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Strong Maehara and Takeuti Type Interpolation Theorems for $L^{2+}_{k,k}$

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0. Introduction.

After Malitz's negative results, a way to deal with interpolation theorems for infinitary languages was opened by Karp with her notion of ω-satisfiability.

Taking advantage of such a generalization of the notion of satisfiability, several interpolation theorems were proved. Karp herself extended to the infinitary language $L_{k,k}$, where k is a strong limit cardinal of cofinality ω, Craig's interpolation theorem. Along the same lines I proved the extension to the same language of the interpolation theorem of Maehara and Takeuti. These two results were not as complete as desired for they needed the use of the standard notion of satisfiability in the definition of the interpolant. The reason for such a limitation is to be found in the way individual constant symbols are thought of in their interpretation. Changing the manner of interpreting constants within the notion of ω-satisfiability Cunningham was able to improve Karp's interpolation theorem as to use ω-satisfiability also in the definition of the interpolant. With Cunningham's approach to constants it seems rather difficult to achieve also an improvement of the extension of the interpolation theorem of Maehara and Takeuti [9].

So I introduced a different way of interpreting constants within the notion of ω-satisfiability, which is closer to Karp's point of view than Cunningham's notion.

Again with this notion I was able to prove a theorem formally equal to that of Cunningham, but not equivalent to it if there are constants in the language, since the notions involved are not the same.

The main reason to introduce this last manner of viewing constants

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was the hope of proving also an interpolation theorem in the style of Maehara and Takeuti with ω-satisfability also in the definition of the interpolant.

As I will show in this paper, it turned out that a combination of different point of view about constants is what is needed to carry out the proof of the wanted result.

Since we will deal with an extension of the interpolation theorem of Maehara and Takeuti, let us first state their original result.

Let S be a valid sentence of L^{2+} in which each occurrence of a variant of $G(A)$ is positive. Then there is a first order formula $C(A)$ whose free variables are all in A such that $\models C(A) \rightarrow G(A)$ and $\models S'$ where S' is the sentence obtained from S substituting for each variant $G(A/f)$ of $G(A)$ occurring in S the corresponding variant $C(A/f)$ of $C(A)$.

1. Preliminaries.

We will work in positive second order infinitary languages $L^{2+}_{k,k}$, where k is a strong limit cardinal of cofinality ω, $k = \cup\{k_n : n \in \omega\}$ where $2^{k_n} \leqslant k_{n+1}$.

We will adopt the notations and conventions set forth in [4], [5] and [6] except for the changes and additions mentioned below.

The occurrence of a subformula within a formula is positive if it is in the scope of an even number of negation symbols. A formula is first order if no predicate variable is quantified in it.

The metavariables are symbols not in our language, k for each number of places, used as the variables but never quantified to build new formulas called metaformulas. Metavariables and metaformulas aren't but a convenient way of dealing with substitutions of variants of subformulas.

In the following L will denote the language we start off with, and without loss of generality we may assume that there are no constants in L; indeed constants are viewed exactly as variables over which we decide not to quantify. For each natural number n, C_n, will denote a set of individual symbols not in L, that we will call individual constants, such that $|C_n| = k_n$ and if $m \neq n$ then $C_m \cap C_n = \emptyset$. The only point in which individual constants are treated differently then individual variables is in the definition of pseudo satisfiability. Similarly, for any natural numbers n and j, C^j_n will denote a set of j-placed predicates not in L, that we will call predicate constants, such that $|C^j_n| = k_n$ for each j and if $m \neq n$ then $C^j_m \cap C^j_n = \emptyset$.

L_n will denote the language obtained from L by adding the constants of $(\cup\{C_m: m \leqslant n\}) \cup (\cup\{C_m^j: m \leqslant n, j \in \omega\})$ whereas L' will denote the language $\cup\{L_n: n \in \omega\}$.

By even subformula of a formula we mean a subformula that occurs positively in the formula. By even immediate subformula of a formula we mean a proper even subformula which is not a proper even subformula of any proper even subformula of the given formula.

The notion of good ω-sequence is slightly changed from [6] in that the clause concerning the existential quantification of individual variables is extended to include predicate variables. The complete definition is now as follows.

An ω-sequence $S = \langle s_n: n \in \omega\rangle$ of sets s_n of sentences is called a good ω-sequence of sets of sentences if

a) $|\cup\{s_n: n \in \omega\}| \leqslant k$, and

b) for all $n > 0$ all the sentences in s_n are of the form $-F(V_F/f)$ where f is a 1-1 place preserving total function from V_F into $C_n \cup (\cup\{C_n^j: j \in \omega\})$ and the sentence $-\forall V_F F$, where V_F is a set of individual or predicate variables, belongs to $\cup\{s_{n'}: n' < n\}$, and

c) there is a natural number n' such that for all $n > n'$ we have that $|s_n| \leqslant |s_{n'}| \leqslant k_{n'}$, and

d) for all $n > 0$ $s_n \subset Stmt(L_n)$.

If $S = \langle s_n: n \in \omega\rangle$ is a good ω-sequence of sets of sentences we let $\boldsymbol{S}$ denote the set of sentences $\cup\{s_n: n \in \omega\}$ and $\boldsymbol{S}^m$ denote the set of sentences $\cup\{s_n: n \leqslant m\}$.

$\mathfrak{M} = \langle M_n: n \in \omega\rangle$, where each M_n is a set and if $m < n$ then $M_m \subseteq M_n$, will stand for an ω-chain of structures adequate for the language L. ω-chains of structures adequate for other languages as L_n will be obtained from $\mathfrak{M}$ adding an assignment $\boldsymbol{a}$ to the constants; such an assignment will usually be bounded for the individual constants (i.e. its range will be included in M_n for some $n \in \omega$), it may be locally bounded (see below) in connection with pseudo satisfiability.

We say that an ω-chain of structures $\mathfrak{M}$ and an assignment $\boldsymbol{a}$ to the constants of a good ω-sequence $S = \langle s_n: n \in \omega\rangle$ of sets of sentences ω-satisfy the good ω-sequence S, $\mathfrak{M}, \boldsymbol{a} \models^\omega S$, if for all $p \in \omega$ we have that $\mathfrak{M}$, $\boldsymbol{a}_p \models^\omega \cup\{s_n: n \leqslant p\}$ where $\boldsymbol{a}_p$ is a bounded assignment which is the restriction of $\boldsymbol{a}$ to the constants occurring in s_n for $n \leqslant p$.

A good ω-sequence S is said to be ω-satisfiable if there are an ω-chain of structures $\mathfrak{M}$ and an assignment $\boldsymbol{a}$ such that $\mathfrak{M}, \boldsymbol{a} \models^\omega S$.

