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Stefano Meda

Rita Pini

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An Abstract Version of Herz' Imbedding Theorem

STEFANO MEDA - RITA PINI (*)

The aim of this paper is to generalize to a quite abstract setting a well-known result by C. Herz ([6], Theorem 5) which can be stated as follows.

THEOREM. *Suppose that* $1/r = 1/p - \alpha/n > 0$, $1 < p < \infty$ *and* $\alpha > 0$. *Then* $\Lambda^{\alpha}_{p,q}(\boldsymbol{R}^n)$ *is continuously embedded in* $L_{r,q}(\boldsymbol{R}^n)$, *whenever* $1 \leqslant q \leqslant \infty$.

The reader is referred to the notation below for every unexplained terminology or symbol. Herz' proof is based on an inversion formula for standard higher order difference operators on $\boldsymbol{R}^n$, which cannot be usefully generalised to other contexts. Our contribution is to provide a different, perhaps simpler, proof of Herz' theorem in the Euclidean context which can be adapted to a very general situation. We assume that G is a locally compact unimodular topological group with σ-finite Haar measure. Furthermore, we suppose that we can find an approximate identity $(\phi_s)_{s>0}$, which possesses the following properties

$$(1) \qquad \begin{cases} \text{(i)} \ \phi_t * \phi_s = \phi_s * \phi_t & \forall s, t > 0; \\ \text{(ii)} \ \|\phi_s\|_{p'} \leqslant C s^{-\gamma/p} & \text{for some } \gamma > 0 \text{ and } \forall \text{p}, \ 1 \leqslant p \leqslant \infty, \end{cases}$$

where p' denotes the index conjugate to p. It is well-known that approximate identities satisfying the above requirements can be found on many groups. For instance, let G be a connected unimodular Lie group and let $(p_t)_{t>0}$ be the family of heat kernels associated to the heat semigroup generated by a (suitable) invariant (sub-)laplacian. Then it is not hard to prove that $(p_t)_{t>0}$ satisfies (i) and (ii) above (see, for instance [4]). Conditions similar to (1) (i), (ii) have been considered by L.

(*) Indirizzo degli AA.: S. MEDA: Dipartimento di Matematica, Università di Trento, 38050 Povo (TN), Italia; R. PINI: Istituto di Matematica, Università di Verona, via dell'Artigliere 19, 37129 Verona, Italia.

De Michele and I. R. Inglis [2] on spaces of homogeneous type (in the sense of R. R. Coifman and G. Weiss). Our method applies to contects different from locally compact groups and gives «multiplier results» related to Theorem 2.2 in [7]. We shall examine these implications elsewhere.

Our paper is organised as follows. We fix some notation in Section 1. In Section 2 we define the Lipschitz spaces $\Lambda^{\alpha}_{p,q}$ associated to an approximate identity satisfying (1) and give a sufficient condition assuring that Lipschitz spaces arising from different approximate identities give rise to equivalent spaces. Finally we prove our main result, which generalises Herz' theorem.

1. Notation.

Let M be a σ-finite measure space and let $|E|$ denote the measure of the measurable subset E of M. We denote by f^* the noncreasing rearrangement of the measurable function f (see [8]). If f is integrable on the sets of finite measure we may also definite its averaged nonincreasing rearrangement f^{**} by the formula

$$f^{**}(t) = \frac{1}{t}\int_0^t du f^*(u) \qquad \forall t > 0\,.$$

The Lorentz space $L_{r,q}$, $1 < r < \infty$, $1 \leqslant q \leqslant \infty$, is the Banach space of all measurable functions f such that

$$\|f\|_{r,q} = \left(\int_0^\infty \frac{dt}{t}(f^{**}(t)\,t^{1/r})^q\right)^{1/q}$$

is finite. It is easy to prove that $L_{r,p}$ is continuously imbedded in $L_{r,q}$ whenever $1 \leqslant p \leqslant q \leqslant \infty$. We refer the reader to [8] for more on the Lorentz spaces.

