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On a Birational Classification of Bundles and Reflexive Sheaves on Surfaces.

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This note has two aims. The first one is the definition (and show how they can be used, of course) of «birational equivalence», «relatively birational model», «birational model», for a pair (S, E) with S smooth projective surface and E rank-2 vector bundle on S. Two pairs (S, E) and (S', E') are called birationally equivalent if there are: a pair (S'', E''), birational morphisms $t: S'' \to S$ and $t': S'' \to S$, finite sets $D \subset S$, $D' \subset S'$ with $t \mid (S \setminus t^{-1}(D))$ and $t' \mid (S' \setminus t'^{-1}(D'))$ isomorphisms (where «$|$» means «restricted to»), and isomorphisms of $E'' \mid (S \setminus t^{-1}(D))$ with $t^*(E) \mid (S \setminus t^{-1}(D))$ and of $E'' \mid (S' \setminus t'^{-1}(D'))$ with $t'^*(E') \mid (S' \setminus t'^{-1}(D'))$. If $((S, E), (S'', E''), t)$ are as above, and $t: S'' \to S$ is a (relatively) minimal model of S'', (S, E) is called a (relatively) minimal model of (S'', E''); note that E is uniquely determined by (S'', E'') and t, since, being reflexive, it is uniquely determined by its restriction to $S \setminus D$, i.e. by $E'' \mid (S \setminus t^{-1}(D))$; if S'' has non-negative Kodaira dimension, then S and t are uniquely determined by S. For much more, see § 1.

The second aim of this note is to show that rank-2 reflexive sheaves on normal projective surfaces can be classified (as in the case of vector bundles on, say, smooth surfaces) if they have «low invariants». These classification lists have two aims: find interesting examples and find the «exceptional cases» for general theorems. I will consider here only a few very simple cases (Theorems 0.1, 0.2, and 0.3) in which not much appears, leaving to other interested mathematicians the task of finding interesting examples (for slightly higher invariants or, perhaps, for the non birational classification in 0.1). As we will see, the local geometry of the singularities (of the surface and of reflexive sheaf) plays a big role in the definition of the numerical invariants. Let X be a (complete)

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normal surface and E a reflexive sheaf on X; for simplicity we will assume rank $(E)=2$ and X defined over the complex number field (since for the classification results 0.1, 0.2, 0.3, we will use the characteristic 0 assumption, to make quotations easier). Let $f\colon S \to X$ be a relatively minimal desingularization of X. Let F be the quotient of $f^*(E)$ by its torsion part, and $G := F^{**}$ its double dual; G is locally free; G will be called the bundle associated to F. Set $\delta(E) :=$ length (G/F); since F has no torsion, we have $0 \leqslant \delta(E) < \infty$ and $\delta(E)=0$ if and only if F is locally free. The relatively minimal model (or the Grassmann model (see § 1)) of (S, G) will be called the relatively minimal model (or the Grassmann model) of (X, E). If E is spanned by its global sections, $t^*(E)$ and F are spanned; however if $\delta(E) > 0$, G is never spanned: every section of G has image contained in the subsheaf F, i.e. the natural inclusion of $H^0(S,F)$ into $H^0(S,G)$ is an isomorphism (see e.g. [K], Cor. 3.7 at p. 53). There is an important case in which $\delta(E)=0$ for every reflexive shef on X: when X has only rational singularities ([K]; Th. 4.5 at p. 80). Set $\mu(E) := c_2(G)$ and $d(E) := \mu(E) - \delta(E)$. Set $\boldsymbol{I} := \mathrm{Ann}(G/F)$ and let Δ be the scheme with support $\mathrm{Supp}(G/F)$ and $\boldsymbol{O}_\Delta = \boldsymbol{O}_S/\boldsymbol{I}$; by abuse of notations, Δ will denote also $\mathrm{Supp}(G/F)$; set $\delta'(E) := \mathrm{length}(\boldsymbol{O}_S/\boldsymbol{I})$. Let (S', G') be the Grassmann model (see § 1) of (S, G); (S', G') is defined, for instance, if E is spanned; set $\lambda(E) := c_2(G')$ and $\delta''(E) := \mu(E) - \lambda(E)$. By construction $\lambda(E) \geqslant 0$. If E is spanned we will check (see Remark 2.1) that $0 \leqslant \delta''(E) \leqslant \delta'(E) \leqslant \delta(E)$.

