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On Almost-Free Modules over Complete Discrete Valuation Rings.

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1. Introduction.

It is well known that torsion-free modules over a complete discrete valuation ring R have many nice properties not possessed by torsion-free modules (even those over incomplete discrete valuation rings) over general domains: indecomposable R-modules have rank one and reduced countably generated R-modules are free. Despite this, it has been established (see e.g. [1] and [4]) that most of the standard pathologies of decomposition associated with Abelian groups also occur in this class of modules. This was achieved by showing that a wide range of R-algebras A could be realized, modulo the ideal of finite rank endomorphism, as the endomorphism algebra of a torsion-free reduced R-module. The investigations in [1], [4] were carried out in ZFC set theory. The main aim of the present work is to establish similar, but in some senses stronger, results working under the additional set-theoretical hypothesis $V = L$. (See [5] or [10] for further details of this and other set-theoretical terms used.) Specifically the modules involved in our realization of an algebra A will always be strongly κ-free quâ A-module; recall that a module is κ-free, for a cardinal κ, if every submodule of cardinality less than κ is contained in a free module and G is said to be strongly κ-free if it is κ-free and every submodule of infinite cardinality less than κ can be embedded in a submodule U of the same cardinality with G/U κ-free.

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In order to give a precise statement of our first main result we fix some notation. R shall always be a (commutative) complete discrete valuation ring with unique maximal ideal pR and A shall denote an arbitrary unital R-algebra which is torsion-free and Hausdorff in the p-adic topology. (All topological references shall be to this topology.) Recall also the notion of inessential used extensively in previous work on the so-called «realization problem»: an R-homomorphism $\phi: F \to G$, between two reduced torsion-free modules F and G is said to be *inessential* if the unique extension $\hat{\phi}$ of ϕ to the completion $\hat{F}$ of F has the property that $\hat{F}\hat{\phi} \leq G$. (We shall normally identify the extension $\hat{\phi}$ with ϕ.) It is well known that set of inessential endomorphism of such a module F form a two-sided ideal Ines F, in the endomorphism ring $E_R(F)$. We can now state:

THEOREM 1 ($V=L$). Let R be a complete discrete valuation ring and let A be an unital, torsion-free, Hausdorff R-algebra. Then, for each regular, not-weakly compact cardinal $\varkappa > |A|$, there exists a strongly $\varkappa$-free A-module H of cardinality $\varkappa$ such that

$$E_R(H) = A \oplus \operatorname{Ines} H.$$

REMARK. The above theorem is stated for a single A-module H but our results have been derived in such a way that the transition to results about the existence of essentially-rigid families of maximal size and essentially-rigid proper classes (cf. [8, 1]) can be easily made. In particular we have derived both Step-Lemmas required for such a construction but in the interests of clarity (and since additional work required is totally standard) we shall be content to state (without proof) the full version of our work in Theorem 3.

Notice that the module constructed in Theorem 1 is only strongly $\varkappa$-free quâ A-module and moreover the ideal of inessentials has not been tightly prescribed. We can overcome these defects easily by imposing further restrictions on the algebra A. Recall (see e.g. [1], [4]) that an R-module M is said to be $\aleph_0$-cotorsion-free provided M does contain a submodule isomorphic to the completion of a free R-module not of infinite rank.

THEOREM 2 ($V=L$). Let R be a complete discrete valuation ring and A a unital R-algebra which is $\aleph_0$-cotorsion-free quâ R-module. Then, for each regular, not-weakly compact cardinal $\varkappa > |A|$, there exists a strongly $\varkappa$-free A-module H of rank $\varkappa$ such that

$$E_R(H) = A \oplus E_0(H)$$

where $E_0(H)$ denotes the ideal of finite rank endomorphisms of H. Moreover H is $\aleph_0$-cotorsion-free quâ R-module. If, in addition, A is free quâ R-module then H is a strongly $\varkappa$-free R-module.

PROOF. Let H denote the A-module constructed in the proof of Theorem 1. Then H is certainly a strongly $\varkappa$-free A-module. If, however, H contains the completion of a free R-module of infinite rank, then certainly H has a submodule isomorphic to the completion of a free R-module of countable rank. Since H is $\varkappa$-free this would imply that such a submodule is contained in a free A-module which is clearly impossible since A is $\aleph_0$-cotorsion-free. So H is $\aleph_0$-cotorsion-free quâ R-module. It only remains to show that every inessential endomorphism has finite rank since clearly $E_0(H) \leq \mathrm{Ines}\, H$.

