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Existence and Uniqueness of a Periodic Solution to the Two-Dimensional Generalized Korteweg-de Vries Equation.

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ABSTRACT - The two-dimensional generalized Korteweg-de Vries equation is considered. Using the Galerkin's method and the Sobolev imbedding theorem we show that a periodic (in x and y) solution exists and is unique if the initial function $g(x, y)$ satisfies certain suitable conditions.

1. Introduction.

The one-dimensional Korteweg-de Vries equation (referred to as the KdV equation henceforth) derived in 1895 [7] has attracted the attention of many researchers since the name «soliton» was coined by Zabusky and Kruskal in 1965 [13]. For survey of the KdV equation we cite a paper by Miura [9], and for the related inverse scattering technique we cite a book by Ablowitz and Segur [1]. The two-dimensional KdV equation (called the KP equation) was first derived by Kadomtsev and Petriashvili in 1970 [6]. Some two-dimensional results involving traveling wave solutions have been obtained by Chen and Wen [3], [4]. Generalizing the one-dimensional result of LeVeque [8] the authors discussed the interaction of two nearly equal solitons in the two-dimensional KdV equation [11]. A uniqueness theorem for two-dimensional periodic solutions was established by the authors [12].

In this paper we consider periodic solutions (in x and y) to the fol-

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lowing initial value problem (referred to as problem (I) henceforth):

(1) $$u_t + [f(u)]_x = \alpha u_{xxx} + \beta \int_0^x u_{yy}(s, y, t)\, ds\,,$$

(2) $$u|_{t=0} = g(x, y)\,,$$

where $u(x+P, y, t) = u(x, y+P, t) = u(x, y, t)$ for all real x, y and for $t \geqslant 0$, constants $\alpha < 0$ and $\beta > 0$. Furthermore, we assume that

(3) $$\int_0^P u(x, y, t)\, dx = 0\,, \qquad \text{for any } y,\ t \geqslant 0\,.$$

In addition, we shall assume $f(u) = Au^r$ for a constant $A > 0$ and either $r \in Z^+$ or $r > 3$. When $r = 2$, eq. (1) is the KP equation, and when $r = 1$ it is linear. Without loss of generality, we shall prove for $r \geqslant 3$ since the cases where $r = 1, 2$ are easier to handle in a similar way.

Using the Galerkin's method and the Sobolev imbedding theorem [2] we shall establish the existence and uniqueness of a periodic solution to problem (I). Guo [5] has proved a uniqueness and existence theorem of periodic solutions for the one-dimensional KdV equation with $|f(u)| \leqslant Au^2 + B$ where A and B are constants. However, there are major errors in the paper which invalidate the proof. Recently, Schwartz [10] showed a theorem on existence and uniqueness of periodic solutions for the two-dimensional KdV equation. Both Guo and Schwartz used the Galerkin's method. The basic ideas we use are similar to that of Guo and Schwartz. But, we use a different function space and the details are quite different. In particular, we employ the Sobolev imbedding theorem which was not used by Schwartz or Guo. Our result is good for $f(u) = Au^r$ with $r \in Z^+$ or $r > 3$ which includes Schwartz's result as a special case (i.e. $A = 1/2$ and $r = 2$).

2. Some notations and definitions.

First, we introduce some notations and definitions:

1) $S = [0, P] \times [0, P] \subset R^2$.

2) $D = \dfrac{\partial}{\partial x}$; $D_y = \dfrac{\partial}{\partial y}$.

3) $(u, v) = \displaystyle\iint_S u(x, y)\, v(x, y)\, dx\, dy\,.$

4) $\|u\|^2 = (u, u)$.

5) W_P^m is a Sobolev space of functions with generalized derivatives up to the m-th order that belong to L_P.

$$\|u\|_m^2 = \sum_{0 \leq k+j \leq m} \|D^k D_y^j u\|^2,$$

where D^k means the k-th derivative in x and D_y^j means the j-th derivative in y.

6) $S_1 = \{v | \, v \in W_2^4(R^2)$ with $v(x+P, y) = v(x, y+P) = v(x, y)\}$.

7) $C_B^j(S) = \{u \in C^j(S);\ D^k D_y^i u$ is bounded on S for $k + i \leq j\}$.

8) Hypothesis A: $f(u) = Au^r$ for constant $A > 0$ and either $r \in Z^+$ or $r > 3$. (Again, without loss of generality, we assume $r \geq 3$.)

9) Hypothesis B: For the same r in Hypothesis A

$$g \in W_2^5(S), \qquad \|g_{xx}\| < \frac{2}{(r-1)\, a_3 T},$$

where a_3 is a constant to be defined in Lemma 2.

We should note that S_1 is a Banach space. Since by the Sobolev imbedding theorem (pp. 95-100, ref. [2]), a convergent sequence in S_1 converges pointwise, it is closed in W_2^4. (I.e. the limit of the sequence is in S_1.)

