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T_3-systems of finite simple groups

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T_3-Systems of Finite Simple Groups.

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1. Introduction.

We present here some further evidence in support of the following conjecture, first formulated by Wiegold in the seventies.

CONJECTURE. Every finite non-abelian simple group has exactly one T_3-system.

Gilman [5] has shown that the conjecture holds for the simple groups $PSL(2, p)$ with p prime, (indeed it was this result that prompted the conjecture), while Evans [4] has done it for certain Suzuki groups. In both cases the action of the automorphism group on the G-defining subgroups is alternating or symmetric, and this too seems likely to reflect a general truth.

The Suzuki groups and the $PSL(2, p)$ are easier to cope with than the alternating groups, no doubt because of the much greater diversity of subgroups in alternating groups. Since $A_5 \simeq PSL(2, 5)$, Gilman's result provides the answer, while A_6 is so small that a simple calculation is sufficient. The aim of this note is to sketch a proof of the following result.

THEOREM. The alternating group A_7 has just one T_3-system, and the action of $\operatorname{Aut} F_3$ on the A_7 defining subgroups is alternating or symmetric.

The methods are elementary throughout. I see no way of establishing the conjecture for the general alternating group A_n.

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2. T_3-systems and a result of Evans.

Let F_n be a free group of finite rank n, and let G be any group. We say that N is a G-defining subgroup of F_n if $N \lhd F_n$ and $F_n/N \simeq G$. Denote the set of all G-defining subgroups of F_n by $\Sigma(G, n)$ and notice that $\Sigma(G, n)$ is not empty if and only if G can be generated by n elements.

For each $\sigma \in \operatorname{Aut} F_n$ and $N \in \Sigma(G, n)$ we clearly have $F_n/N\sigma \simeq G$ so that $N\sigma \in \Sigma(G, n)$. In this way we obtain an action of $\operatorname{Aut} F_n$ on $\Sigma(G, n)$, the orbits of which are called the T_n-systems of G. ([5] and [3]).

When we investigate T-systems of a specific group G, it is found to be rather difficult to work directly with the action of $\operatorname{Aut} F_n$ on $\Sigma(G, n)$. B. H. Neumann and H. Neumann [7] introduced the notion of generating G-vectors which enabled them to define an equivalent action of $\operatorname{Aut} F_n$ which is more manageable. The details with respect to T_3 are now given following the argument indicated in [4].

Let G be a 3-generator group. A generating G-vector of length 3 is defined to be an ordered triple (g_1, g_2, g_3) where $\langle g_1, g_2, g_3\rangle = G$. The set of all generating G-vectors of length 3 is denoted by $V(G, 3)$.

Fix a set of free generators x_1, x_2, x_3 for F_3 and let E be the set of epimorphisms from F_3 to G. Define an action of $\operatorname{Aut} F_3 \times \operatorname{Aut} G$ on E by

$$\rho(\sigma, \alpha) = \sigma^{-1}\rho\alpha \tag{2.1}$$

where $\rho \in E$ and $(\sigma, \alpha) \in \operatorname{Aut} F_3 \times \operatorname{Aut} G$.

We can identify $\operatorname{Aut} F_3$ and $\operatorname{Aut} G$ with their copies in $\operatorname{Aut} F_3 \times \operatorname{Aut} G$ and speak of the action of $\operatorname{Aut} F_3$ or $\operatorname{Aut} G$ on E. We clearly have

(2.2) *ρ_1 and ρ_2 lie in the same* $\operatorname{Aut} G$*-orbit of E if and only if* $\ker \rho_1 = \ker \rho_2$.

Suppose that $\ker \rho = N$. Then $\ker \rho\alpha = N$ too, and so we can associate N with the $\operatorname{Aut} G$-orbit of E that contains ρ, *viz.* $\{\rho\alpha \colon \alpha \in \operatorname{Aut} G\}$. Notice that for all $\sigma \in \operatorname{Aut} F_3$ we have $\ker(\rho(\sigma, 1)) = \ker(\sigma^{-1}\rho) = N\sigma$. Hence $N\sigma$ is associated with the $\operatorname{Aut} G$-orbit of E containing $\rho(\sigma, 1)$. Moreover, $N \in \Sigma(G, 3)$ if and only if $N = \ker \rho$ for some $\rho \in E$. Therefore

(2.3) *The action of* $\operatorname{Aut} F_3$ *on* $\Sigma(G, 3)$ *is equivalent to its action on the* $\operatorname{Aut} G$*-orbits of* E.