Now, as observed in the proof of Lemma 5, § 3, cf $\varkappa > \omega$ implies $\hat{H} = \bigcup_{\alpha<\varkappa} \hat{H}_\alpha$ and so if ϕ is inessential then $\hat{H}_\alpha \phi$ is a submodule of H and has cardinality $< \varkappa$. Hence $\hat{H}_\alpha \phi$ is of finite rank since it is simultaneously complete and $\aleph_0$-cotorsion-free. Thus if $S = \{\nu \in \varkappa \mid \phi \restriction H_\nu$ has finite rank$\}$, then S is unbounded in $\varkappa$ and so it follows from [8, Lemma 5] that ϕ has finite rank. The final conclusion of the theorem is immediate since A is then free quâ R-module.

We remark that if $\varkappa = |A| = \aleph_1$ then the above result can be derived in ZFC as e.g. in [1] (cf. remarks in § 1 of [8]). As mentioned earlier, we now state (without proof) a more general version of the above theorem which can be easily derived from the construction given in this paper.

THEOREM 3 ($V = L$). Let R be a complete discrete valuation ring and A a unital R-algebra which is free quâ R-module. Then, for each regular not-weakly compact cardinal $\varkappa > |A|$,

(i) there exists strongly $\varkappa$-free R-modules $H_\varkappa^\alpha$ ($\alpha < 2^\varkappa$) of rank $\varkappa$ such that $E_R(H_\varkappa^\alpha) = A \oplus E_0(H_\varkappa^\alpha)$,

(ii) if $\phi\colon H_\varkappa^\alpha \to H_\lambda^\beta$ is a homomorphism and $(\alpha, \varkappa) \neq (\beta, \lambda)$ then ϕ has finite rank.

We conclude this introduction with two further remarks:

(*a*) The condition that $\varkappa$ be regular, not-weakly compact cannot be weakened: if $\varkappa$ is singular or weakly compact then a strongly $\varkappa$-free module is free, see e.g. [5].

(*b*) As noted in [8] it follows from an easy modification of results of Eklof [5] on $\aleph_1$-separable groups in $ZFC + MA + \neg CH$ that the results obtained above are independent of ZFC.

2. The algebraic preliminaries.

Before developing the usual Step-Lemmas associated with a construction in $V = L$, we introduce some additional notation and conventions. Let F be a free A-module, $F = \bigoplus_{i \in I} e_i A$ and $x \in F$. The support of x (with respect to the given decomposition for F), $[x]$, is defined by $[x] = \{i \in I \mid a_i \neq 0 \text{ where } x = \sum e_i a_i\}$. Clearly $[x]$ is a finite subset of I. Moreover if $y \in \hat{F}$ then it is well known that y may be represented as $y = \sum e_i a_i$, where $\{a_i\}$ is a null sequence of elements of $\hat{A}$ and so the support of y may be similarly defined. In this case $[y]$ is a countable subset of I. More generally if X is a subset of F we may define $[X] = \bigcup_{x \in X} [x]$. If $\phi \in E_R(F)$ and G is an A-submodule of F then we define the ϕ-closure of G as follows:

Let $I_0 = [G]$, $I_{n+1} = I_n \cup [\{e_j \phi : j \in I_n\}]$ and set $I_\omega = \bigcup_{n < \omega} I_n$.

Then the ϕ-closure of G is defined by $G^{c\phi} = \bigoplus_{i \in I_\omega} e_i A$. Clearly $G \leq G^{c\phi}$ and the latter is a canonical summand of F which is invariant under ϕ. Moreover if G has infinite rank then $G^{c\phi}$ has rank equal to $\mathrm{rk}(G)$.

Algebraic terminology follows the standard works of Fuchs [7] with the exception that maps are written on the right and the symbol $\sqsubset$ is used to denote a direct summand. Notice that if C is a submodule of G then we write C_* for the pure submodule of G generated by C.

Let F be a free A-module, $F = \bigoplus_{i \in I} e_i A$, with a strictly increasing chain of summands $\{F_n\}$, say $F_{n+1} = F_n \oplus D_n$. Then an element $y \in \hat{F}$ is said to be a *branch* (relative to the chain of summands) if there exist basis elements $e_n \in D_n$ such that $y = \sum e_n p^n$. A pair of elements (z, y) is said to be *branch-like* (relative to the chain of summands) if y is a branch and $z = x + y$ where $x \in \hat{F}$ and $[x] \cap [y] = \emptyset$. Note that x will then have the form $x = \sum_{k \in K} e_k a_k$ where $\{a_k\}$ is a null sequence of elements of $\hat{A}$. Finally we say that the free module F has a chain of summands of type I if, for each element $y \in \hat{D}$, where $D = \oplus D_n$, there exists a branch $y' \in \hat{D}$ with $[y] \cap [y'] = \emptyset$. It seems worth remarking on the similarity between the branch elements introduced above and the concept of branch used in *ZFC* constructions such as [1]; an important difference in the $V = L$ construction is the necessity preserving a given of summands.

