcating from $u(t)$ as λ crosses some special value. We follow the approach used in [6] to prove the existence of asymptotically stable sets bifurcating from a given invariant set after a sudden change of stability, together with Lefschetz fixed point theorem for polyhedra [9]. The main result presented here is the following.

THEOREM. *Let n be odd and $u(t)$ be a periodic solution of E_λ, for $\lambda \in [0, \lambda^\#)$. If $u(t)$ is asymptotically stable for $\lambda = 0$ and negatively*

(*) Indirizzo dell'A.: Dipartimento di Matematica, Università di Trento, I-38050 Povo (TN), Italy.

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asymptotically stable for $\lambda > 0$, then $\lambda = 0$ is a point of bifurcation from $u(t)$ for a family of periodic solutions.

A result similar to the above theorem holds also when, for $\lambda = 0$, there exist even-dimensional submanifolds D_λ of R^{n+1} containing $u(t)$, that are locally invariant with respect to the systems E_λ. This is the case when an odd number of the characteristic multipliers of $u(t)$ leave simultaneously the unit circle as λ becomes positive, while the other ones stay inside the open unit circle for any $\lambda \in [0, \lambda^{\#})$. A similar situation has been considered by several authors for the case of an even number of characteristic multipliers crossing the imaginary axis (see [3], and [2] for more recent results). As a consequence of the above theorem, we obtain an analogous statement for autonomous systems in even-dimensional spaces. In this case we assume that $f \in \mathcal{C}^2([0, \lambda^{\#}) \times \times U, R^n)$:

COROLLARY. *Let n be even, E_λ autonomous and $u(t)$ a non-trivial periodic solution of E_λ, for $\lambda \in [0, \lambda^{\#})$. If $u(t)$ is orbitally asymptotically stable for $\lambda = 0$ and orbitally negatively asymptotically stable for $\lambda > 0$, then $\lambda = 0$ is a point of bifurcation from $u(t)$ for a family of non-trivial periodic solutions.*

Also in this case, a more general statement holds in presence of invariant manifolds of even dimension containing $u(t)$. For autonomous systems, the bifurcating periodic solutions do not have, in general, the same period as $u(t)$. However, the method used here does not seem to be useful to prove the existence of period-doubling bifurcation.

In a forthcoming paper, part of the results presented here will be extended to the study of bifurcation from infinity. The author wishes to thank Professors Loud and Sell of the University of Minnesota for many useful conversations.

2. Definitions and results.

We assume that, for any $\lambda \in [0, \lambda^{\#})$, the differential system

$$(S_\lambda) \qquad \begin{cases} \dot{x} = f(\lambda, t, x) \equiv f(\lambda, t + T, x), \\ \dot{t} = 1, \end{cases}$$

where $f \in \mathcal{C}^2([0, \lambda^{\#}) \times R \times U, R^n)$, U open subset of R^n, defines a dynamical system on $R \times U$. We can always reduce to such a situation, by possibly altering the vector field out of a neighbourhood of the periodic solution involved. By possibly performing a change of variables, we may also assume that $u(t)$ coincides with the zero solution, so that $f(\lambda, t, 0) \equiv 0$. We denote by $\phi_\lambda(t, \tau, x)$ the solution of (S_λ) such that $\phi_\lambda(\tau, \tau, x) = x$. We recall the definition of bifurcation given in [6], for a one-parameter family of flows $\pi_\lambda(t, x)$.

DEFINITION 1. Let X be a locally compact metric space with distance d, and C the set of all proper, non-empty, compact subsets of X. Let us consider a map $K: [0, \lambda^\#) \to C$, $\lambda \mapsto K_\lambda$, such that:

i) $\forall\lambda \in [0, \lambda^\#)$, K_λ is π_λ-invariant,

ii) $\max\{d(x, K_0), x \in K_\lambda\} \to 0$ as $\lambda \to 0$,

then $\lambda = 0$ is said to be a bifurcation point for the map K if there exists $\lambda^* \in (0, \lambda^\#)$, and a second map $M: (0, \lambda^*) \to C$, $\lambda \mapsto M_\lambda$, satisfying the conditions:

i)′ $\forall\lambda \in (0, \lambda^*)$, M_λ is π_λ-invariant and $K_\lambda \cap M_\lambda = \emptyset$,

ii)′ $\max\{d(x, K_0), x \in M_\lambda\} \to 0$ as $\lambda \to 0$.

In the same paper, the following theorem has been proved.