THEOREM 0.1. *Let X be a complex complete normal surface and E a rank-2 spanned reflexive sheaf on X. Assume $\lambda(E)=1$. Then the bundle associated to E is birational to $(\boldsymbol{P}^2, U)$, with U direct sum of two line bundles of degree 1. Furthermore, if we assume also that E is ample and $\delta(E) \leqslant 1$, then $(X, E) \cong (\boldsymbol{P}^2, U)$.*

THEOREM 0.2. *Let X be a complex complete normal surface and E a rank-2 ample and spanned reflexive sheaf on X. If $\delta(E)=1$, then $\mu(E) > 1$.*

Recall that the invariant $s(Y)$, Y rank-2 bundle was defined in [B2]; see § 1 for more on this notion; if (X, E) is as above, and (S', G') is its Grassmann model, we will set $s(E) := s(G')$; see [B3] for a definition of $c(Y)$, when Y is spanned.

THEOREM 0.3. *Let X be a complex complete normal surface and E a rank-2 spanned reflexive sheaf on X. If $s(E) = 2\lambda(E) > 0$, then X is rational. If $s(E) = \lambda(E) > 0$ and (S', G') is the Grassmann model of (X, E), then there is a morphism $\pi\colon S' \to \boldsymbol{P}^2$, π generically of degree 2,* *with* $G' \cong \pi^*(U)$, U *direct sum of two degree* 1 *line bundles. If* $s(E) = 2\lambda(E) - 1 \geqslant 5$, *then* X *is rational. If* $c(G') = 2s(G') - 2 \geqslant 6$, *then* $h^1(\boldsymbol{O}_{S'}) = 0$.

1. Fix a smooth complete surface S, a rank-2 vector bundle on S and a linear subspace $W \subseteq H^0(S, E)$. Assume that there is a non empty open set $U \subseteq S$ such that W spans $E|U$. Let Gr (or $Gr(2, W)$) be the Grassmannian of 2-dimensional linear subspaces of W; let J be the tautological rank-2 quotient bundle on Gr. W induces a morphism j: $U \to Gr$. Minimally resolving the singularities of the closure in $S \times Gr$ of the graph of this morphism, we find a smooth complete surface S' and morphisms a: $S' \to S$, b: $S' \to Gr$ with a birational, and such that $(S', b^*(J))$ is birational to (S, E); set $E' := b^*(J)$; (S', E') depends on W but not on the choice of U; (S', E') will be called the Grassmann model of $(S, E; W)$ (or of (S, E) if $W = H^0(S, E)$). We note that we may pass from (S'', E'') to $(S'', t^*(E))$ (and viceversa) (in the case of smooth surfaces, but with E of any rank) with a finite number of «elementary transformation» (an operation introduced and studied very much by Maruyama (see [M]) and applied e.g. in [B] and [Br]). For higher dimensional varieties (and for singular surfaces) it seems that on a birational model one should consider reflexive sheaves, maybe with controlled singularities.