LEMMA 1. If F is a free A-module with a strictly ascending chain of summands $\{F_n\}$ and $y \in \hat{F}$ is a branch, then $F' = \langle F, yA \rangle_* \leq \hat{F}$ is a free A-module and $F_n \sqsubset F'$ for all n.

PROOF. Let $F_{n+1} = F_n \oplus D_n$, hence $F = F_0 \oplus \bigoplus_{n<\omega} D_n$ and $D_n \neq 0$. Suppose that $y = \sum_n e_n p^n$ is a branch. We may write $D_n = C_n \oplus \langle e_n A \rangle$; observe that $F_n = F_0 \oplus \bigoplus_{j<n} C_j \oplus \bigoplus_{j<n} \langle e_j A \rangle$. Let $F^* = F_0 \oplus \bigoplus_{n<\omega} C_n$ so that $F = F^* \oplus B$ where $B = \bigoplus_{j<\omega} \langle e_j A \rangle$. Define elements $y^n \in \hat{B}$ by $y^n = \sum_{j \geq n} e_j p^{j-n}$; note that $y^0 = y$.

We shall show that $X = F^* \oplus \bigoplus_{n<\omega} \langle y^n A \rangle$ is equal to F'. That the sum in X is direct is a simple (and standard) exercise in elementary linear algebra. Moreover since $y^n - py^{n+1} = e_n \in F$, it is immediate that $F \leq X$. The purification of $\langle F, yA \rangle$ ensures that X is contained in F' and so we establish the reverse inclusion. If $g \in F'$ then $p^N g = f + ya$ for some $a \in A$, $f \in F$ and $N < \omega$. But now $p^N y^N = y - (e_0 p^0 + \ldots + e_{N-1} p^{N-1})$ and so $p^N (g - y^N a) \in F$. It follows by purity of F in $\hat{F}$ that $g = y^N a + f_0$ for some $f_0 \in F$. Since $F \leq X$ this gives $F' \leq X$ and hence equality. It only remains to prove that each F_n is a direct summand of $X = F'$. It clearly suffices to show that $\bigoplus_{j<n} \langle e_j A \rangle$ is a direct summand of $\bigoplus_{n<\omega} \langle y^n A \rangle$. We establish this by showing that

$$\bigoplus_{j<\omega} \langle y^j A \rangle = \bigoplus_{j<n} \langle e_j A \rangle \oplus \bigoplus_{j \geq n} \langle y^j A \rangle .$$

The sum on the *RHS* of the above expression is direct since if $\sum_{j<n} e_j a_j = \sum_{j \geq n} y^j b_j$, then it follows immediately by examining supports that the a_j, b_j are zero. Also it is clear (compare the earlier argument relating to F) that the $RHS \subseteq LHS$. So we complete the proof of the lemma by establishing that for $i < n$, $y^i \in RHS$. Observe that $y^{n-1} - py^n = e_{n-1}$ and so $y^{n-1} \in RHS$. But then we have $y^{n-2} = py^{n-1} + e_{n-2}$ and so $y^{n-2} \in RHS$. Continuing this process completes the proof.

LEMMA 2. If F is a free A-module with a strictly ascending chain of summands $\{F_n\}$ and the pair (z, y) is branch-like, then $F' = \langle F, zA \rangle_* \leq \hat{F}$ is a free A-module and F_n is a direct summand of F' for all $n \in \omega$.

PROOF. Using the same notation as in Lemma 1, suppose $F = F^* \oplus B$, $z = x + y$ with $y \in \hat{B}$ and $x \in \hat{F}^*$. Let $F^* = \tilde{F} \oplus \bigoplus_{k \in K} \langle f_k A \rangle$ (and then $x = \sum_{k \in K} f_k a_k$) where K is a countable set and we have used basis elements f_k purely for notational convenience. Clearly we may identify K with ω. Since $\{a_k\}$ is a null sequence, for each $n < \omega$ there is an integer N_n such that $p^n \mid a_j$ for all $j \geq N_n$. Set $x^0 = x$ and for each $n \geq 1$, set $x^n = \sum_{j > N_n} f_j a_j p^{-n} + u^n$, where u^n is still to be defined. The definition of

u^n requires us to look at the «fine structure» of the algebra A. Since A is an R-algebra and R is a complete discrete valuation ring, there is a free R-module S such that $S \leqslant_* A \leqslant_* \widehat{S}$. Thus we may write $a_j = \sum s_{jv} r_v p^{n_{jv}}$ where v ranges through a countable set (again without loss we may take ω), $n_{jv} \to \infty$ as $v \to \infty$, $r_v \in R$ and the s_{jv} are basis elements of S. Now for each n, there exists v_n such that $n < n_{jv}$ for all $v \geqslant v_n$. Define $a_j^n = \sum_{v \geqslant v_n} s_{jv} r_v t_v p^{n_{jv} - n}$ and observe that for each n,

$$(*) \qquad p^n a_j^n = a_j - \sum_{v < v_n} s_{jv} r_v p^{n_{jv}} = a_j - c_j^n \qquad \text{where } c_j^n \in A .$$