THEOREM 1. *Let X be connected and π_λ be a continuous family of flows on X. Let $\lambda > 0$ and $K: [0, \lambda^\#) \to C$ be a map as in Definition* 1. *If K_0 is π_0-asymptotically stable and K_λ is π_λ-negatively asymptotically stable for $\lambda \in (0, \lambda^\#)$, then $\lambda = 0$ is a bifurcation point for K. Furthermore, the map M and λ can be chosen so that $\forall\lambda \in (0, \lambda^*)$, M_λ is π_λ-asymptotically stable.*

REMARK 1. There exists a neighbourhood U_0 of K_0 such that the set M_λ is the largest π_λ-invariant compact set contained in U_0 and disjoint from K_0, for $\lambda \in (0, \lambda^*)$. If V_0 is a Liapunov function associated to the asymptotic stability of K_0 with respect to (S_0), then U_0 can be taken of the form $V_0^{-1}([0, b_0])$, for some $b_0 > 0$.

Let S^1 be the set $\{x \in R^2 \colon |x| = 1\}$. We associate to (S_λ) a vector field v_λ on a subset $S^1 \times U$ of the cylinder $C := S^1 \times R^n$, in the usual way. A curve $\gamma_\lambda(t)$ in $S^1 \times U$ is an integral curve of v_λ if and only if there is a solution $\phi_\lambda(t, \tau, x)$ of (S_λ) such that $\gamma_\lambda(t) = \pi \circ \phi_\lambda(t, \tau, x)$, where π is the canonical projection of R^{n+1} onto $S^1 \times R^n$. The set $\Gamma = \pi(R \times \times \{0\})$ is a cycle, whose stability properties are strictly related to those of $u(t)$. In fact, Γ is stable (asymptotically stable) with respect to the flow on the cylinder if and only if $u(t)$ is stable (asymptotically stable) with respect to E_λ. This suggests an alternative proof of Theorem 3.1 in [7]. If $u(t)$ is asymptotically stable for $\lambda = 0$ and negatively asymptotically stable for $\lambda > 0$, then the same is true for Γ with respect to the flows on the cylinder. Since Γ is compact, we may apply Theorem 1 to deduce the existence of a family of asymptotically stable compact invariant sets M_λ, bifurcating from Γ. The set $N_\lambda = \pi^{-1}(M_\lambda)$ is an asymptotically stable periodic invariant set for (S_λ), with compact t-section. Moreover, $d(N_\lambda, u(t))$ tends to zero as λ tends to zero, so that N_λ bifurcates out of $u(t)$ as it changes stability.

If V_0 is a T-periodic Liapunov function for $u(t)$, existing by the converse theorems about asymptotic stability, then $V_0 \circ \pi^{-1}$ is a Liapunov function for Γ. By Remark 1, there exists b_0 such that M_λ is the largest invariant compact set disjoint from Γ, contained in $(V_0 \circ \pi^{-1})^{-1}([0, b_0]) = \pi(V_0^{-1}([0, b_0]))$. In order to prove the existence of bifurcating periodic solutions, we show that $V_0^{-1}([0, b_0])$ contains a periodic solution $u_\lambda(t) \neq u(t)$, so that $\pi \circ u_\lambda$ is an integral curve of v_λ, disjoint from Γ, contained in $\pi(V_0^{-1}([0, b_0]))$. This implies that $\pi \circ u_\lambda(t) \subseteq \subseteq M_\lambda$, so that $u_\lambda(t) \subseteq N_\lambda$.

THEOREM 2. *Let n be odd and $u(t)$ be a periodic solution of E_λ, for $\lambda \in [0, \lambda^{\#})$. If $u(t)$ is asymptotically stable for $\lambda = 0$ and negatively asymptotically stable for $\lambda > 0$, then $\lambda = 0$ is a point of bifurcation from $u(t)$ for a family of periodic solutions. Moreover, for any $\lambda \in (0, \lambda^*)$, either the bifurcating set N_λ contains a harmonic solution of E_λ, or it contains infinitely many subharmonic solutions of E_λ.*