Let f: $\boldsymbol{S} \to T$ be a smooth projective morphism of relative dimension 2 and $\boldsymbol{E}$ a rank-2 vector bundle on $\boldsymbol{S}$; assume for simplicity T reduced and connected. For any $t \in T$, set $S(t) := f^{-1}(t)$ and $E(t) := \boldsymbol{E} \,|\, S(t)$. We would like to say something about the variation of the relatively minimal models and of the Grassman models of $(S(t); E(t))$ when t varies in T. Note that for smooth surfaces the plurigenera and the Kodaira dimension are constant in smooth, connected families of surfaces ([Ii]). First of all, if the Kodaira dimension of any $S(t)$ is negative, we have to prescribe in a coherent way what exceptional curves blew-down on each fiber (and in what order); this trouble does not arise if any $S(t)$ has non negative Kodaira dimension where it is sufficient to use Kodaira's invariance under small deformations of exceptional curves. We will always assume that this trouble can be handled for f without stratifying T, i.e. we will assume the existence of $\boldsymbol{C} \subset \boldsymbol{S}$, with $f \,|\, \boldsymbol{C}$ flat, and such that for every $t \in T$, $C(t) := \boldsymbol{C} \cap S(t)$ is the configuration we can blow-down to obtain a relatively minimal model; essentially we assume the existence of $(\tau, \boldsymbol{S}', f)$ with τ: $\boldsymbol{S} \to \boldsymbol{S}'$ fibrewise contraction of the exceptional curves, f: $\boldsymbol{S}' \to T$ smooth, smooth, and $f' \circ \tau = f$. Ser $\Delta := \tau(\boldsymbol{C})$. By assumption $f|\Delta$ is finite and étale, and τ induces an isomorphism over T between $\boldsymbol{S}' \setminus \boldsymbol{C}$ and $\boldsymbol{S}' \setminus \Delta$. Use this isomorphism to see $\boldsymbol{E}|(\boldsymbol{S}' \setminus \boldsymbol{C})$ as a bundle on $\boldsymbol{S}' \setminus \Delta$; take any coherent exyension of it to $\boldsymbol{S}'$ and call $\boldsymbol{E}'$ its double dual. Set $S'(t) := f'^{-1}(t)$, $E'(t) := \boldsymbol{E}' \mid S'(t)$. By the existence of flattening stratifications ([Mu], p. 56) and the semicontinuity theorem, there is a partition of T into finitely many disjoint locally closed strata, such that: (i) the restriction of $\boldsymbol{E}'$ to the counterimage of any stratum, say A, is f'-flat; (ii) for every $j \geqslant 0$, $h^i(S'(t), E'(t))$ is constant for $t \in A$, $f'_*(\boldsymbol{E}' \mid f'^{-1}(A))$ is locally free (with $h^0(S'(t), E'(t))$, $t \in A$, as rank) and commutes with base change. Over an open dense stratum $\boldsymbol{E}'$ is locally free; over lower dimensional strata we take the double dual of the restriction of $\boldsymbol{E}'$ to the counterimage of the stratum, and continue. Similarly, we get a stratification on which it is defined a Grassman model. Now we consider one such stratum, i.e. we assume that $f_*(\boldsymbol{E})$ is locally free and that the natural map $f^*f_*(\boldsymbol{E}) \to \boldsymbol{E}$ is suriective (under these assumptions everything works for higher dimensional $S(t)$) if rank $(E(t)) = \dim(S(t))$; then (considering the relative Hilbert scheme of f), we get by the definition of $s(E(t))$ in the introduction of [B2] and the semicontinuity of the fiber of, say, proper morphisms between algebraic varieties, the lower semicontinuity of $s(E(t))$; in particular, if $n := \dim(S(t))$ and $s(E(o))$ is maximal (i.e. equal to $nc_n(E(o))$) then this is true for all t in a neighborhood U of o in T. This is interesting because if $s(E(t))$ is maximal, then $S(t)$ is rational ([B2], Lemma 1.1), and for higher dimensional varieties it seems extremely difficult to say something about the set of rational varieties in a flat family.

For a smooth surface S of general type with K_S not ample the pluricanonical model, X, of S has mild singularities (rational double points); one should consider also suitable pairs (X, F) with F rank-2 reflexive sheaves on X as possible birational model of (S, E). Luckily, there is a complete description of the restriction of all F to a formal neighborhood of $\operatorname{Sing}(X)$ in terms of line bundles on the $\boldsymbol{P}^1$ of the fundamental cycles of $S \to X$ (see [K], p. 81, bemerkung 4.6; note that (with the notations of loc. cit.) for rational double points we have $0 \leqslant d_{ij} \leqslant 1$ for all i, j).