Define $u^n = \sum_{j \leqslant N_n} f_j a_j^n$ and observe the following relationship:

$$p^{n+1} x^{n+1} - p^n x^n = \sum_{j > N_{n+1}} f_j a_j + p^{n+1} u^{n+1} - \sum_{j > N_n} f_j a_j - p^n u^n =$$

$$= - \sum_{N_n + 1}^{N_{n+1}} f_j a_j + \sum_{j \leqslant N_{n+1}} f_j p^{n+1} a_j^{n+1} - \sum_{j \leqslant N_n} f_j p^n a_j^n =$$

$$= - \sum_{N_n + 1}^{N_{n+1}} f_j a_j + \sum_{N_n + 1}^{N_{n+1}} f_j a_j^{n+1} + \sum_{j \leqslant N_n} f_j (p^{n+1} a_j^{n+1} - p^n a_j^n) .$$

On applying $(*)$ to these expressions we see that

$$p^{n+1} x^{n+1} - p^n x^n = - \sum_{N_n + 1}^{N_{n+1}} f_j c_j^{n+1} + \sum_{j \leqslant N_n} f_j (c_j^n - c_j^{n+1}) \in \bigoplus_{j \in K} f_j A < F^* .$$

Now claim $F' = \langle F, zA \rangle_* \leqslant \widehat{F}$ is free and equal to $X = F^* \oplus \bigoplus_{n < \omega} z^n A$ where $z^n = x^n + y^n$. We show this in a number of simple steps:

(i) The sum in X is direct. It suffices clearly to show that $\bigoplus_{k \in K} \langle f_k A \rangle \oplus \bigoplus_{n < \omega} \langle z^n A \rangle$ really is direct. Now $\sum f_n a_n + \sum z^n \alpha_n = 0$ (where $a_n, \alpha_n \in A$) implies $\sum f_n a_n + \sum x^n \alpha_n + \sum y^n \alpha_n = 0$. But the supports of the terms y^n lie in B while other terms have supports in F^* and so $\sum y^n \alpha_n = 0$. Thus, as observed in Lemma 1, $\alpha_n = 0$ for all n. This in turn forces $a_n = 0$ for all n and so the sum is direct.

(ii) $F \leqslant X$: Clearly it suffices to show that $B \leqslant X$. Now, for each n, $e_n = y^n - p y^{n+1}$ and so $z^n - p z^{n+1} = e_n + (x^n - p x^{n+1})$. However as observed above $(x^n - p x^{n+1}) \in F^*$ and hence $e_n \in X$.

(iii) $z^n \in F'$ for each n: Note that $p^n z^n = p^n y^n + p^n x^n$ and $p^n y^n = y - s_n$ where $s_n \in F$. Moreover $x - p^n x^n = \sum_{j \leqslant N_n} f_j a_j - p^n \sum_{j \leqslant N_n} f_j a_j^n = \sum_{j \leqslant N_n} f_j (a_j - p^n a_j^n) = \sum_{j \leqslant N_n} f_j c_j^n = \bar{f} \in F$. Thus $p^n z^n = (y - s_n) + x - \bar{f} \in \langle F, zA \rangle$ and so $z^n \in \langle F, zA \rangle_*$.

(iv) $F' \leqslant X$: If $g \in F'$ then $p^n g = f + za$, some $f \in F$, $a \in A$. But $z = p^n z^n + (s_n + \bar{f})$ and so $p^n(g - z^n a) = f + (s_n + \bar{f})a \in F$. Hence $g - z^n a \in F$ and so, since $F \leqslant X$ by (ii), $g \in X$ follows from (iii).

It only remains to show that $F_n \sqsubset F'$ for all n. To do this it suffices to show that $\bigoplus_{i<n} \langle e_i A \rangle \sqsubset X$ for each n. Claim

$$F' = X = F^* \oplus \bigoplus_{i<n} \langle e_i A \rangle \oplus \bigoplus_{i \geqslant n} \langle z^i A \rangle .$$

Observe firstly, by a simple support argument, that this sum is direct and that the *RHS* is certainly in X. The proof will be complete if we show $z^i \in RHS$ for $i < n$. However

$$z^{n-1} - pz^n = (x^{n-1} - px^n) + (y^{n-1} - py^n) = e_{n-1} + f$$

where $f \in F^*$ and so $z^{n-1} = pz^n + e_{n-1} + f \in RHS$. Similarly $z^{n-2} - pz^{n-1} = e_{n-2} + f_1$ etc. This completes the proof of Lemma 2.