Also the notion of seq-consistency property is changed in condition C6) to take care of the second order symbols. Now it is as follows.

Σ is a seq-consistency property for L with respect to $\{C_n: n \in \omega\}$ and to $\{C_n^j: n \in \omega, j \in \omega\}$ if Σ is a set of good ω-sequences $S = \langle s_n: n \in \omega\rangle$ of sets s_n of sentences such that all of the following conditions hold.

C0) If Z is an atomic sentence then either $Z \notin \boldsymbol{S}$ or $-Z \notin \boldsymbol{S}$ and if Z is of the form $-(t=t)$, t a constant, then $Z \notin \boldsymbol{S}$.

C1) Suppose $|I|<k$,

a) if $\{c_i = d_i: i \in I\} \subset s_0$ and $\langle s_n: n \in \omega\rangle = S \in \Sigma$ with c_i and d_i constants, then the good ω-sequence $S' = \langle s_n': n \in \omega\rangle$ such that $s_0' = s_0 \cup \{d_i = c_i: i \in I\}$ and $s_n' = s_n$ for $n>0$, $n \in \omega$, belongs to Σ;

b) if $\{Z_i(c_i),\ c_i = d_i: i \in I\} \subset \boldsymbol{S}^m$ for some $m \in \omega$ and $\langle s_n: n \in \omega\rangle = S \in \Sigma$ and Z_i are atomic or negated atomic sentences and c_i and d_i are constants, then the good ω-sequence $S' = \langle s_n': n \in \omega\rangle$ such that $s_0' = s_0 \cup \{Z_i(d_i): i \in I\}$ and $s_n' = s_n$ for $n>0$, $n \in \omega$, belongs to Σ.

C2) If $\{--F_i: i \in I\} \subset \boldsymbol{S}^m$ for some $m \in \omega$, and $\langle s_n: n \in \omega\rangle = S \in \Sigma$ and $|I|<k$, then the good ω-sequence $S' = \langle s_n': n \in \omega\rangle$ such that $s_0' = s_0 \cup \{F_i: i \in I\}$, $s_n' = s_n$ for $n>0$, $n \in \omega$, belongs to Σ.

C3) If $\{\&\boldsymbol{F}_i: i \in I\} \subset s_0$ and $|I|<k$ and there is $m' \in \omega$ such that $0<|\boldsymbol{F}_i|<k_{m'}$ for all $i \in I$ and $\langle s_n: n \in \omega\rangle = S \in \Sigma$ then the good ω-sequence $S' = \langle s_n': n \in \omega\rangle$ such that $s_0' = s_0 \cup \{F: F \in \boldsymbol{F}_i, i \in I\}$, $s_n' = s_n$ for $n>0$, $n \in \omega$, belongs to Σ.

C4) If $\{\forall \boldsymbol{v}_i F_i: i \in I\} \dot{\subset} s_0$ and $|I|<k$ and $\boldsymbol{v}_i$ is a set of individual variables and there is $m' \in \omega$ such that $0<|\boldsymbol{v}_i|<k_{m'}$ for all $i \in I$, and $\langle s_n: n \in \omega\rangle = S \in \Sigma$, and n' is the natural number mentioned in *c)* of the definition of good ω-sequence relative to S, then the good ω-sequence $S' = \langle s_n': n \in \omega\rangle$ such that $s_0' = s_0 \cup \{F_i(\boldsymbol{v}_i/f): f$ is a total function from $\cup\{\boldsymbol{v}_i: i \in I\}$ into $\cup\{C_h: h \leqslant n'+1, i \in I\}$, $s_n' = s_n$ for all $n>0$, $n \in \omega$, belongs to Σ.

C5) If $\{-\&\boldsymbol{F}_i: i \in I\} \subset \boldsymbol{S}^m$ for some $m \in \omega$ and $|I|<k$ and there is $m' \in \omega$ such that $0<|\boldsymbol{F}_i|<k_{m'}$ for all $i \in I$, and $\langle s_n: n \in \omega\rangle = S \in \Sigma$, then there is $g \in \times\{\boldsymbol{F}_i: i \in I\}$ such that the good ω-sequences $S' = \langle s_n': n \in \omega\rangle$ such that $s_0' = s_0 \cup \{-g(i): i \in I\}$, $s_n' = s_n$ for all $n>0$, $n \in \omega$, belongs to Σ.

C6) If $\{-\forall \boldsymbol{V}_i F_i: i \in I\} \subset \boldsymbol{S}^m$ for some $m \in \omega$ and there is m' the least natural number such that $m \leqslant m'$ and $|I|<k_{m'}$ and $0<|\boldsymbol{V}_i|<k_{m'}$ for all $i \in I$ and $\boldsymbol{S}^m \subset Stmt(L_{m'})$, and $\langle s_n: n \in \omega\rangle = S \in \Sigma$, then there is an ω-partition $P = \langle I_p: p \in \omega\rangle$ of I such that for any set $\{f_p: p \in \omega\}$ of 1-1 place preserving total functions f_p from $\cup\{\boldsymbol{V}_i: i \in \boldsymbol{I}_p\}$ into

$((C_{m'+p} \cup (\cup\{C^j_{m'+p}: j \in \omega\})) - \{c: c$ is a constant occurring in $S\})$ the good ω-sequence $S' = \langle s'_n: n \in \omega\rangle$ such that $s'_n = s_n$ for $n \leqslant m'$ and for all $p \in \omega$

$$s'_{m'+p+1} = s_{m'+p+1} \cup \{-F_i(\boldsymbol{V}_i/f_p): i \in I_p\}$$

belongs to Σ.

As Baldo showed in [1], the model existence theorem can be proved also for this notion of seq-consistency property, i.e. if S belongs to a seq-consistency property then S is ω-satisfiabile. (But the proof of Baldo's first interpolation theorem is not correct.)

We now introduce some new notions that will be usefull to deal with constants in the proof of the interpolation theorem.

We say that $S = \langle s_n: n \in \omega\rangle$ is a pseudo good ω-sequence of sets of sentences if it satisfies all the conditions for being a good ω-sequence of sets of sentences except for the requirement that all the sentences of s_0 are within a language L_j for some natural number j.

We say that a pseudo good ω-sequence S in L' is pseudo satisfied by the ω-chain of structures $\mathcal{M} = \langle M_n: n \in \omega\rangle$ adequate for L and by the assignment $\boldsymbol{a}$ to the constants of L' if for all φ occurring in S we have that $\mathcal{M}, \boldsymbol{a} \models^{\omega} \varphi$ and furthermore $\boldsymbol{a}$ is such that for each sentence φ occurring in S if $c_\varphi = \{c: c \in L'$ and c occurs in φ and c is an individual constant$\}$ then there is a natural number n such that $\boldsymbol{a}[c_\varphi] \subset M_n$. Such an assignment $\boldsymbol{a}$ is not bounded, but we could call it locally bounded. We also use the notation $\mathcal{M}, \boldsymbol{a} \models^p S$ to say that S is pseudo satisfied by $\mathcal{M}$ and $\boldsymbol{a}$.