2. Here we will prove the results stated in the introduction. Fix X (complete normal surface), E, S, f, F, and G as in the «second aim part» of the introduction. Let (S', G') be the Grassmann model of (X, E) (i.e. of (S, G)).

REMARK 2.1. If E is spanned by its global sections, then $\delta''(E) := := \mu(E) - \lambda(E) \geqslant 0$.

PROOF. Choose a general $s \in H^0(X, E)$ and let Z be its zero locus on X_{reg}. Then s induces $s' \in H^0(S, F)$ vanishing on Z (with abuse of notations) and $s'' \in H^0(S, G)$ (identifying $H^0(S, F)$ with $H^0(S, G)$) with scheme of zeroes containing Z. Since a general section of G' (or even a general element in the vector space $W \cong H^0(S, F)$ of sections of G' coming from (S, G)) does not vanishes on the exceptional locus (which has dimension 1) of $S' \to S$ (a dimensional count and the fact that W spans G') s'' induces $\boldsymbol{s} \in H^0(S', G')$ with zero locus exactly Z. ■

REMARK 2.2. Since by definition G/F is annihilated by $\boldsymbol{I}$, we have $\delta(E) \geqslant \delta'(E)$. It should be possible to have a strict inequality here; this must always occur when Δ is not locally a complete intersection: however we have not checked any non trivial example.

We recall that a sheaf E over a complete variety X is calle ample if the tautological line bundle on $\boldsymbol{P}(E)$ is ample. We need only the following weaker property of ample sheaves (see [B4], Remark 1.1): for every curve $C \subset X$ and for the normalization u: $C' \to C$ any quotient bundle of $u^*(E|C)$ has degree >0; we could take this as definition of ampleness; indeed if E is locally free and spanned, this property is known to be equivalent (in characteristic zero) to the ampleness of E; in general, without the spannedness assumptions, it is strictly weaker than the ampleness property (even for line bundles on smooth surfaces).

PROOF OF 0.1. (i) First assume E ample. Set $L := \det(G)$, $K := K_S$ and $\boldsymbol{O} := \boldsymbol{O}_S$.

(a) Note that L is spanned by its global sections outside Δ and it is nef and big. By the assumption on $\lambda(E)$, for a general $P \in S$ there is $s \in H^0(G)$ with $(s)_{0|(S \setminus \Delta)} = \{P\}$ as a scheme; set $Z := (s)_0$; s induces the following exact sequence

$$0 \to \boldsymbol{O} \to G \to L \otimes \boldsymbol{I}_Z \to 0. \tag{1}$$

(b) By the generality of P, we get (see [B1], Lemma 1.1, or [B2], Lemma 1.1) that S' is rational; hence S is rational. In particular $h^1(\boldsymbol{O}) = 0$.

(c) Let $T \subset S$ be an integral curve with $\dim(f(T)) > 0$. Here we will check that $LT \geqslant 2$ and that if $LT = 2$ then T is smooth and rational and $G|T$ is the sum of two degree 1 line bundles. Let r: $R \to T$ be the normalization of T; let M be the quotient of the pull-back on T of E by its torsion part. By the assumption of $f(T)$ we have an inclusion (induced by the inclusion $F \to F^{**} = G$) of M into $r^*((G|T))$, with cokernel of finite length. Hence this inclusion induces an inclusion of the corresponding determinant line bundles. Thus $TL = \deg(\det(r^*((G|T)) \geqslant \deg(M) \geqslant 2$ (because M is ample). Assume $TL = 2$ (hence $\deg(M) = 2$). By [W], 3.2.1, if $p_a(T) = 0$ (and its proof if $p_a(T) \geqslant 0$, or see [B4], Lemma 1.2, for both a statement and a proof), T is smooth and rational and M is as wanted.