REMARK. The reader familiar with the realization problem in *ZFC* will see an immediate relationship between the concept of «branch-like» introduced above and the techniques used by Corner and Göbel to produce the «Recognition Lemma» in § 3 of [1] or the earlier concept of λ-high element used by Dugas and the present authors in § 2 of [4].

STEP-LEMMA A. Let F be a free A-module with a strictly ascending chain of summands $\{F_n\}$ of type I. Then if ϕ: $F \to G$, a torsion-free A-module, is a homomorphism which is not inessential then there exists a free A-module F' containing F such that

(i) F'/F is a torsion-free divisible rank one A-module isomorphic to $Q \otimes A$,

(ii) ϕ does not extend to a homomorphism : $F' \to G$,

(iii) $F_n \sqsubset F'$ for all $n < \omega$.

PROOF. If there exists a branch y in $\hat{F}$ such that $y\phi \notin G$ then we choose $F' = \langle F, yA \rangle_* \leqslant \hat{F}$ and the result follows from Lemma 1 and the observation that F'/F is divisible of rank 1. Hence we assume that $y\phi \in G$ for all branches y in $\hat{F}$. If there is an element $x \in \hat{F}_0$ such that $x\phi \notin G$ then, writing $F_{n+1} = F_n \oplus D_n$ and $D = \bigoplus_{n<\alpha} D_n$, choose a branck-like pair (z, y) where $y \in \hat{D}$ is a branch. Now, by our assumption on branches, $z = x + y$ has the property that $z\phi \notin G$. So if we set $F' = \langle F, zA \rangle_* \leqslant F$ the result follows from Lemma 2. If no such x exists in

$\widehat{F}_0$ then, since $F = F_0 \oplus D$, we conclude that there is an element $x \in \widehat{D}$ with $x\phi \notin G$. Since the chain of summands is of type I we may choose a branch $y \in \widehat{D}$ with $[x] \cap [y] = \emptyset$. But then if we set $z = x + y$ and choose $F' = \langle F, zA \rangle_* \leq \widehat{F}$, the result follows from Lemma 2.

LEMMA 3. Let F be a free A-module of infinite rank and assume that $\phi \in E(F) \setminus A \oplus \operatorname{Ines} F$, then there exists a canonical summand P of F such that

(i) $\operatorname{rk}(P) \leq \aleph_0$;

(ii) $\phi_P = \phi \restriction P \in E(P)$;

(iii) $\phi_P \notin A \oplus \operatorname{Ines} P$.

PROOF. Suppose not and let B be any countable rank summand of F. Set $B_0 = B^{c\phi}$, the ϕ-closure of B. Clearly B_0 is a canonical summand of F and satisfies (i) and (ii). So $\phi \restriction B_0 \in A \oplus \operatorname{Ines} B_0$. Observe that this means there is a *unique* $a \in A$ such that $\widehat{B}_0(\phi - a) \leq B_0$. (If two such elements a_1, a_2 existed then subtraction would force $\widehat{B}_0(a_2 - a_1)$ to lie in B_0 which is impossible.) Since $\phi - a \notin \operatorname{Ines} F$, there exists $x \in \widehat{F}$ such that $x(\phi - a) \notin F$. Let $B_1 = \langle B_0, x \rangle^{c\phi}$, a canonical summand of F which clearly satisfies (i) and (ii). By assumption then $\phi \restriction B_1 \in A \oplus \operatorname{Ines} B_1$. This however leads to a contradiction since we can establish that $\widehat{B}_1(\phi - b) \not\leq B_1$ for *all* $b \in A$. To see this suppose, on the contrary, that $\widehat{B}_1(\phi - b) \leq B_1$ for some $b \in A$. However since $B_0 \leq B_1$, $\widehat{B}_0(\phi - b) \leq \leq B_1 \cap \widehat{B}_0 = B_0$. The uniqueness of the element a then forces $b = a$. But then $x \in \widehat{B}_1$ and $x(\phi - b) = x(\phi - a) \notin B_1$—contradiction. This establishes the lemma.

Any summand P satisfying the conditions (i) to (iii) in Lemma 3 will be called a ϕ-canonical summand of F.

LEMMA 4. If F is a free A-module of infinite rank then $A \oplus \operatorname{Ines} F$ is pure in $E(F)$.