A function $u(x, y, t)$ which is periodic in x and y with period P and satisfies (3) is called a *classical solution* of problem (I) if all the derivatives of $u(x, y, t)$ in eq. (1) are continuous and $u(x, y, t)$ satisfies eq. (1) identically. We define a *generalized solution* of the problem (I) in the domain $Q_T = S \times [0, T]$ to be the function $u(\cdot, t) \in W_2^4(S)$, $u_t \in L_2$, satisfying the following conditions:

$$(u_t, v) + (Df(u), v) - \alpha(D^3 u, v) - \beta(F_{yy}, v) = 0 \tag{4}$$

for any $v \in S_1$, where $F(x, y, t) = \int_0^x u(s, y, t)\, ds$

$$u(x+P, y, t) = u(x, y+P, t) = u(x, y, t), \tag{5}$$

for any (x, y, t) with $t \geq 0$,

$$u(x, y, 0) = g(x, y) \tag{6}$$

and

(7) $$\int_0^P u(x,y,t)\,dx = 0\,, \qquad \forall y,\ t \geq 0\,.$$

We want to prove that the problem (I) has a unique classical solution. To do so, we use the Garlerkin's method. Since Sobolev spaces are separable Hilbert spaces (Adams [2], p. 47), we can pick out a trigonometric orthonormal basis $\{\phi_j(x,y)\}$ for $W_2^4(S)$, where $\phi_j(x,y)$ are periodic with period P in x and y.

For any given $g(x,y) \in W_2^4$, there exist real constants c_j such that $\sum_{j=1}^{n} c_j \phi_j \to g(x,y)$ in W_2^4 norm as $n \to \infty$. We shall look for approximate solutions $u_n(x,y,t)$ of the problem (I). We call $u_n = \sum_{j=1}^{n} c_{jn}(t)\,\phi_j$ a *Galerkian solution* if u_n satisfies

(8) $$(u_{nt}, v) + (Df(u_n), v) - \alpha(D^3 u_n, v) - \beta(D^{-1} u_{nyy}, v) = 0$$

for any $v \in S_1$, where

(9) $$c_{jn}(0) = c_j\,, \qquad j = 1, 2, 3, \ldots, n\,.$$

Before we proceed further we note that c_{jn} are uniquely determined for a given $g(x,y)$ and for the $f(u)$ satisfying Hypothesis A.

In view of the orthonormality of $\{\phi_j\}$ if we choose $v = \phi_j$ in eq. (4) for $j = 1, 2, \ldots, n$, then it becomes a system of ordinary differential equations of the form

(10) $$\frac{dC}{dt} = \boldsymbol{h}(t, \boldsymbol{C})$$

with initial condition

(11) $$\boldsymbol{C}(0) = \boldsymbol{B}$$

where $\boldsymbol{C}(t)$, $\boldsymbol{B}$ are n-column vectors whose transposes are

$$\boldsymbol{C}^T = (c_{1n}, c_{2n}, \ldots, c_{jn}, \ldots, c_{nn})\,, \qquad \boldsymbol{B}^T = (c_1, c_2, \ldots, c_n)\,,$$

and $\boldsymbol{h}(t, \boldsymbol{C})$ is a power function in $c_{1n}, c_{2n}, \ldots, c_{nn}$ of degree r. In other words, $c_{jn}(t)$ must satisfy the initial value problem eqs. (10) and (11). C_{jn} are continuous in t. The partial derivatives of $\boldsymbol{h}$ with respect to c_{jn} ($j = 1, 2, \ldots, n$) are power functions in c_{jn} of degree at most $r - 1$. Since r is fixed, these derivatives are continuous and bounded if $|c_{jn}|$ are bounded for $t \geq 0$. Therefore c_{jn} ($j = 1, 2, \ldots, n$) exist for $t \geq 0$ and are uniquely determined if we can show that $|c_{jn}|$ are bounded.

The boundedness of $|c_{jn}|$ can be established from the result, $\|u_n(t)\| \leq \|g\| \leq q_1$ (see Lemma 1 below), and the Parseval's formula, $\sum_{j=1}^{n} c_{jn}^2 = \|u_n(t)\|^2$. Hence in this way the Galerkin's solutions are uniquely determined for a given $g(x, y)$.

3. Preliminary lemmas.

To establish the existence of solutions to Problem (I), we shall show u_n converge (as $n \to \infty$) uniformly in S, to a solution of Problem (I). To this end we need the following Lemmas. In view of the fact that the techniques used in the proof of many lemmas are similar and some of them are quite lengthy, we shall outline some of the proofs. Many details will be omitted. Readers interested in details may write to one of the authors.

LEMMA 1. If the Hypothesis A is true, then the following estimate holds for the Galerkin's solutions u_n

$$\|u_n(t)\|^2 = \|u_n(0)\|^2 \leq q_1,$$

where the constant q_1 is independent of n.

PROOF. Choosing $v = \phi_j(x, y)$ in eq. (8) for $j = 1, 2, 3, \ldots, n$, multiplying each equation by $C_{jn}(t)$ and then summing over j from 1 to n, we get

$$\iint_S (u_{nt} + Df(u_n) - \alpha D^3 u_n - \beta D^{-1} D_y^2 u_n) u_n \, dx \, dy = 0 .$$

Now we need to examine every term in the above equation.

$$\iint_S u_n u_{nt} \, dx \, dy = \frac{1}{2} \frac{d}{dt} \|u_n\|^2 ,$$

$$\iint_S u_{nxxx} u_n \, dx \, dy = \int_0^P u_{nxx} u_n \Big|_{x=0}^{x=P} dy - \iint_S u_{nx} u_{nxx} \, dx \, dy$$

$$= - \iint_S \frac{1}{2} [(u_{nx})^2]_x \, dx \, dy = 0 ,$$

$$\iint_S (D^{-1} u_{nyy})\, u_n \,dy\,dx = \int_0^P (D^{-1} u_{ny})\, u_n \Big|_{y=0}^{y=P} dx - \iint_S (D^{-1} u_{ny})\, u_{ny}\, dx\,dy$$

$$= -\iint_S \frac{1}{2}[(D^{-1} u_{ny})^2]_x \,dx\,dy = 0\,,$$

and since $f(u_n) = A(u_n)^r$,

$$\iint_S rA(u_n)^{r-1} u_{nx} u_n \,dx\,dy = rA/(r+1)\iint_S \frac{\partial}{\partial x}(u_n)^{r+1}\,dx\,dy = 0\,.$$