(*d*) Assume $\mathscr{E}(E) = 0$ (i.e. by 2.2 $\mathscr{E}'(E) = 0$, i.e. G spanned and $S = S'$); note that this is the case if X has only rational singularities. Thus Z is reduced, say $Z = \{P\}$ with P general. G is spanned and $c_2(G) = 1$. By the classification of such pairs (S, G) ([B]) there is a birational morphism $t: S \to \boldsymbol{P}^2$ such that $G \cong t^*(U)$ with U the sum of two degree 1 line bundles. The restriction of G to any positive dimensional fiber of t is trivial. By step (*b*) π factors through t and we have $X \cong \boldsymbol{P}^2$.

(*e*) Let $T \subset S$ be an integral curve with $T \cap \mathrm{Supp}(G/F) \neq \emptyset$. Here we show that $LT > 0$. With the notations of part (*c*), M is spanned and the inclusion of M into $r^*(G|T)$ is not an isomorphism since the inclusion of $F|T$ into $G|T$ cannot be an isomorphism (use that G is locally free, standard exact sequences and the assumption on T).

(*f*) Now assume $\mathscr{E}(E) = 1$. Now Z is reduced and $Z = \{P, x\}$ with $\{x\} = \mathrm{Supp}(G/F)$ (hence $\pi(x) \in \mathrm{Sing}(X)$), while we may take as P any general point of S. Moving P and using the Cayley-Bacharach property, we see that $h^0(K \otimes L \otimes \boldsymbol{I}_{\{x\}}) = 0$. Thus $h^0(K \otimes L) \leqslant 1$. We need the notion of a-minimality for a pair (S, L) with S. Gorestein surface and $L \in \mathrm{Pic}(S)$, L nef and big (see [AS1] or [AS2]). Since in our case S is smooth, our (S, L) is a-minimal if it contains no smooth rational curve T with $T^2 = -1$ and $TL = 0$: since S is a minimal desingularization of X, no such curve exists by step (*c*). First assume $h^0(K \otimes L) = 1$. Since (S, L) is a-minimal, $(K \otimes L)^N$ is spanned for some $N > 0$ ([AS2], 0.8.3). Since $q(S) = 0$, L has sectional genus 1. We obtain $K \otimes L \cong \boldsymbol{O}_S$. Let T be an exceptional curve of the first kind on S. Since $L \cong K^{-1}$, by part (*c*) $\pi(T)$ is a point, contradicting again the fact that S is a minimal desingularization of X. Thus S is a relatively minimal rational surface. Since $\mathscr{E} > 0$, E is not a bundle; thus S is not $\boldsymbol{P}^2$. Since L is nef, S is not a surface F_e with $e > 2$. Hence we may assume $S \cong F_2$. On F_2 we may contract only the negative section, D. Since $DK = 0$, this contradicts step (*e*).

Now assume $h^0(K \otimes L) = 0$. Since $q(S) = 0$, (S, L) has sectional genus 0 and $h^0((K \otimes L)^N) = 0$ for every $N > 0$. Since S is a-minimal, by [AS1] (or [AS2], 0.8.1) S is either $\boldsymbol{P}^2$ or a quadratic or (S, L) is a scroll. In the first two cases π cannot contract any curve, contradiction. If (S, L) is a scroll, the contradiction comes from step (*c*).

(ii) Now drop the ampleness assumption for E. The result follows from [B1], Theorem 0.1, applied to (S', G'). ∎

PROOF OF 0.2. We have the exact sequence (1) with Z reduced, say $Z = \{x\}$, and Z the support of G/F. By (1) and the Cayley-Bacharach property, we see that x is a base point of $K \otimes L$. By [R], Theorem 1 (a), if $L^2 \geqslant 5$ there is a curve T containing x with either $LT = 0$ or $LT = 1$ and $T^2 = 0$. The first case is impossible by step (e) in the proof of 0.1. If $T^2 = 0$, then f cannot contract T; thus the second case contradicts step (c) in the proof of 0.1. ∎

PROOF OF 0.3. These assertions follows at once from the existence of the Grassmann model and, respectively, [B2], Lemma 1.1, [B2], Cor. 4.2, [B3], Lemma 1.1, and [B3], Prop. 6.1. ∎

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