PROOF. Since F has infinite rank, the sum $A + \operatorname{Ines} F$ is certainly direct. Assume $p^n \phi \in A \oplus \operatorname{Ines} F$ for some n. Thus there is an $a \in A$ with $p^n \phi - a$ inessential i.e. $\widehat{F}(p^n \phi - a) \leq F$. Since the term on the *LHS* of this inequality is complete and F is free of infinite rank, we conclude $\widehat{F}(p^n \phi - a) \leq F$ is a finite rank A-module. Choose a canonical summand xA of F such that $xA \cap \widehat{F}(p^n \phi - a) = 0$. But then $x(p^n \phi) = x(p^n \phi - a) + + xa$ and a simple support argument shows $p^n \mid a$. So $p^n \phi - a = p^n(\phi - - a')$, where $a' \in A$. But now $\widehat{F}(\phi - a') \leq F$ since F is pure in $\widehat{F}$, and so $\phi \in A \oplus \operatorname{Ines} F$ as required.

STEP-LEMMA B. Let F be a free A-module with a strictly ascending chain of summands $\{F_n\}$ and suppose $\phi \in E(F) \setminus A \oplus \operatorname{Ines} F$. Then if F has a ϕ-canonical summand of infinite corank, there exists a free A-module F' containing F such that

(i) F'/F is a torsion-free, divisible rank one A-module isomorphic to $Q \otimes A$,

(ii) ϕ does not extend to an endomorphism of F',

(iii) $F_n \sqsubset F'$ for all $n < \omega$.

PROOF. By assumption we may write $F = P \oplus B$ where P is a ϕ-canonical summand and B is of infinite rank. To prove the lemma it suffices (see Lemmas 1 and 2) to find either a branch y in $\widehat{B}$ with $y\phi \notin \langle B, yA \rangle_* \leqslant \widehat{B}$ or a branch-like pair (z, y) such that $z\phi \notin \langle F, zA \rangle_* \leqslant \widehat{F}$.

If we can find such a branch y then we choose this and we are finished. If not, then for every branch y in $\widehat{B}$ there exists a pair $(n, a) \in \omega \times A$ such that $y(p^n \phi - a) \in B$. However it follows from Lemma 4 that $(p^n \phi - a) \notin \operatorname{Ines} P$ for any pair (n, a) and so we can deduce that there exists an element $x_{na} \in \widehat{P}$ with $x_{na}(p^n \phi - a) \in \widehat{P} \setminus P$. Now consider the branch-like pair (z, y) where $z = y + x_{na}$. We claim $z\phi \notin \langle F, zA \rangle_*$. For if not, there exists a pair $(m, c) \in \omega \times A$ such that $z(p^m \phi - c) \in F$. By absorbing appropriate powers there is no loss in generality in assuming $m = n$. Thus we have $(y + x_{na})(p^n \phi - c) \in B \oplus P$ and since $y(p^n \phi - a) \in B$, we get on subtraction, that $y(a - c) + x_{na}(p^n \phi - c) \in B \oplus P$. However since y is a branch in B, $y(a - c) \in \widehat{B} \setminus B$. But

$$x_{na}(p^n \phi - c) = x_{na}(p^n \phi - a) + x_{na}(a - c) \in \widehat{P}$$

and $[x_{na}(p^n \phi - c)]$ is disjoint from $[y(a - c)]$ and so we must conclude that $a = c$ which gives $x_{na}(p^n \phi - a)$ in $B \oplus P \cap \widehat{P} = P$—contradiction. This completes the proof of Step-Lemma B.

3. The proof of Theorem 1.

Having obtained the algebraic step-lemmas we are now in a position to carry out the by now standard construction of the required A-module. It is only at this point that we make use of $V = L$; in fact we only use Jensen's Diamond Principle $\Diamond$ which is known to hold in $V = L$. (See [5] or [10].) The construction is via induction and is similar to previous proofs in [2], [3], [6], [9] and especially [8]. Let $\varkappa$ be the given regular, not-weakly compact cardinal $> |A|$ and let $H = \bigcup_{\alpha < \varkappa} H_\alpha$ be a $\varkappa$-filtration of a set H of cardinality $\varkappa$ such that $|H_0| = |A|$. Choose, as we may since we are assuming $V = L$, a sparse stationary subset $E \subsetneq \{\lambda < \varkappa | \, cf(\lambda) = \omega\}$ and partition E into pairwise disjoint stationary subsets, $E = E_e \cup E_k \cup \bigcup_{\alpha < \varkappa} E_\alpha$. (For a proof of the permissibility of this see e.g. [10].) Then assuming $\Diamond_\varkappa(E_\alpha)$ for $\alpha \in \{e, k\} \cup \varkappa$, we derive a sequence of Jensen functions and Jensen sets

$$\{\phi_\alpha : H_\alpha \to H_\alpha \,|\, \alpha \in E_e\}, \qquad \{U_\alpha \subsetneq H_\alpha \,|\, \alpha \in E_k\}$$

and sets of the form $\{(\phi_\alpha, +_\alpha, \cdot_\alpha) \subsetneq H_\alpha^7 \times A | \, \alpha \in E_\alpha\}$; these latter sets are supposed to guess the additive and scalar multiplicative structure on H_α and homomorphisms of these modules. (Thus $+_\alpha \subsetneq H_\alpha^3$ etc.; see [8] for further details.)