The map $\pi: E \to V(G, 3)$ given by

(2.4) $\rho\pi = (x_1\rho, x_2\rho, x_{3\rho})$ *is a bijection. Furthermore,* π *enables us to carry over the action of* $\operatorname{Aut} F_3 \times \operatorname{Aut} G$ *on* E *to an action on* $V(G, 3)$.

This is given by

$$\rho\pi(\sigma, \alpha) = \sigma^{-1}\rho\alpha\pi. \tag{2.5}$$

The action of $\operatorname{Aut} F_3 \times \operatorname{Aut} G$ on $V(G, 3)$ given by (2.5) is equivalent to its action on E. Therefore the action of $\operatorname{Aut} F_3$ on the $\operatorname{Aut} G$-orbits of $V(G, 3)$ is equivalent to its action on the $\operatorname{Aut} G$-orbits of E. Combining this last remark with (2.3) gives the following fundamental result.

(2.6) *The action of* $\operatorname{Aut} F_3$ *on the* $\operatorname{Aut} G$*-orbits of* $V(G, 3)$ *is equivalent to its action on* $\Sigma(G, 3)$.

Let us now examine in greater detail the actions of $\operatorname{Aut} F_3$ and $\operatorname{Aut} G$ on $V(G, 3)$. Here we again identify $\operatorname{Aut} F_3$ and Aut G with their copies in $\operatorname{Aut} F_3 \times \operatorname{Aut} G$.

Suppose throughout that (g_1, g_2, g_3) is a typical element of $V(G, 3)$. By (2.4) there exists $\rho \in E$ with $(g_1, g_2, g_3) = \rho\pi = (x_1\rho, x_2\rho, x_3\rho)$. The action of $\operatorname{Aut} G$ on $V(G, 3)$ is now easily given explicitly; by (2.5) we have $(g_1, g_2, g_3)(1, \alpha) = \rho\pi(1, \alpha) = (x_1\rho\alpha, x_2\rho\alpha, x_3\rho\alpha) = (g_1\alpha, g_2\alpha, g_3\alpha)$. Moreover since $\langle g_1, g_2, g_3\rangle = G$ we have $(g_1, g_2, g_3) = (g_1\alpha, g_2\alpha, g_3\alpha)$. if and only if $\alpha = 1$. Hence

(2.7) *The action of* $\operatorname{Aut} G$ *on* $V(G, 3)$ *is given by* $\alpha: (g_1, g_2, g_3) \to$ $\to (g_1\alpha, g_2\alpha, g_3\alpha)$ *for all* $\alpha \in \operatorname{Aut} G$ *and all* $(g_1, g_2, g_3) \in V(G, 3)$.

We next consider the action of $\operatorname{Aut} F_3$ on $V(G, 3)$. For all $\sigma \in \operatorname{Aut} F_3$ we have $(g_1, g_2, g_3)(\sigma, 1) = \rho\pi(\sigma, 1) = \sigma^{-1}\rho\pi = (x_1\sigma\rho, x_2\sigma\rho, x_3\sigma\rho)$ from (2.5). Suppose that

$$\begin{cases} x_1\sigma^{-1} = w_1(x_1, x_2, x_3), \\ x_2\sigma^{-1} = w_2(x_1, x_2, x_3), \\ x_3\sigma^{-1} = w_3(x_1, x_2, x_3), \end{cases} \tag{2.8}$$

where $w_1(x_1, x_2, x_3)$ is a word in (x_1, x_2, x_3). Now

$$(x_1\sigma^{-1}\rho, x_2\sigma^{-1}\rho, x_3\sigma^{-1}\rho) = (w_1\rho, w_2\rho, w_3\rho) =$$

$$= (w_1(g_1, g_2, g_3), w_2(g_1, g_2, g_3), w_3(g_1, g_2, g_3))$$

where $\sigma \in \operatorname{Aut} F_3$ and w_1, w_2, w_3 are given by (2.8). Therefore