We say that the pseudo good ω-sequence S is pseudo satisfiable if there is an ω-chain of structures $\mathcal{M}$ adequate for the language L and a locally bounded assignment $\boldsymbol{a}$ to the constants of L' such that $\mathcal{M}$, $\boldsymbol{a} \models^p S$.

If S is a good ω-sequence of sets of sentences, we can compare the three notions of $\mathbf{S}$ being ω-satisfiable, S being ω-satisfiable and S being pseudo satisfiable. We have that the ω-satisfiability of $\mathbf{S}$ implies the ω-satisfiability of S which in turn implies the pseudo satisfiability of S.

Furthermore the three notions coincide if S has only a finite number of sentences. Also S is pseudo satisfiable if and only if $\mathbf{S}$ is pseudo satisfiable.

But the good ω-sequence $S = \langle s_n: n \in \omega\rangle$ where $s_n = \emptyset$ for $n > 0$ and $s_0 = \{F_j: j \in \omega\}$ is pseudo satisfiable but not ω-satisfiable, and the good ω-sequence $S = \langle s_j: j \in \omega\rangle$ where $s_j = \{F_j\}$ for $j > 0$ and $s_0 = \{\exists\{v_i: i \leqslant j\} F'_j: j \in \omega\}$ is ω-satisfiable but $\mathbf{S}$ is not ω-satisfiable, where

F_j is the sentence

$$- - [(\forall v_{j+1} - P(v_{j+1}, v_{j+1})) \&$$

$$\& (\forall v_{j+2} \forall v_{j+3} (P(v_{j+2}, v_{j+3}) \vee P(v_{j+3}, v_{j+2}) \vee v_{j+2} = v_{j+3})) \&$$

$$\& (\forall v_{j+4} \forall v_{j+5} \forall v_{j+6} ((P(v_{j+4}, v_{j+5}) \& P(v_{j+5}, v_{j+6})) \rightarrow P(v_{j+4}, v_{j+6}))) \&$$

$$\& (\forall v_{j+7} \exists v_{j+8} P(v_{j+7}, v_{j+8})) \&$$

$$\& (\forall \{v_{j+9+n}: n \in \omega\} - \& \{P(v_{j+9+n}, v_{j+10+n}: n \in \omega)\}) \&$$

$$\& (\& \{P(c_i, c_{i+1}): i < j\})]$$

and F_j' is the formula obtained from F_j replacing the variables v_i, $i \leqslant j$, for the constants c_i, $i \leqslant j$.

We remark that.

1) If $\boldsymbol{F} = \{F_{i,j}: i \in I, j \in J_i\}$ ($|I| < k$, $|J_i| < k$ for all $i \in I$) is a set of sentences in the first set of a pseudo good ω-sequence S which is not pseudo satisfiable, then also the pseudo good ω-sequence S' obtained replacing $\{\&\{F_{i,j}: j \in J_i\}: i \in I\}$ for $\boldsymbol{F}$ in S is not pseudo satisfiable.

2) If $g \in \times\{\{F_{i,j}: j \in J_i\}: i \in I\}$ ($|I| < k$ and $|J_i| < k$ for all $i \in I$, $F_{i,j}$ a sentence) and S is a pseudo good ω-sequence such that the set of sentences $\{\vee\{F_{i,j}: j \in J_i\}: i \in I\} \subset \boldsymbol{S}^m$ for some natural number m, and S_g is obtained from S adding to its first set the set of sentence $\{g(i): i \in I\}$, then if S_g is not pseudo satisfiable for all g then so it is S.

3) If $\{\forall \boldsymbol{v}_i F_i: i \in I\}$ is a subset of the first set of a pseudo good ω-sequence S and $|I| < k$ and there is a natural number m' such that $0 < |\boldsymbol{v}_i| < k_{m'}$ for all $i \in I$ and for some natural number n' the pseudo good ω-sequence S' obtained from S adding to its first set the set of sentences

$$\{F_i(\boldsymbol{v}_i/f): f \text{ a total function from } \cup \{\boldsymbol{v}_i: i \in I\} \text{ into } \cup \{C_h: h \leqslant n'\}, i \in I\}$$

is not pseudo satisfiable then so it is S.

4) Assume that $\{-\forall \boldsymbol{V}_i F_i: i \in I\}$ is a subset of $\boldsymbol{S}^m$ for some pseudo good ω-sequence S, and that there is $m' \in \omega$ such that $0 < |\boldsymbol{V}_i| < k_{m'}$ for all $i \in I$ and $|I| < k_{m'}$ and for any ω-partition $P = \langle I_p: p \in \omega\rangle$ of I let $S_{P,f}$ be the pseudo good ω-sequence obtained from S adding to its $m' + p + 1st$ set the set of sentences $\{-F_i(\boldsymbol{V}_i/f_p): i \in I_p\}$ where f_p is a 1-1 place preserving total function from $\cup\{\boldsymbol{v}_i: i \in I_p\}$ into $((C_{m'+p} \cup (\cup\{C^j_{m'+p}: j \in \omega\})) - \{c: c \text{ is a constant occurring in } \boldsymbol{S}\})$ and $f = \cup\{f_p: p \in \omega\}$. If for all ω-partitions P there is an f such that $S_{P,f}$ is not pseudo satisfiable, then so it is also S.

2. The statement of the main interpolation theorem.

First let us state clearly the notion of substitution of variants of a formula for the occurrences of variants of a subformula within a certain sentence or a set of sentences or a set of sets of sentences.

Let $G(A)$ be a metaformula of $L^{2+}_{k,k}$ in which there is a bound predicate variable, and $G(A)$ has no free variables and no bound metavariables; here A is the set of all metavariables in $G(A)$ and $|A| < k$. Let A^* be the subset of A of the predicate metavariables, i.e. the metavariables with superscript different from zero.

For the time being, suppose that $G(A)$ is of the form $--G'(A)$ for a convenient metaformula $G'(A)$.

G and A will be fixed throughout this and the next sections.

By a variant of $G(A)$ we mean a formula $G(A/\alpha)$ where α is a place preserving function from the whole of A into the union of the constants and the variables. We remark that a variant determined uniquely the function α.