Combining the above results, we have

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|^2 = 0\,.$$

Hence $\|u_n(T)\|^2 = \|u_n(0)\|^2$ for any $T \geqslant 0$. Since $\sum_{j=1}^{n} c_j \phi_j \to g$ in $W_2^4(S)$ norm as $n \to \infty$, if we choose $\varepsilon = 1$, then there is a N such that for any $n > N$,

$$\|u_n(0) - g\|_4 < 1\,,$$

especially,

$$\|u_n(0) - g\| < 1\,,$$

thus $\|u_n(0)\| < \|g\| + 1$ for $n > N$.

Hence we can find a constant q_1 (depending on g only) such that

$$\|u_n(t)\|^2 = \|u_n(0)\|^2 \leqslant q_1 \qquad \text{for all } n. \qquad \text{Q.E.D.}$$

LEMMA 2. Suppose Hypotheses A and B are satisfied. Then the estimate holds

$$\|u_{nxx}\| \leqslant q_2\,,$$

where q_2 is a consist which is independent of n.

PROOF. First we notice that if we apply the Sobolev inequalities here, for some constants c_1, c_2, and c_3, we have

$$\|u_{nx}\|_\infty \leqslant \|u_n\| + c_1 \|u_{nxx}\|\,,$$

and

$$\|u_n\| \leqslant \frac{1}{2}\|u_n\| + c_2\|u_{nxx}\|,$$

thus

$$\|u_{nx}\|_\infty \leqslant c_3\|u_{nxx}\|.$$

Next we follow the same routine as that at the beginning of the proof of Lemma 1 by choosing $v = \phi_{jxxxx}$ to get

$$\iint_S (u_{nt} + f'\,u_{nx} - \alpha u_{nxxx} - \beta D^{-1}u_{nyy})\,u_{nxxxx}\,dx\,dy = 0.$$

Again, after examing each term in the above equation, we obtain the following relations

$$\iint_S u_{nt}\,u_{nxxxx}\,dx\,dy = -\iint_S u_{ntx}\,u_{nxxx}\,dx\,dy$$

$$= \iint_S u_{ntxx}\,u_{nxx}\,dx\,dy = \frac{1}{2}\frac{d}{dt}\|_{nxx}\|^2,$$

$$\left|\iint_S f'\,u_{nx}\,u_{nxxxx}\,dx\,dy\right| = \left|-\iint_S [f''(u_{nx})^2 + f'\,u_{nxx}]\,u_{nxxx}\,dx\,dy\right| = G.$$

Then, we use integration by parts in x again to get

$$G = \left|\iint_S [u_{nxx}(f'''(u_{nx})^3 + 2f''\,u_{nx}\,u_{nxx}) + f''u_{nx}(u_{nxx})^2/2]\,dx\,dy\right|$$

$$\leqslant A_3\|u_n\|_\infty^{r-3}\|u_{nx}\|_\infty^2\|u_{nx}\|\|u_{nxx}\| + 5/2A_2\|u_n\|_\infty^{r-2}\|u_{nx}\|_\infty\|u_{nxx}\|^2$$

$$\leqslant a_1\|u_{nxx}\|^{r+1} + a_2\|u_{nxx}\|^{r+1} = a_3\|u_{nxx}\|^{r+1},$$

where we have made use of integration by parts repeatedly, the periodicity property of u_n and its derivatives, and the Sobolev inequalities, $A_3 = Ar(r-1)(r-2)$, and $A_2 = Ar(r-1)$.

$$\iint_S u_{nxxx}\,u_{nxxxx}\,dx\,dy = 0,$$

and

$$\iint_S D^{-1} u_{nyy} u_{nxxxx} \, dx \, dy = 0 \,,$$

by use of integration by parts in x and y. Combining the above results and letting $Y = \|u_{nxx}\|^2$ we have derived the inequality

$$\frac{dY}{dt} \leqslant a_3 Y^{(r+1)/2} \,.$$

Hence we have

$$Y^{(1-r)/2}(T) \geqslant Y^{(1-r)/2}(0) - \frac{r-1}{2} a_3 T \,.$$

Now, we choose $Y(0) = \|D^2 g\|$ to be smaller than

$$\frac{2}{(r-1) a_3 T} \,.$$

Then

$$\frac{1}{[Y(0)]^{(r-1)/2}} - \frac{r-1}{2} a_3 T = b_1 > 0 \,.$$

Therefore, $Y(T) \leqslant [1/b_1]^{2/(r-1)} = q_2$. Then for $0 \leqslant t \leqslant T$, q_2 is independent of n. Hence $\|u_{nxx}\| \leqslant q_2$. Q.E.D.

LEMMA 3. Suppose Hypotheses A and B are satisfied. Then the following estimates hold:

$$\|u_{nx}\| \leqslant q_3 \,, \qquad \|u_n\|_\infty \leqslant q_4 \,, \qquad \|u_{nx}\|_\infty \leqslant q_5 \,,$$

where q_3, q_4 and q_5 are constants independent of n.

PROOF. We choose $\varepsilon = 1$ in the Sobolev inequalities, then there are constants C_1, C_2 and C_3 such that

$$\|u_{nx}\|_\infty \leqslant \|u_{nx}\| + C_3 \|u_{nxx}\| \,, \qquad \|u_{nx}\| \leqslant \|u_n\| + C_1 \|u_{nxx}\| \,,$$

and

$$\|u_n\|_\infty \leqslant \|u_n\| + C_2 \|u_{nx}\| \,.$$

Then Lemmas 1 and 2 give us the desired results. Q.E.D.