PROOF. By the converse theorems on asymptotic stability [10, ch. VI], there exist an open set $U_0 \subseteq U$, a T-periodic Liapunov function $V \in \in \mathcal{C}^1(R \times U_0, R)$, increasing functions $\alpha, \beta, \gamma \in C^0(R, R)$ such that $\alpha(0) = = \beta(0) = \gamma(0) = 0$ and

$$\alpha(|x|) \leqslant V(t, x) \leqslant \beta(|x|); \tag{1}$$

$$\dot{V}(t, x) \leqslant -\gamma(|x|); \tag{2}$$

where $\dot{V}(t, x)$ denotes the derivative of V along the solutions of (S_0). We denote by $\dot{V}_\lambda$ the derivative of V along the solutions of (S_λ). By the periodicity of V and f_λ, and the continuity of ∇V and f_λ, for any $b_0 > 0$ there exists $\lambda(b_0) \in (0, \lambda^{\#})$, $l(b_0) \in (0, b_0)$ such that for any $\lambda \in (0, \lambda(b_0))$ we have $\dot{V}_\lambda < 0$ on the set $V^{-1}([l(b_0), b_0])$.

Let A_0 be the region of attraction of $u(t) \equiv 0$ with respect to the flow π_0, and A_λ be the region of negative attraction of $u(t)$, for $\lambda \in (0, \lambda^*)$. For small positive values of λ, $A_\lambda \subset V^{-1}([0, b_0])$. Let $B_0 \equiv B[O, r_0]$ and $C_0 \equiv$

$\equiv B[O, \rho_0]$ satisfy $R \times C_0 \subseteq V^{-1}([0, b_0]) \subseteq R \times B_0$. There exists $\bar{\lambda} \in (0, \lambda^*)$ such that $N_\lambda \cup A_\lambda \subset R \times C_0$ for any $\lambda \in (0, \bar{\lambda})$. We can take $C_\lambda \equiv B[O, \rho_\lambda]$ such that $R \times C_\lambda \subseteq A_\lambda$. Let us set $m_\lambda := \inf\{|\pi_\lambda(t, x)|: \ |x| = \rho_\lambda, t \geqslant 0\}$, and $B_\lambda \equiv B[O, m_\lambda/2]$. The bifurcating set N_λ attracts uniformly the spherical shell $C_0 \setminus C_\lambda$. Let n_λ be a positive integer such that $\pi_\lambda(nT, C_0 \setminus C_\lambda) \subseteq C_0 \setminus C_\lambda$ for any $n > n_\lambda$. Let $H_\lambda(t, x)$, $t \in [0, 1]$, be a homotopy contracting homeomorphically the shell $B_0 \setminus B_\lambda$ onto $C_0 \setminus C_\lambda$. Then, for any $n > n_\lambda$, the map $H_\lambda^{-1}(1, \cdot) \circ \pi_\lambda(nT, \cdot) \circ H_\lambda(1, \cdot)$ maps $B_0 \setminus B_\lambda$ into itself. Since it is homotopic to the identity map, its Lefschetz number is the Euler characteristic of $B_0 \setminus B_\lambda$. The dimension of the space is odd, so that we have $\chi(B_0 \setminus B_\lambda) = 2$. By Lefschetz fixed point theorem, this implies the existence of a fixed point $y_{\lambda, n} \in B_0 \setminus B_\lambda$. Then the point $x_{\lambda, n} = H_\lambda(y_{\lambda, n})$ is a fixed point of the map $\pi_\lambda(nT, \cdot)$.

Let us assume that N_λ does not contain harmonic solutions. In this case N_λ contains infinitely many subharmonic solutions. Indeed, let us take n and m relatively prime and greater than n_λ. Then, for some integers r, s, we have $rm + sn = 1$. If $x_{\lambda, m} = x_{\lambda, n}$, then

$$\pi(T, x_{\lambda, n}) = \pi((rm + sn)T, x_{\lambda, n}) = \pi(rmT, \pi(snT, x_{\lambda, n})) =$$

$$= \pi(rmT, x_{\lambda, n}) = \pi(rmT, x_{\lambda, m}) = x_{\lambda, m} = x_{\lambda, n},$$

contradicting the absence of harmonic solutions. ■

COROLLARY 1. *Let n be even, E_λ autonomous and $u(t)$ a non-trivial periodic solution of E_λ, for $\lambda \in [0, \lambda^\#)$. If $u(t)$ is orbitally asymptotically stable for $\lambda = 0$ and orbitally negatively asymptotically stable for $\lambda > 0$, then $\lambda = 0$ is a point of bifurcation from $u(t)$ for a family of non-trivial periodic solutions.*