If $G(A/\alpha)$ is a variant of $G(A)$ occurring in F, let

$$\alpha^* = \{(a, v): (a, v) \in \alpha \text{ and } v \text{ is a constant}\}.$$

Let $S = \langle s_n: n \in \omega \rangle$ be a good ω-sequence of sets s_n of sentences. Let $S = \cup\{s_n: n \in \omega\}$. We say that the occurrences of the variants of $G(A)$ are adequate in S is there is a set Q_S with $|Q_S| < k$ and for each $q \in Q_S$ there is a place preserving function α_q from the whole of A into the union of the constants and the variables, say $B_q = \alpha_q(A)$, such that each occurrence of a variant K of $G(A)$ either as a formula or as a subformula in S determines the corresponding q such that K is $G(A/\alpha_q)$, and $\cup\{\alpha_q^{-1}$ restricted to $\cup\{\alpha_{q'}(A^*): q' \in Q_S\}: q' \in Q_S\}$ is a function and if $G(A/\alpha_{q'})$ and $G(A/\alpha_{q''})$ are two occurrences of variants of $G(A)$ in the same formula of S then $\alpha^*_{q'} = \alpha^*_{q''}$.

Suppose that S, A, α_q, Q_S are as defined above and that the occurrencies of the variants of $G(A)$ are adequate in S. We now define S^*. This is going to be an ω-sequence of sets of metaformulas obtained from S by substituting some of its sentences as follows.

If the formula E occurs in S and E is a subformula of $-G(A/\alpha_q)$, a variant of $-G(A)$, for some $q \in Q_S$, then E may be thought of as $E^*(A/\alpha_q)$ where E^* is the corresponding submetaformula of $G(A)$ as E is a subformula of $G(A/\alpha_q)$.

If $E = E^*(A/\alpha_q)$ is $-G'(A/\alpha_q)$ then E has to be replaced by E^* to obtain S^*. If $E = E^*(A/\alpha_q)$ is an immedite even subformula of an even subformula $E' = E'^*(A/\alpha_q)$ of $-G'(A/\alpha_q)$ in S which was replaced according to this clause by E'^* in S^* then also E may be replaced by E^* to obtain S^*. All other sentences in S left unchanged.

In doing so different subformulas of different variants of $G(A)$ may be replaced by the same submetaformula, and the same subformula E of a variant of $G(A)$ may be replaced by different submetaformulas E^* of $G(A)$ if there are several q's such that $E^*(A/\alpha_q) = E$.

Let $Q_{E,S} = \{q: E = E^*(A/\alpha_q)$ and $E^* \in S^*$ and $E \in S\}$. Remark that $Q_S = \cup\{Q_{E,S}: E \in S\}$.

Let R_S be the relation that associates to each $q \in Q_S$ the metaformula E^* in S^* such that $E^*(A/\alpha_q)$ is in S: $R_S = \{(q, E^*): E^* \in S^*,\ q \in Q_S,\ E^*(A/\alpha_q) \in S\}$.

For the S, S^* and R_S that we will consider it will be the case that if a predicate variable occurs free in $E^* \in S^*$ then there is at most one q such that $(q, E^*) \in R_S$ (exactly one if E^* is not E) and if the same predicate variable occurs free in two metaformulas E_1^* and E_2^* in S^* and $(q_1, E_1^*) \in R_S$ and $(q_2, E_2^*) \in R_S$ then $q_1 = q_2$.

Let H be a metaformula such that all its metavariables are in A.

By S_H we mean the ω-sequence of set of sentences obtained from the good ω-sequence S of sets of sentences by substituting for each occurrence of a variant $G(A/\alpha_q)$ of $G(A)$ either as a formula or as subformula in S the corresponding variant $H(A/\alpha_q)$ of H.

We remark that Q_{S_H} is the same as Q_S.

When dealing with interpolation theorems in a language with sequents one uses partitions of the sequents. This is also true for the approach with consistency properties, as we will choose. Thus let us state clearly the definition of a good partition.

By a good partition (S_1, S_2^*) for a good ω-sequence $S = \langle s_n: n \in \omega\rangle$ of sets of sentences (with a related ω-sequence S^* of sets of metaformulas as explained above) we mean a pair of ω-sequences $(S_1, S_2^*) = (\langle s_{1,n}: n \in \omega\rangle, \langle s_{2,n}^*: n \in \omega\rangle)$ of sets of metaformulas such that $s_{1,n} \subset s_n$, no variant of $-G'(A)$ occurs in $s_{1,n}$ as a formula by itself (it may occur as subformula) $s_n \supset s_{2,n} \supset s_n - s_{1,n}$, each formula in $s_{2,n}$ is either $-G'(A)$ or an even immediate subformula of an even subformula E of a variant of $-G'(A)$ with E in S_2 and $s_{2,n}^* = \{E^*: E^*(A/\alpha_q) \in s_{2,n},\ q \in Q_{s_2},\ (q, E^*) \in R_{S_2}\}$, and no predicate variable occurs free in both S_1 and S_2^*.

Remark that all occurrences of metavariables are free in the metaformulas of S_2^*. Remark also that given S, and hence S^*, there are several good partitions for S according to whether a metaformula $E^* \in S^*$ (which is not $-G'(A)$) is in S_2^* or $E^*(A/\alpha_q)$ is in S_1.

Let $S_{2q}^* = \langle s_{2q,n}^*: n \in \omega\rangle$ where

$$s_{2q,n}^* = \{E: E^* \in s_{2,n}^* \quad \text{and} \quad E^*(A/\alpha_q) \in s_{2,n}\}.$$

Remark that if $s_{2,0}^*$ is not empty then $-G'(A)$ is in $s_{2,0}^*$.

Note that $Q_{S_2} = \{q: q \in Q_S$ and there is E^* such that $E^* \in s^*_{2,n}$ for some $n \in \omega$ and $(q, E^*) \in R_{S_2}\}$.

By a good partition (S_{H1}, S^*_{H2}) for S_H we mean $((S_1)_H, (S^*_2)_H)$ for a good partition (S_1, S^*_2) for S.

Remark that $Q_{S_{H2}} = Q_{S_2}$ and $(S^*_2)_H = S^*_2$ so that (S_{H1}, S^*_{H2}) may also be denoted as (S_{H1}, S^*_2).

By an interpolant for a good ω-sequence $S = \langle s_n: n \in \omega \rangle$ of sets of sentences with respect to $G(A)$ we mean a first order formula H satisfying

(A) In H there are neither free variables nor constants and all the metavariables in H are free and in A,

(B) S_H is not pseudo satisfiable,

(C) $\{H(A/f), -G(A/f)\}$ is not ω-satisfiable, where f is a 1-1 place preserving total function from A into the constants that do not occur in either H or $G(A)$.

Suppose that (S_1, S^*_2) is a good partition for S.