LEMMA 4. Suppose Hypotheses A and B are satisfied. Then the estimate is true

$$\|u_{nt}\| \leqslant q_6 ,$$

where q_6 is a constant independent of n.

PROOF. Choosing $v = \phi_j$, $j = 1, \ldots, n$ and differentiating eq. (8) with respect to t, we have

$$\frac{d}{dt} \iint\limits_S (u_{nt}\,\phi_j + f'\,u_{nx}\,\phi_j - \alpha u_{nxxx}\,\phi_j - \beta D^{-1} u_{nyy}\,\phi_j)\,dx\,dy = 0 .$$

Let $u_{nt} = v_n$. Multiplying the resulting equation by $c'_{jn}(t)$ then summing over j from 1 to n, we obtain

$$\iint\limits_S (v_{nt} + f''\,u_{nx}\,v_n + f'\,v_{nx} - \alpha v_{nxxx} - \beta D^{-1} v_{nyy})\,v_n\,dx\,dy = 0 .$$

Estimates can be made as before. The rest of the proof is omitted. Q.E.D.

LEMMA 5. Suppose Hypotheses A and B are satisfied. Then the following estimates hold:

$$\|u_{nxxx}\| \leqslant q_7 , \qquad \|u_{nxy}\|^2 \leqslant q_8 ,$$

where q_7 and q_8 are constants independent of n.

PROOF. Following the same routine as in the proof of Lemma 4, we can get the following equation

$$\iint\limits_S (u_{nt} + f'\,u_{nx} - \alpha u_{nxxx} - \beta D^{-1} u_{nyy})\,u_{nxxx}\,dx\,dy = 0 .$$

Again, the rest of the proof is omitted. Q.E.D.

LEMMA 6. Suppose Hypotheses A and B are satisfied. Then the following estimate is true:

$$\|u_{ny}\|^2 \leqslant q_9 ,$$

where q_9 is a constant independent of n.

PROOF. Again, as in the proof of Lemma 3, we use Sobolev inequality to get

$$\|u_{ny}\| \leq 1/2\|u_{ny}\| + C\|u_{nyx}\| \quad \text{for some constant } C.$$

Then Lemma 5 ensures that

$$\|u_{ny}\| \leq 2C\sqrt{q_8} = q_9. \quad \text{Q.E.D.}$$

LEMMA 7. Suppose Hypotheses A and B are satisfied. Then the following estimate is true:

$$\|u_{nxt}\| \leq q_{10},$$

where q_{10} is a constant independent of n.

PROOF. Differentiating eq. (8) with respect to t, choosing $v = \phi_{jxx}$, $j = 1, \ldots, n$, using integration by parts, and letting $w_n = v_{nx} = u_{nxt}$, the resulting equation becomes

$$\iint_S [w_{nt} + 2f''(u_n)\, u_{nx} w_n + f'(u_n)\, w_{nx} + f'''(u_n)\, v_n (u_{nx}^2)$$

$$+ f''(u_n)\, v_n u_{nxx} - \alpha w_{nxxx} - \beta v_{nyy}]\, w_n \, dx\, dy = 0.$$

The rest of the proof is routine. Q.E.D.

LEMMA 8. Suppose Hypotheses A and B are satisfied. Then the following estimates hold:

$$\|u_{nyy}\| \leq q_{11}, \qquad \|u_{nxxy}\| \leq q_{12},$$

where constants q_{11} and q_{12} are independent of n.

PROOF. In eq. (8), we let $v = \phi_{jyyx}$, $j = 1, 2, \ldots, n$, and use integration by parts with respect to x. Then the resulting equation is

$$\iint_S [Du_{nt} + f''(u_n)\, u_{nx}^2 + f'(u_n)\, u_{nxx} - \alpha u_{nxxxx} - \beta u_{nyy}]\, \phi_{jyy} \, dx\, dy = 0.$$

We multiply the above equation by $c_{jn}(t)$ and sum over j from 1 to n to get

$$\iint_S [u_{nxt} + f''(u_n)u_{nx}^2 + f'(u_n)u_{nxx} - \alpha u_{nxxxx} - \beta u_{nyy}]\, u_{nyy}\, dx\, dy = 0\,.$$

The rest of the proof is omitted. Q.E.D.

LEMMA 9. Suppose Hypotheses A and B are satisfied. Then the following estimate holds:

$$\|u_{nxxt}\|^2 \leq q_{13}\,,$$

where the constant q_{13} is independent of u_n.

PROOF. Following the same procedure as in the proof of Lemma 4, choosing $v = c_{jn}\phi_{jxxx}$, $j = 1, 2, \ldots, n$, and letting $z_n = w_{nx} = v_{nxx} = u_{nxxt}$, we obtain the following equation

$$\iint_S (z_{nt} + 3f'''(u_n)u_{nx}^2 w_n + 3f''(u_n)u_{nxx}w_n + 3f''(u_n)u_{nx}z_n$$
$$+ f'(u_n)z_{nx} + f^{(4)}(u_n)u_{nx}^3 v_n + 3f'''(u_n)u_{nx}u_{nxx}v_n$$
$$+ f''(u_n)v_n u_{nxxx} - \alpha z_{nxxx} - \beta v_{nyyx})\, z_n\, dx\, dy = 0\,.$$

The rest is rather routine. Q.E.D.