(2.9) *The action of* $\operatorname{Aut} F_3$ *on* $V(G,3)$ *is given by*

$$\sigma: (g_1, g_2, g_3) \to (w_1(g_1, g_2, g_3), w_2(g_1, g_2, g_3), w_3(g_1, g_2, g_3))$$

where $\sigma \in \operatorname{Aut} F_3$ *and* w_1, w_2, w_3 *are given by* (2.8).

We continue, using the following result, a convenient reference for which is [6] Chapter 3.

(2.10) $\operatorname{Aut} F_3$ *is generated by the automorphisms given below, where* $1 \leqslant i, k \leqslant 3$, $i \neq k$ *and unmentioned generators of* F_3 *are fixed.*

$$P(i, k): x_i \to x_k, \qquad x_k \to x_i ,$$

$$\sigma(i): x_i \to x_i^{-1} ,$$

$$L(i, k): x_i \to x_k x_i ,$$

$$R(i, k): x_i \to x_i x_k .$$

These are called the elementary automorphisms of F_3. Their effect on $(g_1, g_2, g_3) \in V(G, 3)$ is to interchange any two entries, invert any entry or multiply any entry by any other on the left or right. This is seen with the aid of (2.9).

As $\operatorname{Aut} F_3$ is generated by elementary automorphisms, the above remark has an important consequence, namely

(2.11) *Two elements of* $V(G,3)$ *lie in the same* $\operatorname{Aut} F_3$*-orbit if and only if one can be transformed into the other by a finite sequence of the following operations:*

— *Interchanging two entries*:
e.g. $(g_1, g_2, g_3) \to (g_1, g_2, g_3)$.

— *Inverting an entry*:
e.g. $(g_1, g_2, g_3) \to (g_1^{-1}, g_2, g_3)$.

— *Multiplying one entry on the left by another*:
e.g. $(g_1, g_2, g_3) \to (g_2 g_1, g_2, g_3)$.

— *Multiplying one entry on the right by another*:
e.g. $(g_1, g_2, g_3) \to (g_1 g_2, g_2, g_3)$.

We say that two elements of $V(G, 3)$ are equivalent if they lie in the same $\mathrm{Aut}\, F_3$-orbit.

An important property of A_7 in our context is that it has *spread* 2 in the sense of Brenner and Wiegold ([1] and [2]). This means that for any pair x, y of non-trivial elements of A_7, there is a third element z such that $\langle x, z\rangle = \langle y, z\rangle = A_7$. The connection with T_3-systems is the following simple but important result of Evans [4].

(2.12) *Let G be any group of spread 2. Then all redundant generating triple are equivalent.*

A redundant generating triple (g_1, g_2, g_3) is one where one of g_1, g_2, g_3 can be omitted and the remaining two elements still generate the group. Thus our strategy will be to show that every generating triple for A_7 is equivalent to a redundant triple.

3. T_3-systems of A_7.

The 2520 elements of A_7 are classified into distinct types of permutations. We shall use the representation of these permutations as products of disjoint cycles, omitting cycles of length one. If an element is a product of disjoint cycles of lengths $r_1, r_2, \ldots, r_k$ where $r_1 > 1$ the we say it is of *type* $r_1, r_2, \ldots, r_k$. The table below gives the number of elements of each type in A_7 and also in each of the maximal subgroups of A_7 which are isomorphic to $PSL(2, 7)$.

Type	7	5	4, 2	3, 3	3, 2, 2	3	2, 2	Ident.	Total
A_7	720	504	630	280	210	70	105	1	2520
$PSL(2, 7)$	48	0	42	56	0	0	21	1	168

There are 15 maximal subgroups of A_7 which are isomorphic to $PSL(2, 7)$. Each element of type 7 of A_7 is in one and only one of these maximal subgroups. This property is also true for each element of type 4, 2 of A_7.