By an interpolation set for S with respect to a good partition (S_1, S^*_2) for it and an interpolant H we mean a set $D = \{D_q: q \in Q_{S_2}\}$ of first order metaformulas such that

(1) for each $q \in Q_{S_2}$ all the metavariables in D_q are free and belong to A, there are neither free variables nor predicate constants in D_q, every individual constant in D_q occurs in S^*_{2q};

(2) the pseudo good ω-sequence S^D_{H1} is not pseudo satisfiable, where $S^D_{H1} = \langle s^D_{H1,n}: n \in \omega \rangle$ with $s^D_{H1,o} = s_{H1,o} \cup \{-D_q(A/\alpha_q): q \in Q_{S_2}\}$ and $s^D_{H1,n} = s_{1,n}$ for $n > 0$;

(3) for each $q \in Q_{S_2}$ $S^{*D_q}_{2,q}(A/f)$ is not pseudo satisfiable, where $S^{*D_q}_{2,q}(A/f) = \langle s^{*D_q}_{2,q,n}(A/f): n \in \omega \rangle$ with

$$s^{*D_q}_{2,q,o}(A/f) = s^*_{2,q,o}(A/f) \cup \{D_q(A/f)\}, \qquad s^{*D_q}_{2,q,n}(A/f) = s^*_{2,q,n}(A/f)$$

for $n > 0$, and f is a 1-1 place preserving total function from A into the constants, and the constants of $f(A)$ do not occur in either S^*_2 or D.

THEOREM. Suppose that the occurrences of variants of $G(A)$ in a good ω-sequence S of sets of sentences, $|S| < k$, are negative and adequate in S. Suppose that S is not ω-satisfiable. Then for each good partition (S_1, S^*_2) for S there is an interpolant H for S with respect to $G(A)$ and an interpolation set for S with respect to (S_1, S^*_2) and H.

3. The proof of the main theorem.

For the proof we are going to use the method of seq-consistency properties.

We will argue by contradiction, so we sill define a set Σ of good ω-sequences of sets of sentences, containing S, for which the theorem fails and show that it is a seq-consistency property, hence S would be ω-satisfiable, contradicting the assumption of the theorem.

Let $\Sigma = \{S$: 1) S is a good ω-sequence of sets of sentences, 2) the occurrences of the variants of $G(A)$ in S are adequate and negative, 3) there is a good partition (S_1, S_2^*) for S such that for all first order metaformulas H, whose metavaraibles are all in A, either H is not an interpolant for S with respect to $G(A)$ or there is no interpolation set for S with respect to (S_1, S_2^*) and $H\}$.

LEMMA. Σ is a seq-consistency property.

We should check all the claused of a seq-consistency property, but here we will consider only the tipical cases, the others being routinely proved.

C0) *a*) Assume that the atomic sentence $Z \in s_0$ and $-Z \in S$, where $S = \langle s_n : n \in \omega \rangle$ is a good ω-sequence of sentences satisfying 1) and 2) in the definition of Σ.

Given any good partition (S_i, S_2^*) for S we have to consider four cases:

*a*1) Z is in S_1 and $-Z$ is in S_1;

*a*2) Z is in S_1 and $-Z$ is in S_2;

*a*3) Z is in S_2 and $-Z$ is in S_1;

*a*4) Z is in S_2 and $-Z$ is in S_2.

In all cases let H be $\forall x(x \neq x)$. This is an interpolant for S with respect to $G(A)$.

The case *a*1) is easy with $\{D_q : q \in Q_{S_2}\}$ as an interpolation set where D_q is $\forall x(x \neq x)$ for each $q \in Q_S$.

In case *a*2) let $-Z$ be $-Z^*(A/\alpha_{q_0})$ for a $q_0 \in Q_{S_2}$. The predicate symbol in Z^* is a metavariables since otherwise it could not occur in both S_2 and S_1. An interpolation set $\{D_q : q \in Q_{S_2}\}$ is obtained letting D_{q_0} be $-Z^*$ and D_q be $\forall x(x \neq x)$ for all $q \neq q_0$, $q \in S_{S_2}$.

Case *a*3) is similar to case *a*2), just interchange the role of $-Z$ with that of Z.

In case *a*4) let Z be $Z_1^*(A/\alpha_{q_1})$ and $-Z$ be $-Z_2^*(A/\alpha_{q_2})$, say Z_1^* is $P'(t_1', \ldots, t_n')$ and Z_2^* is $P''(t_1'', \ldots, t_n'')$. If either P' or P'' is a constant then

$P' = P''$ and $q_1 = q_2$; and an interpolation set $\{D_q\colon q \in Q_{S_2}\}$ is obtained letting D_{q_1} be $\wedge\{t_i' = t_i''\colon i = 1, \ldots, n\}$ and D_q be $\forall x(x \neq x)$ for all $q \neq q_1$, $q \in Q_{S_2}$. If both P' and P'' are metavariables then $P' = P''$. If again $q_1 = q_2$ then the last interpolation set still works. Othervise $q_1 \neq q_2$. In this last case let D_{q_1} be $-Z_1^*$, D_{q_2} be Z_2^* and D_q be $\forall x(x \neq x)$ for all $q \in Q_{S_2}$ such that $q \neq q_1$ and $q \neq q_2$ and $\{D_q\colon q \in Q_{S_2}\}$ will be an interpolation set.

Thus if $S \in \Sigma$ then either Z does not occur in S or $-Z$ does not occur in S.

b) Here there are two cases: *b*1) when $-(t = t)$ occurs in S_1, and *b*2) when $-(t = t)$ occurs in S_2, t a constant and (S_1, S_2^*) a good partition of S a good ω-sequence of sentences satisfying 1) and 2) of the definition of Σ. In both cases $\forall x(x \neq x)$ is an interpolant for S with respect to $G(A)$. Furthermore $\{D_q\colon q \in Q_{S_2}\}$ where D_q is $\forall x(x \neq x)$ for all $q \in Q_{S_2}$ is an interpolation set in the case *b*1). In case *b*2) say that $-(t = t)$ is $-(\delta_1 = \delta_2)(A/\alpha_{q_0})$ for a $q_0 \in Q_{S_2}$, where each one of δ_1, δ_2 is either a metavariable or t. Let D_{q_0} be $\delta_1 = \delta_2$ and D_q be $\forall x(x \neq x)$ for all $q \neq q_0$, $q \in Q_{S_2}$: this $\{D_q\colon q \in Q_{S_2}\}$ is again an interpolation set.

Thus we have checked C0).