2. The main result.

Assume that G is a locally compact unimodular group with σ-finite Haar measure and $(\phi_s)_{s>0}$ is an approximate identity satisfying (1). For every integer n, we set

$$P_n = \phi_{2^{-n}} - \phi_{2^{-n+1}}\,.$$

For every positive real number α let $\mathcal{E}^{\alpha}_{p,q}$, $1 \leq p < \infty$, $1 \leq q \leq \infty$, denote the set of all compactly supported continuous functions f on G such that

$$N^{\alpha}_{p,q}(f) = \begin{cases} \left(\sum_{n \in \boldsymbol{Z}} \|f * P_n\|_p^q 2^{n\alpha q} \right)^{1/q} & \text{if } 1 \leq q < \infty \\ \sup_{n \in \boldsymbol{Z}} \|f * P_n\|_p 2^{n\alpha} & \text{if } q = \infty \end{cases} \tag{2}$$

if finite. Set $\mathcal{Z} = \{f \in \mathcal{E}^{\alpha}_{p,q} : N^{\alpha}_{p,q}(f) = 0\}$. It is not hard to show that $\mathcal{Z} = \{0\}$. We shall show that

$$f = \sum_{n \in \boldsymbol{Z}} f * P_n * Q_n , \tag{3}$$

where $Q_n = \phi_{2^{-n}} + \phi_{2^{-n+1}}$, the convergence being in the uniform sense. Suppose now that $N^{\alpha}_{p,q}(f) = 0$; then $\|f * P_n\|_p = 0$ and $f * P_n = 0$ a.e., whence $f = 0$ by (3). It remains to show that (3) holds. A straightforward computation gives that

$$\sum_{-N}^{N} f * P_n * Q_n = f * \phi_{2^{-N}} * \phi_{2^{-N}} - f * \phi_{2^{N+1}} * \phi_{2^{N+1}}$$

by (1) (i). Since $\varphi_N = \phi_{2^{-N}} * \phi_{2^{-N}}$ is an approximate identity on G as $N \to \infty$ and f is a compactly supported continuous function, we have that $f * \varphi_N \to f$ in the uniform topology. Furthermore, for every $p < \infty$ we get that

$$\|f * \varphi_{-N-1}\|_\infty \leq \|\phi_{2^{N+1}}\|_{p'} \|f * \phi_{2^{N+1}}\|_p \leq$$

$$\leq C\, 2^{-\gamma(N+1)/p} \|f\|_p \|\phi_{2^{N+1}}\|_1 = C\, 2^{-\gamma(N+1)/p} \|f\|_p ,$$

whence $\|f * \varphi_{-N-1}\|_\infty \to 0$ as $N \to \infty$, thereby proving (3). The above calculations show that $N^{\alpha}_{p,q}$ is a norm on $\mathcal{E}^{\alpha}_{p,q}$. The Lipschitz space $\Lambda^{\alpha}_{p,q}$ is defined as the completion of $\mathcal{E}^{\alpha}_{p,q}$ with respect to the norm $N^{\alpha}_{p,q}$.

Lipschitz spaces have been considered by several authors. The reader is referred to [11] and [15] for a rather complete treatement of the properties of such spaces. For a more coincise introduction to the subject, see [13].

Remark. (*a*) The Lipschitz space $\Lambda^{\alpha}_{p,q}$ depends on the choice of the approximate identity $(\phi_s)_{s>0}$. Let, for instance, G be $\boldsymbol{R}^m$; consider the canonical family $(\phi_k)_{k \in \boldsymbol{Z}}$ (see [1], p. 139) and the associated Lipschitz space $\Lambda^{\alpha}_{p,q}$. Let f be the inverse Fourier transform of the characteristic function of the interval $[1, 2]$. Then it is easy to check that f is in $\Lambda^{\alpha}_{2,1}$ for every $\alpha > 0$. Indeed

$$\|f * \phi_k\|_2 = \|\hat{f}\hat{\phi}_k\|_2 = 0$$

if $k \neq 0, 1$, while

$$\|f * \phi_k\|_2 \leqslant \|f\|_2 \|\phi_k\|_1 = \|f\|_2$$

if $k = 0, 1$.

Let $(\tilde{\phi}_s)_{s>0}$ be the family defined by the rule

$$\tilde{\phi}_s(s) = s^{-1}\tilde{\phi}(x/s),$$

$\tilde{\phi}$ denoting $1/2$ the characteristic function of the interval $[-1, 1]$. Then

$$(\tilde{P}_n)^\wedge(\xi) = \frac{\sin(2^{-n}\xi)}{2^{-n}\xi} - \frac{\sin(2^{-n+1}\xi)}{2^{-n+1}\xi}.$$

We claim that f is not in $\tilde{\Lambda}^{\alpha}_{2,1}$ if $\alpha \geqslant 2$. Indeed, let n be a large positive integer. Then

$$\|f * \tilde{P}_n\|_2 = \left(\int_1^2 d\xi \left|\frac{\sin(2^{-n}\xi)}{2^{-n}\xi} - \frac{\sin(2^{-n+1}\xi)}{2^{-n+1}\xi}\right|^2\right)^{1/2} =$$

$$= 2^{n-1}\left(\int_1^2 d\xi\, |2\sin(2^{-n}\xi) - \sin(2^{-n+1}\xi)|^2\right)^{1/2} \geqslant$$

$$\geqslant C\, 2^n \left(\int_1^2 d\xi\, |2^{1-3n} - 2^{3-3n}|^2\right)^{1/2} \geqslant C\, 2^{-2n}$$

(C independent on n), whence the claim.