PROOF. Let us set

$$\Gamma := \{x \in R^{2n}: \ x = u(t), t \in [0, T]\}.$$

By a suitable change of variables in as neighbourhood of Γ [4, Ch. XI], [5, Section VI.1], we can transform (S_λ) into a system of the form

$$(\Sigma'_\lambda) \qquad \begin{cases} \dot{r} = F_\lambda(\theta, r), & F_\lambda \in \mathcal{C}^1(R \times U, R^{2n-1}), \\ \dot{\theta} = G_\lambda(\theta, r), & G_\lambda \in \mathcal{C}^1(R \times U, R), \end{cases}$$

where U is an open subset of R^{2n-1}, $F_\lambda(\theta, r)$, $G_\lambda(\theta, r)$ are T-periodic in θ and

$$(4) \qquad 1 - \eta_\lambda \leqslant G_\lambda(\theta, r) \leqslant 1 + \eta_\lambda,$$

for some $0 < \eta_\lambda < 1$. Since the change of variables depends only on $u(t)$ and f, the map $\lambda \mapsto (\Sigma'_\lambda)$ defines a continuous family of dynamical systems in $R \times U$. We reparametrize the system (Σ'_λ) in order to get

$$(\Sigma''_\lambda) \qquad \begin{cases} \dot{r} = F_\lambda(\theta, r)/G_\lambda(\theta, r), \\ \dot{\theta} = 1. \end{cases}$$

The continuity of G_λ with respect to λ, and the inequality (4) for $\lambda = 0$ ensure that we still have a continuous family of dynamical systems. The zero solution of (Σ''_λ), considered as a periodic system in R^{2n-1}, is asymptotically stable if and only if Γ is orbitally asymptotically stable with respect to E_λ. Theorem 2 applied to (Σ''_λ) yields the existence of a family of periodic solutions bifurcating from the zero solution of (Σ''_λ). This implies the existence of a family of non-trivial periodic solutions of E_λ bifurcating from Γ. ■

The periodic solutions of the above corollary do not have necessarily the same period as the original one.

For sake of simplicity, we have not stated Theorem 2 in its more general form. In Theorem 1 the set K_λ may vary with λ, and π_λ depend only continuously on λ. Moreover [10, remark, page 104], locally lipschitzian differential systems have $\mathcal{C}^1$ Liapunov functions when the zero solution is uniformly asymptotically stable. Since we are concerned with periodic systems, the asymptotic stability of the zero solution is sufficient to ensure its uniform asymptotic stability, and the existence of Liapunov functions of class $\mathcal{C}^1$. This allows to consider a more general class of differential systems:

$$(E'_\lambda) \qquad x' = f(\lambda, t, x) \equiv f(\lambda, t + T(\lambda), x).$$

The proof of next theorem can be easily derived from what above and from Theorem 2, so that we do not report it.

THEOREM 2'. *Let $\lambda^\#$ be a positive number and U an open subset of R^n, with n odd. Let $T \in \mathcal{C}^0([0, \lambda^\#), R)$, $f \in \mathcal{C}^0([0, \lambda^\#) \times R \times U, R^n)$, $u \in \mathcal{C}^0([0, \lambda^\#) \times R, R^n)$, be such that $T(0) \neq 0$, f is locally lipschitzian in (t, x) for any $\lambda \in [0, \lambda^\#)$, $u(\lambda, t)$ is a $T(\lambda)$-periodic solution of E'_λ. If $u(0, t)$ is asymptotically stable, and $u(\lambda, t)$ is negatively asymptotically stable for $\lambda > 0$, then $\lambda = 0$ is a point of bifurcation from $u(0, t)$ for a family of periodic solutions of E'_λ.*

We say that a periodic solution $u(t)$ of E_λ is properly subharmonic if, for some positive integer $k > 1$, $u(t) \equiv u(t + kT)$, and $u(t) \neq u(t + hT)$ for any integer $0 < h < k$. In next proposition we show that if $u(t)$ is properly subharmonic, then the bifurcating periodic solutions are properly subharmonic, too.

COROLLARY 2. *Let n be odd and $u(t)$ be a properly subharmonic solution of E_λ. If $u(t)$ is asymptotically stable for $\lambda = 0$ and $u(t)$ is negatively asymptotically stable for $\lambda > 0$, then $\lambda = 0$ is a point of bifurcation for a family of properly subharmonic solutions of E_λ.*