LEMMA 10. Suppose Hypotheses A and B are satisfied. Then the following estimates hold:

$$\|u_{nxyy}\| \leq q_{14}\,, \qquad \|u_{nxxxy}\|^2 \leq q_{15}\,,$$

where the constants q_{14} and q_{15} are independent of n.

PROOF. In eq. (8), choosing $v = c_{jn}(t)\phi_{jyyxxx}$, and using integration by parts twice with respect to x, we have the equation

$$\iint_S (u_{ntxx} + f'''(u_n)u_{nx}^3 + 3f''(u_n)u_{nx}u_{nxx} + f'(u_n)u_{nxxx}$$
$$- \alpha u_{nxxxxx} - \beta u_{nyyx})\, c_{jn}(t)\,\phi_{jyyx}\, dx\, dy = 0\,.$$

We sum over j from 1 to n to obtain the identity

$$\iint_S [u_{ntxx} + f'''(u_n) u_{nx}^3 + 3f''(u_n) u_{nx} u_{nxx} + f'(u_n) u_{nxxx} - \alpha u_{nxxxxx} - \beta u_{nyyx}] u_{nyyx} \, dx \, dy = 0 .$$

The rest is routine. Q.E.D.

LEMMA 11. Suppose Hypotheses A and B are satisfied. Then the following estimate holds:

$$\|u_{nty}\|^2 \leqslant q_{16} ,$$

where the constant q_{16} is independent of n.

PROOF. Similar to the proof of Lemma 4, letting $u_{nt} = v_n$, and $v = \phi_{jy}$, $j = 1, \dots, n$, we can derive the following equation:

$$\iint_S v_{ny} (v_{nty} + f'''(u_n) u_{nx} u_{ny} v_n + f''(u_n) u_{nxy} v_n + f''(u_n) u_{nx} v_{ny} + f'' v_{nx} u_{ny} + f'(u_n) v_{nxy} - \alpha v_{nxxxy} - \beta D^{-1} v_{nyyy}) \, dx \, dy = 0 .$$

The rest is omitted. Q.E.D.

LEMMA 12. Suppose Hypotheses A and B are satisfied. Then the following estimate holds:

$$\|u_{ntxy}\| \leqslant q_{17} .$$

PROOF. Let $w_{ny} = u_{nxyt} = z_n$. Then following the same procedure as in the proof of Lemma 4, we'll deal with the following equation:

$$\begin{aligned}\iint_S z_n (z_{nt} &+ 2f'''(u_n) u_{nx} u_{ny} w_n + 2f''(u_n) u_{nxy} w_n + 2f''(u_n) u_{nx} z_n \\ &+ f''(u_n) u_{ny} w_{nx} + f'(u_n) z_{nx} + f^{(4)}(u_n) u_{ny} v_n u_{nx}^2 \\ &+ 2f'''(u_n) v_n u_{nx} u_{nxy} + f'''(u_n) v_{ny} u_{nx}^2 + f''(u_n) v_{ny} u_{nxx} \\ &+ f'''(u_n) u_{ny} v_n u_{nxx} + f''(u_n) v_n u_{nxxy} - \alpha z_{nxxx} - \beta v_{nyyy}) \, dx \, dy = 0 .\end{aligned}$$

The rest is omitted. Q.E.D.

LEMMA 13. Suppose Hypotheses A and B are satisfied. Then we have the following estimates:

$$\|u_{nyyy}\| \leqslant q_{18}, \qquad \|u_{nyyxx}\| \leqslant q_{19},$$

where constant q_{18} and q_{19} are independent of n.

PROOF. We choose $v = c_{jn}\phi_{jyyyy}$, $j = 1, 2, \ldots, n$ in eq. (8), then integrate it by parts with respect to y and sum over the resulting equations from $j = 1$ to n to obtain

$$\iint_S u_{nyyy}(u_{ntxy} + f'''(u_n)u_{ny}u_{nx}^2 + 2f''u_{nx}u_{nxy} + f''u_{ny}u_{nxx} + f'u_{nxxy} - \alpha u_{nxxxxy} - \beta u_{nyyy})\,dx\,dy = 0.$$

The rest is omitted. Q.E.D.

LEMMA 14. Suppose Hypotheses A and B are satisfied. Then we have the following estimate:

$$\|u_{nxxxx}\|^2 \leqslant q_{20}.$$

PROOF. Choosing $v = c_{jn}\phi_{jxxxxx}$, $j = 1, 2, \ldots, n$ in eq. (8), integrating by parts in x, and then summing over the resulting equations from $j = 1$ to n, we obtain

$$\iint_S u_{nxxxx}(u_{ntx} + f''(u_n)u_{nx}^2 + f'(u_n)u_{nxx} - \alpha u_{nxxxx} - \beta u_{nyy})\,dx\,dy = 0.$$

We omit the details. Q.E.D.

LEMMA 15. Suppose Hypotheses A and B are satisfied. Then

$$\|u_{nxyyy}\|^2 \leqslant q_{21},$$

where constant q_{21} is independent of n.