It is clear that it is not hard to produce functions which are in $\Lambda^{\alpha}_{p,q}$ but not in $\tilde{\Lambda}^{\alpha}_{p,q}$ for any admissible α, p, q. We omit details.

Although the Lipschitz space $\Lambda^{\alpha}_{p,q}$ depends on the choice of the approximate identity, we do not stress this dependence, to avoid combersome notation.

(*b*) For suitable choices of the approximate identity $(\phi_s)_{s>0}$, the Lipschitz spaces $\Lambda^{\alpha}_{p,q}$ defined above are equivalent to Lipschitz spaces defined by higher order difference operators in a variety of contexts.

If G is $\boldsymbol{R}^n$ the equivalence is a classical result (see, for example [1], Theorem 6.3.1).

If G is $\boldsymbol{R}^n$, but nonisotropic diagonal dilations are considered, the result is implicit in [12].

If G is the Heisenberg group $\boldsymbol{H}^n$ the result is essentially contained in [9], Remark (*b*), p. 44. A generalization of his argument covers the case of stratified groups as well (see also [5]).

If G is a compact connected semisimple Lie group the equivalence is proved in [10], Theorem 1.9 (see also [4]).

It is interesting to find sufficient conditions which ensure that Lipschitz spaces, defined by different approximate identities, give rise to equivalent spaces. Let $(\tilde{\phi}_s)_{s>0}$ be another approximate identity and define $\tilde{P}_n$, $\tilde{N}^\alpha_{p,q}$ and $\tilde{\Lambda}^\alpha_{p,q}$ consitently with the previous notation.

PROPOSITION 1. *Suppose that for some $\eta > \alpha$ there exists a positive constant A such that*

$$\|\phi_{2^{-n}} * \tilde{\phi}_{2^{-j}}\|_1 \leqslant A\, 2^{\eta(n-j)}, \qquad \forall n, j \in \boldsymbol{Z}. \tag{4}$$

Then $\Lambda^\alpha_{p,q}$ is continuously embedded in $\tilde{\Lambda}^\alpha_{p,q}$.

PROOF. Clearly it suffices to prove that there exists a positive constant C such that $\tilde{N}^\alpha_{p,q}(f) \leqslant C N^\alpha_{p,q}(f)$ for all compactly supported continuous functions f. We give details only in the case $q = \infty$. The case $q = \infty$ is, perhaps, easier. Set $Q_n = \phi_{2^{-n}} + \phi_{2^{-n+1}}$. Then a routine argument (see (3)) shows that

$$f = \sum_{-\infty}^{\infty} f * P_n * Q_n,$$

the convergence being in the uniform sense. Therefore

$$\tilde{N}^\alpha_{p,q}(f) \leqslant \Bigg(\sum_{j\in\boldsymbol{Z}} 2^{j\alpha q} \Bigg(\sum_{n\in\boldsymbol{Z}} \|f * P_n\|_p \|Q_n * \tilde{P}_j\|_1 \Bigg)^q \Bigg)^{1/q}.$$

Since $\|Q_n * \tilde{P}_j\|_1 \leqslant \|Q_n\|_1 \|\tilde{P}_j\|_1 \leqslant C$ (independent of j and n; $C = 4$ does work), we have that

$$\|Q_n * \tilde{P}_j\|_1 \leqslant C \min(1, 2^{\eta(n-j)})$$

by (4), whence

$$\tilde{N}^{\alpha}_{p,q}(f) \leqslant C\left\{\left(\sum_{j\in \mathbf{Z}} 2^{j\alpha q}\left(\sum_{n\geqslant j}\|f*P_n\|_p\right)^q\right)^{1/q}+\right.$$

$$\left.+\left(\sum_{j\in \mathbf{Z}} 2^{j\alpha q}\left(\sum_{n<j}\|f*P_n\|_p 2^{\eta(n-j)}\right)^q\right)^{1/q}\right\} = C\,(I+II)\,.$$