PROOF. By performing a change of variables $X = x - u(t)$, we transform E_λ into a kT-periodic system

$$\dot{X} = F(\lambda, t, X). \tag{5}$$

The function $u(t)$ is transformed into the zero solution of (5), whose stability properties are the same as those of $u(t)$. Hence there exists a family of periodic solutions $w_\lambda(t)$ bifurcating from the zero solution of (5). The functions $u_\lambda(t) := w_\lambda(t) + u(t)$ are periodic solutions of E_λ bifurcating from $u(t)$. To prove that their minimal period cannot be smaller than kT, let us set,

$$D := \inf \{d(u(hT), u(0)) \colon h = 1, \ldots, k-1\}.$$

There exists $\lambda^* \in [0, \lambda^\#)$ such that the bifurcating set M_λ is contained in the set

$$\{(t, x) \in R^{n+1} \colon d(x, u(t)) < D/4\}.$$

For any periodic solution $u_\lambda(t)$ contained in M_λ, and for any $h = 1, \ldots, k-1$, we cannot have $u_\lambda(t) \equiv u_\lambda(t + hT)$, since $d(u_\lambda(0), u_\lambda(hT)) > D/2$. ■

We conclude this section by considering the system E_λ in connection to its variational system:

$$\dot{x} = A(\lambda, t) \cdot x := \partial_x f(\lambda, t, 0). \tag{6.λ}$$

By Liapunov theorem [8, page 452], there exist linear periodic change of variables, depending on λ, that transform E_λ into

$$\dot{x} = B(\lambda) \cdot x + h(\lambda, t, x), \quad h(\lambda, t, 0) \equiv 0, \quad \partial_x h(\lambda, t, 0) \equiv 0. \tag{7.λ}$$

If $\mu(\lambda)$ is an eigenvalue of $B(\lambda)$, then $\exp(\mu(\lambda))$ is a characteristic multiplier of (7.λ). This allows us to prove the existence of periodic solutions bifurcating from $u(t)$ when some of the characteristic multipliers leave simultaneously the unit circle, while the other ones remain inside it.

Corollary 3. *Let k be an odd integer. Let (6.0) have $n-k$ characteristic multipliers in the open unit circle. Assume that for $\lambda > 0$, (6.λ) has k characteristic multipliers out of the unit circle. If 0 is asymptotically stable with respect to (S_0), then $\lambda = 0$ is a point of bifurcation from $u(t)$ for a family of periodic solutions.*

Proof. Without loss of generality, we assume that $u(t) \equiv 0$. There exists $\lambda' \in (0, \lambda^{\#})$ such that, for $\lambda \in (0, \lambda')$, the matrix $B(\lambda)$ has $n-k$ eigenvalues with negative real part and k eigenvalues with positive real part. Hence there is a unique periodic unstable manifold $D_\lambda \subset R^{n+1}$ of dimension $k+1$ containing $u(t)$. The differential system induced by (S_λ) on D_λ has the form

$$(8.\lambda) \qquad \begin{cases} \dot{y} = Y(\lambda, t, y), \\ \dot{t} = 1, \end{cases}$$

so that we may consider it as a periodic differential system defined on an open subset of R^k. Since the zero solution is negatively asymptotically stable with respect to (8.λ), there exists a positive definite periodic function V_λ with positive definite derivative $\dot{V}_\lambda$ along the solutions of (8.λ). Moreover, by the asymptotic stability of 0 with respect to (S_0), there exists a Liapunov function, whose restriction V_0 to D_λ is a periodic $\mathcal{C}^1$ function. We may use the properties of V_0 and V_λ to prove the existence of a periodic solution $u_\lambda(t)$ on D_λ as in Theorem 2. In [7, Thm. 3.2] it was proved the existence of a family of asymptotically stable periodic invariant sets $N_\lambda \subseteq D_\lambda$, bifurcating from $u(t)$ as λ becomes positive. Since N_λ attracts any solution contained in $D_\lambda \cap V_0^{-1}((0, b_0])$, for b_0 small enough, we have that $u_\lambda(t)$ is contained in N_λ. This proves the thesis. ∎

As above, an analogous result holds for autonomous systems. We recall that a non-trivial periodic solution of an autonomous system has always 1 as a characteristic multiplier.

Corollary 4. *Let E_λ be an autonomous system, and $u(t)$ a non-trivial periodic solution of E_λ, for $\lambda \in [0, \lambda^{\#})$. Let the characteristic multiplier 1 be simple for $\lambda > 0$ and the characteristic multipliers of $u(t)$ different from 1 be as in Corollary 3. If $u(t)$ is orbitally asymptotically stable with respect to (E_0), then $\lambda = 0$ is a bifurcation point from $u(t)$ for a family of non trivial periodic solutions.*

Proof. It is sufficient to apply Corollary 3 to the transformed system of Corollary 1. ∎

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