PROOF. By choosing $v = c_{jn}\phi_{jxxyyyyyy}$, $j = 1, 2, \ldots, n$ in eq. (8), making use of integration by parts in x and y, and then following the same routine as in the proof of Lemma 14, we can obtain the following equation

$$\iint_S (u_{ntxyyy} + f^{(5)} u_{ny}^3 u_{nx}^2 + 3f^{(4)} u_{ny} u_{nyy} u_{nx}^2 + 6f^{(4)} u_{ny}^2 u_{nx} u_{nxy}$$

$$+ f''' u_{nyyy} u_{nx}^2 + 6f''' u_{nyy} u_{nx} u_{nxy} + 6f''' u_{ny} u_{nxy}^2$$

$$+ 6f''' u_{ny} u_{nx} u_{nx} + 6f'' u_{nxy} u_{nxyy} + 2f'' u_{nx} u_{nxyyy}$$

$$+ f^{(4)} u_{ny}^3 u_{nxx} + 3f''' u_{ny} u_{nyy} u_{nx} + 3f''' u_{ny}^2 u_{nxxy}$$

$$+ f'' u_{nyyy} u_{nxx} + 3f'' u_{nyy} u_{nxxy} + 3f'' u_{nyy} u_{nxxy} + 3f'' u_{ny} u_{nxxyy}$$

$$+ f' u_{nxxyyy} - \alpha u_{nxxxxyyy} - \beta u_{nyyyyy}) u_{nxyyy} \, dx\, dy = 0 .$$

The rest is routine. Q.E.D.

LEMMA 16. Suppose Hypotheses A and B are satisfied. Then

$$\|u_{nyyyy}\|^2 \leqslant q_{22} ,$$

where the constant q_{22} is independent of n.

PROOF. The proof is routine. Q.E.D.

LEMMA 17. Suppose Hypotheses A and B are satisfied. Then

$$\|u_{nxxxxx}\|^2 \leqslant q_{23} ,$$

for some constant q_{23} which is independent of n.

PROOF. We choose $v = c_{jn}\, \phi_{jxxxxxxxx}$, $j = 1, 2, \ldots, n$ in eq. (8) and the follow the same routine as in the proof of Lemma 15 to obtain

$$\iint_S u_{nxxxxx} (u_{ntxx} + f''' u_{nx}^3 + 3f'' u_{nx} u_{nxx} + f' u_{nxxx}$$

$$- \alpha u_{nxxxxx} - \beta u_{nyyx}) \, dx\, dy = 0 .$$

The rest is omitted. Q.E.D.

LEMMA 18. Suppose Hypotheses A and B are satisfied. Then

$$\|u_{nxxxxy}\| \leqslant q_{24} ,$$

where q_{24} is some constant which is independent of n.

PROOF. Choosing $v = \phi_{jxxxxxxyy}$, $j = 1, 2, \ldots, n$, and following the same routine as in the proof of Lemma 15, we can transform eq. (8) into

$$\iint_S u_{nxxxxxy}(u_{ntxy} + f''' u_{ny} u_{nx}^2 + 2f'' u_{nx} u_{nxy} + f'' u_{ny} u_{nxx} + f' u_{nxxy} - \alpha u_{nxxxxxy} - \beta u_{nyyy})\, dx\, dy = 0 .$$

The rest is omitted. Q.E.D.

LEMMA 19. Suppose Hypotheses A and B are satisfied. Then

$$\|u_{ntyy}\|^2 \leqslant q_{25} ,$$

where q_{25} is some constant which is independent of n.

PROOF. Refering to the equation in the proof of Lemma 11 we can start from

$$\iint_S (v_{ntyy} + f^{(4)} u_{nx} u_{ny}^2 v_n + 2f''' u_{nxy} u_{ny} v_n + f''' u_{nx} u_{nyy} v_n + 2f''' u_{nx} u_{ny} v_n + f'' u_{nxyy} v_n + 2f'' u_{nxy} v_{ny} + f'' u_{nx} v_{nyy} + f''' u_{ny}^2 v_{nx} + f'' u_{nyy} v_{nx} + 2f'' u_{ny} v_{nxy} + f' v_{nxyy} - \alpha v_{nxxxyy} - \beta D^{-1} v_{nyyyy}) v_{nyy} = 0 ,$$

and then choose $v = c_{jn} \phi_{jyyyy}$, $j = 1, 2, \ldots, n$, and follow the same routing as in the proof of Lemma 11. Q.E.D.

LEMMA 20. Suppose Hypotheses A and B are satisfied. Then

$$\|u_{nxxxyy}\| \leqslant q_{26} ,$$

where q_{26} is some constant which is independent of n.

PROOF. Choosing $v = c_{jn} \phi_{jxxxyyyy}$, $j = 1, 2, \ldots, n$ in eq. (8), and following the same routine as in the proof of Lemma 15, we can obtain the resulting equation:

$$\iint_S (u_{ntyy} + f''' u_{ny}^2 u_{nx} + f'' u_{nyy} u_{nx} + 2f'' u_{ny} u_{nxy} + f' u_{nxyy} - \alpha u_{nxxxyy} - \beta D^{-1} u_{nyyyy}) u_{nxxxyy} \, dx \, dy = 0 .$$

The rest is omitted. Q.E.D.

4. Uniqueness and existence theorems.

THEOREM 1. (*a*) If $u(x, y, t)$ is a classical solution of problem (I), then it is also a generalized solution.

(*b*) Let u be a function with the needed continuous derivatives as described in eq. (1) and u be a generalized solution of eq. (4). Then u is a classical solution of problem (I).

PROOF. (*a*) Assume u is a classical solution of problem (I). We want to show:

$$\iint_S (u_t + f'(u) u_x - \alpha u_{xxx} - \beta F_{yy}) v \, dx \, dy = 0 \quad \text{for any } v \in S_1 .$$

We carry out integration by parts with respect to x for the second and third term, and with respect to y for the last term. Making use of the periodicity conditions for u and v, we obtain eq. (4). Therefore, u is a generalized solution.