C1) Part *a*) is routine, whereas part *b*) requires some work. So assume that $|I| < k$, that $S = \langle s_n\colon n \in \omega\rangle \in \Sigma$, and that $\{Z_i(c_i), c_i = d_i\colon i \in S\} \subset S^m$ for some $m \in \omega$ where c_i, d_i are constants, and $Z_i(c_i)$ is an atomic or negated atomic sentence in which c_i may occur. Let $S' = \langle s_n'\colon n \in \omega\rangle$ be such that $s_0' = s_0 \cup \{Z_i(d_i)\colon i \in I\}$ and $s_n' = s_n$ for $n > 0$. Clearly S' satisfies 1) and 2) of the definition of Σ. Let (S_1, S_2^*) be a good partition for S without interpolant or interpolation set.

Let $I_1 = \{i\colon Z_i(c_i) \text{ occurs in } S_1\}$, $I_2 = I - I_1$, c_i^* be the symbol occurring in $Z_i^*(c_i^*)$ in the places corresponding to those where c_i occurs in Z_i, q_i the element of Q_{S_2} such that $Z_i(c_i) = (Z_i^*(c_i^*))(A/\alpha_{q_i})$ if it exists, $I_{2q} = \{i \in I_2\colon q_i = q\}$, $I_3 = \{i\colon c_i = d_i \text{ occurs in } S_1\}$. Let q_i' be the element of Q_{S_2} such that $(c_i = d_i)$ is $(c_i = d_i)^*(A/\alpha_{q_i'})$, which we denote also by $(\gamma_i' = \delta_i')(A/\alpha_{q_i'})$, if it exists. Let $I_{4q} = \{i\colon q_i' = q\}$.

Let $S_1' = \langle s_{1,n}'\colon n \in \omega\rangle$ where $s_{1,0}' = s_{1,0} \cup \{Z_i(d_i)\colon i \in I_i\}$ and $s_{1,n}' = s_{1,n}$ for $n > 0$.

Let $S_{2q}'^* = \langle s_{2q,n}'^*\colon n \in \omega\rangle$ where $s_{2q,n}'^* = s_{2q,n}^*$ for $n > 0$ and $s_{2q,0}'^* = s_{2q,0}^* \cup \{(Z_i(d_i))^*\colon i \in I_{2q}\}$ with $(Z_i(d_i))^* = Z_i^*(d_i^*)$ and $d_i^* = \delta_i$ if $(q, (c_i = d_i)^*) \in R_{S_2}$ and $(c_i = d_i)^*$ is $\gamma_i = \delta_i$ with $\delta_i \neq d_i$, while $d_i^* = d_i$ otherwise.

Let $S_2'^* = \langle s_{2,n}'^*\colon n \in \omega\rangle$ where $s_{2,n}'^* = \cup\{s_{2q,n}'^*\colon q \in Q_{S_2}\}$.

$(S_1', S_2'^*)$ is a good partition for S'.

Assume that $S' \notin \Sigma$. Hence there is an interpolant H' for S' and an interpolation set $D' = \{D'_q : q \in Q_{S'_2}\}$ for S' with respect to (S'_1, S'^*_2) and H'.

Let $\boldsymbol{d}_q$ be the set of the individual constants occurring in D'_q but not in S^*_{2q}. These constants should be some of the d_i when $d_i = d^*_i$ and $i \in I_{2q} - I_{4q}$.

Let φ_0 be a function from $C = \cup\{C_n : n \in \omega\}$ the set of the individual constants, into the terms such that φ_0 restricted to $\boldsymbol{d}_q$ is a 1-1 function into variables occurring neither in S' nor in D' and φ_q restricted to $(C - \boldsymbol{d}_q)$ is the identity. Let $\boldsymbol{x}_q = \varphi_q[\boldsymbol{d}_q]$. Let $D''_q = D'_q(\boldsymbol{d}_q/\varphi_q)$.

Let D_q be

$$\exists \boldsymbol{x}_q((D''_q \,\&\, (\&\{c^*_i = \varphi_q(d_i) : i \in I_{2q} - I_{4q}\})) \vee (\vee\{(c_i \neq d_i)^* : i \in I_{4q} - I_{2q}\})).$$

With some work it is possible to show that H' is an interpolant also for S with respect to $G(A)$ and that $D = \{D_q : q \in Q_{S_2}\}$ is an interpolation set for S with respect to (S_1, S^*_2) and H', contradicting the fact that $S \in \Sigma$.

Thus also S' has to belong to Σ and C1) is checked.

C2) We consider this usually easy case because this time it is the only point where the interpolation set is used to manifacture an interpolant due to our choice of $G(A)$ as $--G'(A)$. So we skip the easy part and assume that $\{--F_i : i \in I\} \subset S^m$ for some $m \in \omega$ and $S = \langle s_n : n \in \omega\rangle \in \Sigma$ and $|I| < k$ and for each $i \in I$ F_i does not occur in S and $--F_i$ is a variant of $-G(A)$.

Since $--F_i$ is a variant of $-G(A)$, for each $i \in I$ there is a $q \in Q_S$, say q_i, such that $--F_i$ is $-G(A/\alpha_{q_i})$.

Let (S_1, S^*_2) be a good partition of S that has no interpolant for S with respect to $G(A)$ or no interpolation set for S with respect to (S_1, S^*_2) and an eventual interpolant. Remark that since $--F_i$ is not a subformula of a variant of $-G'(A)$ then $--F_i \in S_1$ for all $i \in I$.

Let $S' = \langle s'_n : n \in \omega\}$ where $s'_n = s_n$ if $n > 0$ while $s'_0 = s_0 \cup \cup\{F_i : i \in I\}$.

Remark that $s'_0 = s_0 \cup \{-G'(A/\alpha_{q_i}) : i \in I\}$.

Let $S'_1 = S_1$, $Q_{S'_2} = Q_{S_2} \cup \{q_i : i \in I\}$. For all $q \in Q_{S'_2}$ let $S'^*_{2q} = \langle s'^*_{2q,n} : n \in \omega\rangle$ where $s'^*_{2q,n} = s^*_{2q,n}$ for $n > 0$ and $s'^*_{2q,0} = s^*_{2q,0} \cup \{F^*_i : q_i = q, i \in I\}$.

Let $S'^*_2 = \langle s'^*_{2,n} : n \in \omega\rangle$ where $s'^*_{2,n} = \cup\{s'^*_{2q,n} : q \in Q_{S'_2}\}$.

Note that $s'^*_{2q,0} = \{-G'(A)\}$ if there is $i \in I$ such that $q = q_i$, while $s'^*_{2q,0} = s^*_{2q,0}$ otherwise.

Remark that (S'_1, S'^*_2) is a good partition of S', and that, for each $i \in I$, S_{2q_i} is empty, for otherwise it would already have F_i in it, whereas $S'^*_{2q_i} = \langle\{-G'(A)\}, \emptyset, \emptyset, \ldots\rangle$.