In order to show that every generating G-vector (g_1, g_2, g_3), is equivalent to a redundant vector we systematically look at all possible cases.

CASE 1. If one of the elements of the triple is of type 7, say g_1 then as we remarked above, it is one and only one of the $PSL(2,7)$ contained in A_7; call this group B.

If $g_2 \in B$ then $\langle g_1, g_2 \rangle \subseteq B$ while if $g_2 \notin B$ then $\langle g_1, g_2) = A_7$ as B is a maximal subgroup. The same holds for g_3.

As (g_1, g_2, g_3) is a generating set for A_7, one of g_2, g_3 is not an element of B and will generate A_7 with g_1. Thus any generating triple containing an element of type 7 is equivalent to a redundant triple.

CASE 2. Suppose that g_1 is of type 5, without loss of generality, (12345) say. If $\langle g_1, g_2 \rangle$ is transitive over the set $\{1,2,3,4,5,6,7\}$ then $\langle g_1, g_2 \rangle = A_7$.

So we look at the cases when $\langle g_1, g_2 \rangle$ and $\langle g_1, g_3)$ are non transitive but of course, g_2 and g_3 between them must move 6 and 7. We need to consider two cases.

i) $g_1 = (12345)$, $g_2 = (\dots)(67)$, $g_3 = (\dots 6)(\dots)(7)$. Then $6g_3 = i$ with $i \neq 6$ and $i \neq 7$ and $7g_3 = 7$ so $6g_2g_3 = 7$ and $7g_2g_3 = i$.

This means that $g_2g_3 = (\dots 67i \dots)(\dots)$ and hence $\langle g_1, g_2g_3 \rangle$ is transitive and so must be A_7.

ii) $g_1 = (12345)$, and let g_2 move 6 but not 7 and g_3 move 7 but not 6. Then g_2g_3 will move 6 and 7 and then $\langle g_1, g_2g_3 \rangle$ is again transitive and so is A_7.

Thus if the generating triple contains an element of type 5 it is equivalent to a redundant triple.

The further cases, with g_1, g_2 and g_3 taking all possible types, are shown in the following table, which indicates the length of the calculation required.

We investigate the cases 3, 4, 5, 6 and 7, using the following consideration.

i) There is a need for transitivity over $\{1,2,3,4,5,6,7\}$.

ii) Any triple equivalent to a triple with an element of type 7 or of type 5 is no problem.

iii) Two elements generating a transitive subgroup of A_7, in which one is of type 3 will generate A_7 ([7], p. 34).

iv) Two elements generating a transitive subgroup of A_7 and each of type 4, 2 in different $PSL(2,7)$ subgroups will generated A_7.

The investigation leads to the conclusion that if the generating triple contains an element of type 4, 2 it is equivalent to a redundant triple.

Case	g_1 type	g_2 type	g_3 type
3	4, 2	4, 2	4, 2 or 3 or 3, 3 or 3, 2, 2 or 2, 2
4	4, 2	3	3 or 3, 3 or 3, 2, 2 or 2, 2
5	4, 2	3, 3	3, 3 or 3, 2, 2 or 2, 2
6	4, 2	3, 2, 2	3, 2, 2 or 2, 2
7	4, 2	2, 2	2, 2
8	3	3	3 or 3, 3 or 3, 2, 2 or 2, 2
9	3	3, 3	3, 3 or 3, 2, 2 or 2, 2
10	3	3, 2, 2	3, 2, 2 or 2, 2
11	3	2, 2	2, 2
12	3, 3	3, 3	3, 3 or 3, 2, 2 or 2, 2
13	3, 3	3, 2, 2	3, 2, 2 or 2, 2
14	3, 3	2, 2	2, 2
15	3, 2, 2	3, 2, 2	3, 2, 2 or 2, 2
16	3, 2, 2	2, 2	2, 2
17	2, 2	2, 2	2, 2

We provide here a proof of some of Case 3 to demonstrate the methods used. The complete proofs of the assertions made here involve a great deal of simple but tedious calculation.