(*b*) The converse is also true. More precisely if $u(x, y, t)$ is a generalized solution with the needed continuous derivatives as described in eq. (1). Now, if eq. (1) does not hold. Then there is a neighborhood of some point, say, $\bar{x}_0 = (x_0, y_0)$ in S, in which $[u_t + f'(u) u_x - \alpha u_{xxx} - \beta F_{yy}]$ is positive since the functions inside the square brackets are continuous in S. In this case we choose v such that $v > 0$ and $v \in S_1$ in this neighborhood and zero elsewhere in S so that

$$\iint_S (u_t + f'(u) u_x - \alpha u_{xxx} - \beta F_{yy}) v \, dx \, dy > 0 ,$$

which is a contradiction. Hence u must satisfy eq. (1). Q.E.D.

THEOREM 2. If Hypothesis A is satisfied, then problem (I) has at most one generalized solution.

Proof. Let u and z be two generalized solutions of the problem (I). For any $\phi \in S_1$, we have, after integration by parts as necessary, the following equation for u and z:

$$\iint_S \{(u_t - z_t)\phi - [f(u) - f(z)]\phi_x - \alpha(u_x - z_x)\phi_{xx}$$
$$+ \beta D^{-1}(u_y - z_y)\phi_y\}\, dx\, dy = 0 .$$

Let $u - z = w$ and choose $\phi = w$, we can transform the above equation into

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 - \iint_S [f(u) - f(z)]\, w_x\, dx\, dy = 0 .$$

Using a method similar to that in the proof of Lemmas 2 and 3, we can obtain the upper bounds for $\|u\|_\infty$ and $\|u_x\|_\infty$, where these bounds are independent of the generalized solutions.

To estimate the last term in the above equation, we use integration by parts, the result of Lemma 1 of [12], and the upper bounds of $\|u\|_\infty$ and $\|u_x\|$. Hence we can derive the inequalities:

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 \leqslant \left| \iint_S w \left\{ f'(u)\, w_x + \frac{1}{P}\int_0^P f''\left[\frac{\tau}{P}u + \left(1 - \frac{\tau}{P}\right)z\right] d\tau\, z_x w \right\} dx\, dy \right|$$

$$\leqslant \iint_S |f''(u)\, u_x| \frac{w^2}{2}\, dx\, dy + \frac{1}{P}\iint_S \left| \int_0^P f''\left[\frac{\tau}{P}u + \left(1 - \frac{\tau}{P}\right)z\right] d\tau \cdot z \right| w^2\, dx\, dy$$

$$\leqslant \frac{1}{2}\max|f''|\,\|u_x\|_\infty \|w\|^2 + \max|f''|\,\|z\|_\infty \|w\|^2 \leqslant b\|w\|^2$$

for some constant b independent of u.

By the Gronwall's inequality, we obtain the desired result, for $T > 0$,

$$\|w(T)\|^2 \leqslant \|w(0)\|^2 \exp(2Tb) = 0 .$$

Hence $u \equiv z$. Q.E.D.

Now, we are ready to prove the existence theorem of problem (I). First, let us explain some of the notations which will be used.

$W_2^4(S) \to C_B^2(S)$ means that each $u \in W_2^4(S)$ can, when considered as a function, be redefined on a set of zero measure in S in such a way that the modified function $\underset{\sim}{u}$ (which equals u in W_2^4) belong to $C_B^2(S)$ and satisfies

$$\|\underset{\sim}{u}\|_{C_B^2(S)} \leq K\|u\|_4 \tag{12}$$

with K independent of u [2].

THEOREM 3. Suppose $g \in W_2^5(S)$ and Hypotheses A and B are satisfied. Then Problem (I) has a unique classical solution $u(\cdot, t)$.

PROOF. By Lemmas 1, 5, 6, 8, 10, 13, 14, 15 and 16 $\{u_n\}$ is uniformly bounded in $W_2^4(S)$. By the Rellich-Kondrachov Theorem ([2], p. 144), $W_2^4(S)$ is compactly imbedded into $C_B^2(S)$. Hence there is a subsequence (without loss of generality, we can still denote this subsequence as $\{u_n\}$) which converges to some $u(\cdot, t)$ in $C_B^2(S)$ (from now on, we shall use the modified function $\underset{\sim}{u}$ instead of using u, and without confusion we denote $\underset{\sim}{u}$ by u), i.e.,

$$\|u_n - u\|_{C_B^2(S)} \to 0 \tag{13}$$

as $n \to \infty$ for each fixed $t \in [0, T]$.

Also, $u_n \to u$ weakly in $W_2^4(S)$ for each fixed $t \in [0, T]$, so

$$\left| \iint_S u_n \phi \, dx \, dy - \iint_S u\phi \, dx \, dy \right| \to 0 \tag{14}$$

as $n \to \infty$ for every $\phi \in S_1$.

Furthermore, $\{u_{nt}\}$ is uniformly bounded in $W_2^2(S)$ by Lemmas 4, 7, 11, 9, 12 and 19. Thus there is a subsequence of $\{u_{nt}\}$ (without loss of generally, we can still denote this subsequence as $\{u_{nt}\}$) such that $u_{nt} \to \overline{u}$ weakly, i.e.

$$\left| \iint_S (u_{nt} - \overline{u}) \phi \, dx \, dy \right| \to 0 \quad \text{as } n \to \infty \text{ for } t \in [0, T]. \tag{15}$$

Now we want to show that $\iint_S u_t \phi \, dx \, dy = \iint_S \overline{u}\phi \, dx \, dy$ for every $\phi \in S_1$. By Lemmas 3, 4 and 6, $u_n(x, y, t) \in W_2^1(Q_T)$. Since W_2^1 is a Banach space and any Banach space is weakly sequentially compact, $u_n \to u$ weakly in $W_2^1(Q_T)$. Hence we obtain

$$\left| \iiint_{Q_T} (u_n \psi_t - u\psi_t) \, dt \, dx \, dy \right| \to 0 \quad \text{as } n \to \infty,$$

for any $\psi_t \in W_2^1(Q_T)$.