For each $\alpha < \varkappa$ we define an A-module structure on H_α and our desired module shall be $H = \bigcup_{\alpha < \varkappa} H_\alpha$.

Our inductive construction proceeds as follows:

1) $H_0 = \oplus A$, a free A-module of rank $\aleph_0$ and each $H_\alpha (\alpha < \varkappa)$ is a free A-module.

2) If α is a limit then $H_\alpha = \bigcup_{\beta < \alpha} H_\beta$.

3) If $\alpha < \beta < \varkappa$ and $\alpha \notin E$ then H_α is a proper summand of H_β.

4) If H_α has been defined let

(4.0) $H_{\alpha+1} = H_\alpha \oplus A$ except in the following cases

(4.1) If $\alpha \in E_e$ and $\phi_\alpha : H_\alpha \to H_\alpha$ is an R-homomorphism not in $A \oplus \operatorname{Ines} H_\alpha$ and H_α has a ϕ_α-canonical summand of corank at least $\aleph_0$, then we choose a sequence $\alpha_n \in \varkappa \setminus E$ which is strictly increasing to α. Since $\alpha_n \notin E$ it follows from (3) that $H_{\alpha_n} \sqsubset H_{\alpha_{n+1}} \sqsubset \dots \sqsubset H_\alpha$. Now apply Step-Lemma B to obtain a free A-module $F' = H_{\alpha+1} \supsetneq H_\alpha$ such that ϕ_α does not extend to an endomorphism of $H_{\alpha+1}$. Notice that Step-Lemma B ensures that (3) remains satisfied at this $(\alpha+1)\,st$ stage since if $\gamma < \alpha$ and $\gamma \notin E$ then there exists α_n with $\gamma < \alpha_n$. Thus $H_\gamma \sqsubset H_{\alpha_n} \sqsubset H_{\alpha+1}$.

(4.2) If $\alpha \in E_k$ and $U_\alpha \subsetneq H_\alpha$ is a submodule then let $\pi : H_\alpha \to H_\alpha / U_\alpha$ be the canonical projection. If π is a homomorphism which is not inessential and H_α has a chain of summands of type I then we apply Step-Lemma A to obtain an extension $H_{\alpha+1}$ of H_α. Moreover if α_n is strictly increasing with limit α then $H_{\alpha_n} \sqsubset H_{\alpha+1}$ and π does not lift to an R-homomorphism from $H_{\alpha+1}$. Then as in (4.1) above, we have ensured that (3) remains satisfied at this stage.

(4.3) If $\alpha \in E_\gamma$ for some $\gamma \in \varkappa$ and $(H_\alpha, +_\alpha, \cdot_\alpha)$ is an A-module such that $\phi_\alpha : H_\alpha \to (H_\alpha, +_\alpha, \cdot_\alpha)$ is a homomorphism which is not inessential, then we construct $H_{\alpha+1}$ via Step-Lemma A as in (4.2) above. Once again (3) is preserved.

Finally we show that, if α is a limit ordinal, then $H_\alpha = \bigcup_{\alpha<\beta} H_\beta$ is free. Since E is sparse, $\alpha \cap E$ is not stationary in α and so there exists a cub $C \subseteq \alpha$ with $C \cap E = \emptyset$. But then $H_\alpha = \bigcup_{\delta \in C} H_\delta$ and since $\delta \in C$, is a summand of H_γ for any $\delta < \gamma < \alpha$ by 3). Thus H_α is the union of a chain of summands, and so is free.

The inductive construction is thus consistent and we obtain the desired A-module $H = \bigcup_{\alpha<\varkappa} H_\alpha$. It follows rather easily easily (see e.g. in [9]) that H is a strongly $\varkappa$-free A-module. It remains to show that $E(H) = A \oplus \operatorname{Ines} H$. Before showing this we derive the following simple, but useful, lemma.

LEMMA 5. Let $F = \bigcup_{\alpha<\varkappa} F_\alpha$ be a $\varkappa$-filtration of a torsion-free A-module F and ϕ: $F \to G$ an R-homomorphism into the torsion-free A-module G. If $S = \{\nu \in \varkappa \mid \phi \restriction F_\nu \text{ is inessential}\}$ is unbounded in $\varkappa$ and $cf(\varkappa) > \omega$, then ϕ is inessential.