Choose ε such that $0<\varepsilon<\alpha$. Then, by Hölder's inequality and Fubini's theorem we have that

$$I \leqslant \left(\sum_{j\in \mathbf{Z}} 2^{j\alpha q} \sum_{n\geqslant j}\|f*P_n\|_p^q 2^{\varepsilon n q}\left(\sum_{n\geqslant j} 2^{-\varepsilon n q'}\right)^{q/q'}\right)^{1/q} \leqslant$$

$$\leqslant C\left(\sum_{n\in \mathbf{Z}}\|f*P_n\|_p^q 2^{\varepsilon n q}\sum_{j\leqslant n} 2^{jq(\alpha-\varepsilon)}\right)^{1/q} \leqslant C\,N^{\alpha}_{p,q}(f)\,.$$

Choose now ε such that: $\alpha<\varepsilon<\eta$. Then, again by Hölder's inequality and Fubini's theorem, we have that

$$II \leqslant \left(\sum_{j\in \mathbf{Z}} 2^{jq(\alpha-\eta)} \sum_{n<j}\|f*P_n\|_p^q 2^{\varepsilon n q}\left(\sum_{n<j} 2^{q' n(\eta-\varepsilon)}\right)^{q/q'}\right)^{1/q} \leqslant$$

$$\leqslant C\left(\sum_{j\in \mathbf{Z}} 2^{jq(\alpha-\varepsilon)}\sum_{n<j}\|f*P_n\|_p^q 2^{\varepsilon n q}\right)^{1/q} =$$

$$= C\left(\sum_{n\in \mathbf{Z}}\|f*P_n\|_p^q 2^{\varepsilon n q}\sum_{j>n} 2^{jq(\alpha-\varepsilon)}\right)^{1/q} \leqslant C\,N^{\alpha}_{p,q}(f)\,.$$

Collecting the above estimate, we obtain the desired result. ■

COROLLARY 1. *Suppose that for some $\eta>\alpha$ there exists a positive constant C such that*

(i) $\|\phi_{2^{-n}} * \tilde{\phi}_{2^{-j}}\|_1 \leqslant C\,2^{\eta(n-j)}$;

(ii) $\|\tilde{\phi}_{2^{-j}} * \phi_{2^{-n}}\|_1 \leqslant C'\,2^{\eta(j-n)}$.

Then $\Lambda^{\alpha}_{p,q}$ and $\tilde{\Lambda}^{\alpha}_{p,q}$ are equivalent spaces.

PROOF. It is an immediate consequence of Proposition 1. ■

We are now ready to prove our main result. Let dx denote a two-sided Haar measure of G.

THEOREM 1. *Suppose that* $1/r=1/p-\alpha/\gamma>0$, $1<p<\infty$, *and* $\alpha>0$; *then for all* q, $1\leqslant q\leqslant\infty$, $\Lambda^{\alpha}_{p,q}$ *is continuously embedded in* $L_{r,q}(G)$.

PROOF. We give details only in the case $q<\infty$. Slight modifications give the case $q=\infty$ as well. It is a well-known fact that if f is integrable over sets of finite measure, then the following formula holds

$$\sup_{|E|\leqslant t}\int_E dx\,|f(x)|=\int_0^t du\,f^*(u)$$

(see, for instance [14], p. 202). Set

$$Q_n=\phi_{2^{-n}}+\phi_{2^{-n+1}}.$$

Let f be in $\mathcal{E}^{\alpha}_{p,q}$. Then, owing to (3), we have that

$$f=\sum_{-\infty}^{\infty}f*P_n*Q_n.$$

Choose an integer m such that $2^{m-1}\leqslant t^{-1/\gamma}<2^m$. Then

$$\int_0^t duf^*(u)=$$

$$=\sup_{|E|\leqslant t}\left\{\int_E dx\left|\sum_{-\infty}^{m-1}f*P_n*Q_n(x)\right|+\int_E dx\left|\sum_m^{\infty}f*P_n*Q_n(x)\right|\right\}=I_t+II_t.$$

Choose ε such that $0<\varepsilon<1/p-\alpha/\gamma$. By Hölder's inequality and (1) (ii) we have that

$$I_t\leqslant t\sum_{-\infty}^{m-1}\|f*P_n\|_p\|Q_n\|_{p'}\leqslant Ct\sum_{-\infty}^{m-1}\|f*P_n\|_p 2^{n\gamma/p}\leqslant$$

$$\leqslant Ct\left(\sum_{-\infty}^{m-1}\|f*P_n\|_p 2^{n(\alpha+\varepsilon\gamma)}\right)\sup_{n\leqslant m-1}2^{-n(\alpha+\varepsilon\gamma-\gamma/p)}=$$

$$=C\sum_{-\infty}^{m-1}\|f*P_n\|_p 2^{n(\alpha+\varepsilon\gamma)}t^{1+\alpha/\gamma-1/p+\varepsilon}.$$