Using integration by parts in t and the fact that $u_n \to u$ weakly in S

for each t in $[0, T]$, we obtain

$$\left| \iiint_{Q_T} (u_{nt}\psi - u_t\psi)\, dt\, dx\, dy \right| \to 0 \quad \text{as } n \to \infty .$$

On the other hand, from eq. (13) we obtain

$$\left| \iiint_{Q_T} (u_{nt}\phi - \bar{u}\phi)\, dx\, dy\, dt \right| \leqslant \int_0^T \left| \iint_S (u_{nt}\phi - \bar{u}\phi)\, dx\, dy \right| dt \to 0$$

as $n \to \infty$, for each $\phi \in S_1$. Thus we apply the theorems of changing order of integration in real analysis to obtain

$$\left| \iiint_{Q_T} (u_{nt}\phi - \bar{u}\phi)\, dt\, dx\, dy \right| \to 0 \quad \text{as } n \to \infty .$$

Since ψ is arbitrarily chosen in $W_2^1(Q_T)$, we choose $\psi = \phi$ in the above expression for each fixed t. Then by the uniqueness of the limit, we must have $u_t\phi = \bar{u}\phi$ a.e.

Next, we want to show that u satisfies eq. (4). We have already known from eqs. (13) and (15) that

$$(u_t, \phi) = \lim_{n\to\infty} (u_{nt}, \phi), \qquad (Du, D^2\phi) = \lim_{n\to\infty} (Du_n, D^2\phi).$$

Since $u_n \to u$ pointwise in S for $t \in [0, T]$, it follows that $u_n^r \to u^r$ pointwise in S for $t \in [0, T]$. Furthermore, using eq. (12) we can get

$$|u_n(x, y; t)| \leqslant K\|u_n\|_4 \leqslant B ,$$

where B is some constant from our previous Lemmas. Thus we can use the Dominated Convergence Theorem to get

$$(u^r, D\phi) = \lim_{n\to\infty} (u_n^r, D\phi), \qquad \text{for any } \phi \in S_1,\ r \in Z^+ \text{ or } r > 3 .$$

And eq. (12) implies

$$|u_{ny}(x, y, t) - u_y(x, y, t)| \to 0 \quad \text{as } n \to \infty$$

uniformly in x and y for $(x, y) \in S$ and fixed $t \in [0, T]$. Hence we can obtain

$$\iint_S \left[\int_0^x |u_{ny}(z, y, t) - u_y(z, y, t)|\, dz \right] \phi_y\, dx\, dy \to 0 \quad \text{as } n \to \infty ,$$

i.e.,

$$(F_y, D_y \phi) = \lim_{n \to \infty} (D^{-1} u_{ny}, D_y \phi) \quad \text{for any } \phi \in S_1 .$$

Thus we can use integration by parts to derive that u satisfies eq. (4).

Now, we want to check the boundary and the initial condition. By the meaning of Sobolev imbedding, namely eq. (12), $u_n(\cdot, t)$ actually converges pointwise to $u(\cdot, t)$. Thus u satisfies the periodic condition, i.e.,

$$u(x+P, y, t) = \lim_{n \to \infty} u_n(x+p, y, t) = \lim_{n \to \infty} u_n(x, y, t) = u(x, y, t),$$

$$u(x, y+P, t) = \lim_{n \to \infty} u_n(x, y+P, t) = \lim_{n \to \infty} u_n(x, y, t) = u(x, y, t).$$

Next, we want to show that u is continuous in x, y, and t. First, we apply Lemmas 3 and 7, and the first Sobolev inequality to deduce that $u(x, y, \cdot) \in W_2^1([0, T])$. Then apply the Sobolev imbedding theorem to $W_2^1([0, T])$ to deduce that u is continuous in t. Secondly, since $u \in C_B^2(S)$, u_x and u_y are bounded in absolute values, for any fixed $(x_0, y_0, t_0) \in Q_T$ and any $(x, y, t) \in Q_T$,

$$\begin{aligned} |u(x,y,t) - u(x_0, y_0, t_0)| &\leq |u(x,y,t) - u(x_0, y, t)| \\ &\quad + |u(x_0, y, t) - u(x_0, y_0, t)| + |u(x_0, y_0, t) - u(x_0, y_0, t_0)| \\ &\leq |u_x(\hat{x}_0, y, t)|\,|x - x_0| + |u_y(x, \hat{y}_0, t)|\,|y - y_0| \\ &\quad + |u(x_0, y_0, t) - u(x_0, y_0, t_0)| . \end{aligned}$$

Therefore we can use the definition of continuous functions to deduce the continuity of u in x, y, and t in Q_T. Thus u also satisfies the initial condition. Therefore u is a generalized solution of problem (I).

Finally, we want to show that $u(\cdot, t)$ is continuous in S. But since $u_{xxx}(\cdot, t) \in W_2^2(S)$ by Lemmas 5, 10, 14, 17, 18, and 20, the Sobolev imbedding theorem ensures the continuity of u_{xxx} in x and y. Then applying Theorem 1, we obtain the desired result. Q.E.D.

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