Let

$$S_1'' = S, \qquad S_{2q_i}''^{*} = \langle \{-G'(A)\}, \emptyset, \emptyset \ldots \rangle \text{ for each } i \in I,$$

$$S_2''^{*} = \langle \{-G'(A)\}, \emptyset, , \ldots \rangle .$$

$(S_1'', S_2''^{*})$ is another good partition of S', and $Q_{S_2''} = \{q_i\colon i \in I\}$.

Arguing by contradiction let us assume that $S' \notin \Sigma$. So let H' be an interpolant for S' with respect to $G(A)$ and $D' = \{D_q'\colon q \in Q_{S_2'}\}$ an interpolation set for S' with respect to $(S_1', S_2'^{*})$ and H'. H' and the D_q''s such that $q = q_i$ for some $i \in I$ are not enough to build an interpolant, say H^0, for S with respect to $G(A)$ because there is no hint why the not pseudo satisfiability of S_{H_0} can be deduced from the not pseudo satisfiability of $S_{H'}'$ and of $S_{H'1}'^{D'}$. That is why also the good partition $(S_1'', S_2''^{*})$ of S' has been introduced. So let H'' be an interpolant for S' with respect to $G(A)$ and $D'' = \{D_{q_i}''\colon i \in I\}$ an interpolation set for $S'$with respect to $(S_1'', S_2''^{*})$ and H''.

Let H be $H' \vee H'' \vee (\vee\{D_{q_i}''\colon i \in I\}) \vee (\vee\{D_{q_i}'\colon i \in I\})$.

With some work it is possible to show that H is an interpolant for S with respect to $G(A)$. It is easy to check part A) of the definition of interpolant, while for part B) it is important to notice that $H'' \vee (\vee\{D_{q_i}''\colon i \in I\})$ is an interpolant for S with respect to $G(A)$ mainly due to part (2) of the definition of an interpolation set for S' with respect to $(S_1'', S_2''^{*})$ and the choice of S_1'' as S. The fact that $\models H' \to H$ and that $\models (H'' \vee (\vee\{D_{q_i}''\colon i \in I\})) \to H$ allows us to conclude the checking of part B). As for part C) it is enough to notice that each of the following sets of sentences $\{H'(A/f), -G(A/f)\}$, $\{H''(A/f), -G(A/f)\}$, $\{D_{q_i}'(A/f), -G(A/f)\}$, $\{D_{q_i}''(A/f), -G(A/f)\}$ with $i \in I$ and f as specified in part (3) of the definition of an interpolation set is not pseudo satisfiable.

Now let D be $\{D_q'\colon q \in Q_{S_2}\}$. It can be shown that D is an interpolation set for S with respect to (S_1, S_2^{*}) and H. But this would contradict the fact that $S \in \Sigma$.

Thus also S' has to belong to Σ and C2) is checked.

C6) This is the point where the way of interpreting the constants plays an important role and it calls for the introduction of the notion of pseudo satisfiability.

Assume that $\{-\forall \boldsymbol{v}_i F_i\colon i \in I\} \subset \boldsymbol{S}^m$ for some $m \in \omega$ and that there is m' the least natural number such that $|I| < k_{m'}$ and $0 < |\boldsymbol{v}_i| < k_{m'}$ for all $i \in I$ and $m \leqslant m'$ and $\boldsymbol{S}^m \subset Stmt(L_{m'})$.

Assume that $S = \langle s_n\colon n \in \omega \rangle \in \Sigma$. Let $P = \langle I_p\colon p \in \omega \rangle$ be an ω-partition of I and $\{f_p: p \in \omega\}$ a set of 1-1 total functions from $\cup\{v_i: i \in I_p\}$ into

$$C_{m'+p} - \{c: c \text{ is a constant occurring in } S\}.$$

Let $S^P = \langle s_n^P: n \in \omega\rangle$ be the ω-sequence such that $S_n^P = s_n$ for all $n \leqslant m'$ and $s_{m'+p+1}^P = s_{m'+p+1} \cup \{-F_i(v_i/f_p): i \in I_p\}$ for all $p \in \omega$.

We want to show that there is an ω-partition P of I such that $S^P \in \Sigma$. Since S^P satisfies 1) and 2) of the definition of Σ, it remains to show that S^P satisfies also 3). Suppose that for all ω-partitions P of I, S^P does not satisfy 3); we will get a contradiction out of this assumption.

Since $S \in \Sigma$ there is a good partition (S_1, S_2^*) for S such that for all first order formulas H either H is not an interpolant for S with respect to $G(A)$ or there is no interpolation set for S with respect to (S_1, S_2^*) and H.

Let $I_1 = \{i: -\forall v_i F_i \text{ occurs in } S_1\}$, $I_2 = I - I_1$. Let q_i be the element of Q_{S_2} such that $(-\forall v_i F_i)^*(A/f_{q_i}) = -\forall v_i F_i$ if it exists. Let $I_{2q} = \{i: i \in I_2 \text{ and } q_i = q\}$.

Let $S_1^P = \langle s_{1,n}^P: n \in \omega\rangle$ where $s_{1,n}^P = s_{1,n}$ if $n \leqslant m'$ and for all $p \in \omega$, $s_{1,m'+p+1}^P = s_{1,m'+p+1} \cup \{-F_i(v_i/f_p): i \in I_p \cap I_1\}$.

Even though in general Q_{S^P} will extend Q_S because new variants of $G(A)$ may be introduced in S^P which did not occur in S, nevertheless $Q_{S_2^P}$ is equal to Q_{S_2} because in S_2^P and in S_2 there are only subformulas of variants of $G(A)$ and all the symbols to be changed to obtain a variant have already been considered introducing the metavariables.

For each $q \in Q_{S_2^P}$ let $S_{2q}^{P*} = \langle s_{2q,n}^{P*}: n \in \omega\rangle$ where $s_{2q,n}^{P*} = s_{2q,n}^*$ if $n \leqslant m'$ and $s_{2q,m'+p+1}^{P*} = s_{2q,m'+p+1}^* \cup \{(-F_i(v_i/f_p))^*: i \in I_p \cap I_{2q}\}$ for all $p \in \omega$.

Let $S_2^{P*} = \langle s_{2,n}^{P*}: n \in \omega\rangle$ where $s_{2,n}^{P*} = \cup\{s_{2q,n}^{P*}: q \in Q_{S_2^P}\}$.

Remark that (S_1^P, S_2^{P*}) is a good partition for S^P.

Let H^P be an interpolant for S^P with respect to $G(A)$ and D^P an interpolation set for S^P with respect to (S_1^P, S_2^{P*}) and H^P.