PROOF. We prove firstly that if $x \in \widehat{F}$ then $x \in \widehat{F}_\alpha$ for some $\alpha < \varkappa$. Now $x \in \widehat{F}$ implies $x = \lim x_n$ where $x_n \in F$ and so $x_n \in F_{\alpha_n}$ for some $\alpha_n < \varkappa$. If the sequence $\{\alpha_n\}$ is unbounded in $\varkappa$ then there exists a strictly increasing subsequence which we may, without loss, label as $\alpha_1 < \alpha_2 < \ldots$. However $cf(\varkappa) > \omega$ and since S is unbounded there exists $\nu \in S$ with $\alpha_n < \nu$ for all n. But then $x \in \widehat{F}_\nu$ as required. But it then follows immediately from the fact that S is unbounded that $\widehat{F}\phi \leqslant G$.

We are now in a position to establish Theorem 1. Clearly $A \oplus \operatorname{Ines} H \subseteq E(H)$. Suppose there exists $\phi \in E(H) \setminus A \oplus \operatorname{Ines} H$. Let $C = \{\alpha < \varkappa \mid H_\alpha \phi \leqslant H_\alpha\}$ and note that C is clearly a cub in $\varkappa$. Denote by C_0, C_1 respectively, the sets

$$\{\alpha < \varkappa \mid \phi \restriction H_\alpha \in A \oplus \operatorname{Ines} H_\alpha\} \text{ and } \{\alpha < \varkappa \mid \phi \restriction H_\alpha \notin A \oplus \operatorname{Ines} H_\alpha$$

$$\text{and } H_\alpha \text{ does not have a } \phi \restriction H_\alpha\text{-canonical summand of corank} \geqslant \aleph_0\}.$$

Claim C_0 and C_1 are bounded. C_1 is bounded since it follows from Lemma 3 that ϕ-canonical summands are of countable rank and the construction in (4.0) ensures that for any $\alpha \in C_1$ an appropriate canonical summand exists at the stage $\alpha + \omega$.

For each $\nu \in C_0$ there exists $a_\nu \in A$ such that $(\phi - a_\nu) \restriction H_\alpha$ is in $\operatorname{Ines} H_\alpha$. Thus if $\nu < \mu \in C_0$, then $(\phi - a_\nu) \restriction H_\nu$ and $(\phi - a_\mu) \restriction H_\mu$ are both inessential and so, by subtraction, we obtain that $a_\nu - a_\mu$ is inessential on H_ν—impossible unless $a_\nu = a_\mu$ (= a say). But then $(\phi - - a) \restriction H_\nu$ is inessential for all $\nu \in C_0$. If C_0 were unbounded then it would follow from Lemma 5 that $\phi - a$ is inessential—contradiction. Thus C_0 is bounded.

Let

$$C^* = C \setminus (C_0 \cup C_1); \qquad C^* \text{ is still a cub in } \varkappa.$$

Now applying $\Diamond_\varkappa(E_e)$ we have that $D_e = \{\nu \in E_e \mid \phi_\nu = \phi \restriction H_\nu\}$ is stationary in $\varkappa$. But then we can conclude that there exists $\alpha \in D_e \cap C^*$. So $\phi_\alpha = \phi \restriction H_\alpha : H_\alpha \to H_\alpha$. By the construction (4.1) ϕ_α does not lift to a homomorphism $\phi' : H_{\alpha+1} \to H_{\alpha+1}$. However we know from the construction of $H_{\alpha+1}$ via Step-Lemma B that $H_{\alpha+1}/H_\alpha$ is divisible. It follows from our construction of H that $H/H_{\alpha+1}$ is a $\varkappa$-free A-module and so $H_{\alpha+1}$ is the closure of H_α in H. This immediately ensures that ϕ will extend uniquely to a homomorphism $H_{\alpha+1} \to H_{\alpha+1}$ and so $\phi \restriction H_\alpha = \phi_\alpha$ lifts —contradiction. Clearly no such ϕ exists and so $E(H) = A \oplus \operatorname{Ines} H$. This completes the proof of Theorem 1.

REMARK. The proof of Theorem 3 (ii) requires a similar argument to the above but using Step-Lemma A. In this case the set C_1 is replaced by $C_2 = \{\alpha < \varkappa \mid \phi \restriction H_\alpha$ is not inessential and H_α does not have a chain of summands of type $I\}$. Again this set is bounded since the construction in (4.0) ensures that an appropriate type I chain can be found at stage $\alpha + \omega$.

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