By Hölder's inequality and (1) (ii) we also have that

$$II_t \leqslant \sup_{|E| \leqslant t} |E|^{1-1/p}\left(\int_E dx \left(\sum_m^{\infty} |f * P_n * Q_n(x)| \right)^p \right)^{1/p} \leqslant$$

$$\leqslant t^{1-1/p}\left(\int_G dx \left(\sum_m^{\infty} |f * P_n * Q_n(x)| \right)^p \right)^{1/p} \leqslant$$

$$\leqslant t^{1-1/p} \sum_m^{\infty} \|f * P_n\|_p \|Q_n\|_1 \leqslant C\, t^{1-1/p} \sum_m^{\infty} \|f * P_n\|_p .$$

Collecting the above estimate we get that

$$\|f\|_{r,q} \leqslant C \left\{ \left(\int_0^{\infty} \frac{dt}{t} \left(t^{1/p-\alpha/\gamma} t^{\alpha/\gamma - 1/p + \varepsilon} \sum_{-\infty}^{m-1} \|f * P_n\|_p 2^{n(\alpha + \varepsilon\gamma)} \right)^q \right)^{1/q} + \right.$$

$$\left. + \left(\int_0^{\infty} \frac{dt}{t} \left(t^{1/p-\alpha/\gamma} t^{-1/p} \sum_m^{\infty} \|f * P_n\|_p \right)^q \right)^{1/q} \right\} = C\{I + II\} .$$

By Hölder's inequality and Fubini's theorem

$$I \leqslant C \left(\int_0^{\infty} dt\, t^{\varepsilon q - 1} \sum_{-\infty}^{m-1} \|f * P_n\|_p^q 2^{n(\alpha + \varepsilon\gamma/2)q} \left(\sum_{-\infty}^{m-1} 2^{n\varepsilon\gamma p'/2} \right)^{q/q'} \right)^{1/q} \leqslant$$

$$\leqslant C \left(\int_0^{\infty} dt\, t^{\varepsilon q/2 - 1} \sum_{-\infty}^{m-1} \|f * P_n\|_p^q 2^{n(\alpha + \varepsilon\gamma/2)q} \right)^{1/q} \leqslant$$

$$\leqslant C \left(\sum_{-\infty}^{\infty} \|f * P_n\|_p^q 2^{nq(\alpha + \varepsilon\gamma/2)} \int_0^{2^{-\gamma n}} dt\, t^{\varepsilon q/2 - 1} \right)^{1/q} = C N_{p,q}^{\alpha}(f) .$$

Choose ε such that $0 < \varepsilon < \alpha$. Then Hölder's inequality and Fubini's theorem imply that

$$II \leqslant C \left(\int_0^{\infty} dt\, t^{-\alpha q/\gamma - 1} \sum_m^{\infty} \|f * P_n\|_p^q 2^{n\varepsilon q} \left(\sum_m^{\infty} 2^{-n\varepsilon q'} \right)^{q/q'} \right)^{1/q} \leqslant$$

$$\leqslant C\left(\int_0^\infty dt\, t^{(\varepsilon-\alpha)q/\gamma-1}\sum_m^\infty \|f*P_n\|_p^q 2^{n\varepsilon q}\right)^{1/q} \leqslant$$

$$\leqslant C\left(\sum_{-\infty}^\infty \|f*P_n\|_p^q 2^{n\varepsilon q}\int_{2^{-\gamma n}}^\infty dt\, t^{(\varepsilon-\alpha)q/\gamma-1}\right)^{1/q} = C N_{p,q}^\alpha(f).$$

A density argument completes the proof. ■

COROLLARY 2. *Assume that* $\alpha > 0$ *and let* k *be in* $\Lambda_{p,q}^\alpha$. *Then the map* $f \mapsto k * f$ *is bounded from* L_{p_0,q_0} *to* L_{p_1,q_1} *provided that*

$$0 < \frac{1}{p_1} = \frac{1}{p_0} + \frac{1}{p} - \frac{\alpha}{\gamma} - 1 < 1, \qquad 1 < \frac{1}{p} - \frac{\alpha}{\gamma} < \infty, \qquad 1 < p_0 < \infty$$

and

$$0 \leqslant \frac{1}{q_1} = \frac{1}{q_0} + \frac{1}{q} \leqslant 1.$$

PROOF. It follows immediately from Theorem 1 and the convolution theorem for Lorentz spaces ([8], Theorem 4.10). ■

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