The ω-partition P of I induces the ω-partitions P_1, P_2 and P_{2q} of I_1, I_2 and I_{2q} respectively defined as follows: $P_1 = \langle I_p^1: p \in \omega\rangle = \langle I_p \cap I_1: p \in \omega\rangle$, $P_2 = \langle I_p^2: p \in \omega\rangle = \langle I_p \cap I_2: p \in \omega\rangle$ and $P_{2q} = \langle I_p^{2q}: p \in \omega\rangle = \langle I_p \cap I_{2q}: p \in \omega\rangle$. Viceversa given ω-partitions P_{2q} of I_{2q} for each $q \in Q_{S_2^P}$ the ω-partition P_2 of I_2 is defined as $\langle \cup\{I_p^{2q}: q \in Q_{S_2^P}\}: p \in \omega\rangle$, and given the ω-partitions P_1 and P_2 of I_1 and I_2 respectively, the ω-partition P of I is defined as $\langle I_p^1 \cup I_p^2: p \in \omega\rangle$. Thus we can write $S^{P_1P_2}$, $S_1^{P_1}$, $S_2^{P_2}$, $S_{2q}^{P_{2q}}$, $D^{P_1P_2}$, $D_q^{P_1P_{2q}}$ instead of S^P, S_1^P, S_2^P, S_{2q}^P, D^P, D_q^P respectively.

Let d_q be the set of individual constants in $D_q^{P_1P_{2q}}$ that occur in $S_{2q}^{P_{2q}*}$ but do not occur in S_{2q}^*. Let φ_q be a 1-1 total function from d_q into variables occurring neither in S^P nor in D^P. Let $x_q = \varphi_q[d_q]$.

Let $D_q'^{P_1P_{2q}}$ be $\forall \boldsymbol{x}_q D_q^{P_1P_{2q}}(\boldsymbol{d}_q/\varphi_q)$.

With some work it can be shown that the set $D'^{P_1P_{2q}} = \{D_q'^{P_1P_{2q}}: q \in Q_{S_2^P}\}$ is an interpolation set for S^P with respect to $(S_1^{P_1}, S_2^{P_2*})$ and H^P.

It should be remarked that the constants in $\boldsymbol{d}_q$ occur in the ω-sequence $S_{2q}^{P_{2q}*}$ but not necessarily within $\cup\{s_{2q,n}^{P_{2q}*}: n < n'\}$ for some $n' \in \omega$ and here the pseudo satisfiability play a role really different than the role of ω-satisfiability.

Now let H be $\vee\{H^P: P$ is an ω-partition of $I\}$.

Using the facts that S_{H^P} is not pseudo satisfiable and that $\models H^P \rightarrow H$, standard arguments show that H is an interpolant for S with respect to $G(A)$.

Finally for each $q \in Q_{S_2^P} = Q_{S_2}$ let D_q be $\vee\{\&\{D_q'^{P_1P_{2q}}: P_{2q}$ is an ω-partition of $I_{2q}\}: P_1$ is an ω-partition of $I_1\}$.

Let $D = \{D_q: q \in Q_{S_2}\}$.

With some work it can be proved that D is an interpolation set for S with respect to (S_1, S_2^*) and H.

But this is impossible since we assumed that $S \in \Sigma$; therefore there is an ω-partition P of I such that $S^P \in \Sigma$ and also C6) is checked.

This completes the proof of the lemma and the main theorem easily follows from it.

4. Conclusion.

Usually the interpolation theorems are stated assuming the validity of a sentence. Here also we can easily obtain from our main theorem the following

Restricted interpolation theorem.

Suppose that F is an ω-valid sentence in a language $L_{k,k}^{2+}$ without constants. Let $G(A)$ be a metaformula, where A is the set of its metavariables, of the type $--G'(A)$ whose variants occur negatively and adequatly in F. Then there is an interpolant H for F, i.e. a first order metaformula without constants or free variables, whose only metavariables are free and in A, such that F_H and $H \rightarrow G$ are ω-valid.

Just let S be the good ω-sequence whose first set is $\{-F\}$ and the others are empty; using the main theorem obtain an interpolant H; this is the interpolant required in this theorem since pseudo validity an ω-validity are equivalent on a single sentence.

In the statement of the last theorem there were two limitations.

First, the metaformula $G(A)$ was supposed of the type $--G'(A)$ and, second, no constant is allowed in the original language $L^{2+}_{k,k}$.

These conditions may be easily dropped.

Interpolation theorem.

Let F be a sentence in a language $L^{2+}_{k,k}$ possibly with individual constants. Since we treat constants as variables, it is natural to assume that there are less than K individual constants in F. Suppose that F is ω-valid and that all the occurrences of variants of $G(A)$ ($G(A)$ any metaformula with A as set of its metavariables) are negative and adequate in F. Then there is an interpolant H for F with respect to $G(A)$, i.e. a first order metaformula without free variables whose only metavariables are free and in A and whose only individual constants occur both in $G(A)$ and in F but not within a variant of $G(A)$, such that F_H and $H \to G$ are ω-valid.

For the proof, first replace the individual constants in $G(A)$ that do not occur elsewhere in F by new variables in a 1-1 manner to get $G'(A)$, and obtain $G''(A)$ by existentially quantifying these variables in front of $G'(A)$. Next replace the variants of $G(A)$ in F by the corresponding variants of $G''(A)$ to get F'. Then replace the individual constants which occur both in $G''(A)$ and elsewhere in F' by still new individual variables, say a set $\boldsymbol{x}$, to obtain F''. Let $G^0(A')$ be obtained form $G''(A)$ replacing, in a 1-1 manner, the variables in $\boldsymbol{x}$ by new metavariables, A' is A union the set of these new metavariables. Then replace, again in a 1-1 manner, the individual constant still left in F'' by a set $\boldsymbol{y}$ of again new variables to obtain $'F$; let F^0 be $\forall(\boldsymbol{x} \cup \boldsymbol{y})'F$, this is a sentence in which there are occurrences of variants of $G^0(A')$. Finally replace the variants of $G^0(A')$ in F^0 by the corresponding variants of $--G^0(A')$.

Now we are in a position where we can apply the restricted interpolation theorem, so we can get an interpolant H^0 for F^0 with respect to $--G^0(A')$.

Let H be obtained from H^0 by replacing the metavariables of $A'-A$ occurring in it by the individual constants from where they came. It is easily checked that H is an interpolant for F with respect to $G(A)$.

The last theorem is in a sense the most complete extension to the languages considered equipped with Karp's notion of satisfiability of the interplation theorem of Maehara and Takeuti.

As usual from the Maehara Takeuti style interpolation theorems one can deduce those in the style of Chang and also of Craig.

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