CASE 3. Let g_1, g_2 and g_3 be each of type 4, 2 and each in a different $PSL(2, 7)$-subgroup of A_7. As an example we consider the following case.

$$g_1 = (3567)(12) \in \langle (1234567), (23)(47) \rangle,$$

$$f_2 \in \langle (2314567), (13)(47) \rangle,$$

$$g_3 \in \langle (2431567), (43)(17) \rangle.$$

If $\langle g_1, g_2 \rangle$ is transitive over $\{1, 2, 3, 4, 5, 6, 7\}$ there is no problem. We also find for the remaining elements g_2 that $g_1 g_2$ or $g_1 g_2^{-1}$ or $g_1 g_2^2$ is of type 7 or type 5 except for $g_2 = (2537)(16)$ or $(1567)(23)$ and their inverses.

If $\langle g_1, g_3 \rangle$ is transitive over $\{1, 2, 3, 4, 5, 6, 7\}$ there is no problem. We also find for the remaining elements g_3 that $g_1 g_3$ or $g_1 g_3^{-1}$ is of type 7 or type 5 expect for $g_3 = (3657)(12)$ or $(3567)(14)$ or $(3576)(24)$ and their inverses.

For these elements or their inverses, $g_2 g_3$ or $g_2 g_3^{-1}$ is of type 7 or type 5.

We see that for the selected g_1 and the subgroups concerned, all the triples are equivalent to redundant triples. This is found to be true whichever of the 14 maximal subgroups are chosen to contain elements g_2 and g_3. Thus any generating triple containing three elements of type 4, 2 each in a different $PSL(2, 7)$ maximal subgroup is equivalent to a redundant triple.

We now consider the case with g_1, g_2 each of type 4, 2 and each in a different $PSL(2, 7)$-subgroup of A_7 with g_3 any element of type 3. We consider the following case.

$$g_1 = (3567)(12) \in (1234567), (23)(47)\rangle,$$

$$g_2 \in \langle(2314567), (13)(47)\rangle.$$

$$g_3 = \text{any element of type 3 in } A_7,$$

When we consider the products of $g_1 g_2$ and $g_1 g_3$ we find problems only occur when $g_2 = (2537)(16)$ or $(1567)(23)$ and $g_3 = (124)$ or (345) or (346) or (347) or (456) or (457).

For these elements we find that either an equivalent triple can be obtained with one element, a product of g_1, g_2 and g_3, which is of type 7 or of type 5, or the triple is not a generating triple.

We again see that for the selected g_1 and the subgroups concerned, all the triples are equivalent to redundant triples. This is found to be true whichever of the maximal subgroups are chosen to contain element g_2. Thus any generating triple containing two elements of type 4, 2 each in a different $PSL(2, 7)$ maximal subgroup with the third element of type 3 is equivalent to a redundant triple.

Case 3, when completed, and then cases 4, 5, 6 and 7 all lead to the same conclusion that the generating triples concerned are all equivalent to a redundant triple.

The information obtained from cases 1 to 7 is used in the other cases in the order as shown in the table and with each case leading to a redundant triple.

The final conclusion is that all the generating G-vectors are equivalent to redundant vectors and consequently A_7 has only one T_3-system.

A further result of Evans [4] can now be used to complete the proof.

(3.1) *Let G be a nonabelian finite simple group with $d(G) = k$. Suppose that $G = \langle g_1, g_2, \ldots, g_k \rangle$ where $g_k^2 = 1$. Then* $\mathrm{Aut}\, F_{k+1}$ *acts as a symmetric or alternating group on at least one of its orbits on* $\Sigma(G, k+1)$.

The alternating group A_7 may be generated by $\langle g_1, g_2 \rangle$ where g_1 is an element of type 7 and g_2 is an element of type 2, 2 which is not in the $PSL(2, 7)$ maximal sub-group containing g_1. For example we have $A_7 = \langle (1234567), (12)(45) \rangle$. We conclude that the action of $\mathrm{Aut}\, F_3$ on the A_7 defining subgroups is alternating or symmetric.

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