Rendiconti
del
Seminario Matematico
della
Università di Padova

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Rendiconti del Seminario Matematico della Università di Padova, tome 86 (1991), p. 213-231
<http://www.numdam.org/item?id=RSMUP_1991__86__213_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 86 (1991)

Multivalued Superposition Operators

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SUNTO - Dato un aperto limitato Ω in $\mathbb{R}^N$, studiamo l'operatore di sovrapposizione (di Nemytskij) multivoco N_F generato da una funzione multivoca $F: \Omega \times \times \mathbb{R} \to 2^{\mathbb{R}}$. Diamo delle condizioni sufficienti per la continuità e la limitatezza di N_F nello spazio S delle funzioni misurabili su Ω, tra due spazi ideali X e Y, e nello spazio C delle funzioni continue su $\overline{\Omega}$. Inoltre, facciamo un confronto tra la chiusura $\overline{F}$ e la convessificazione $F^{\square}$ della funzione F, da una parte, e la chiusura $\overline{N}_F$ e la convessificazione $N_F^{\square}$ del relativo operatore N_F, dall'altra.

Let Ω be a bounded domain in the Euclidean space and f some real function on $\Omega \times \mathbb{R}$. The *superposition operator* (or *Nemytskij operator*) N_f generated by the function f associates, by definition, with each function $x = x(t)$ on Ω another function $y = y(t)$ according to the rule

$$y(t) = N_f(x)(t) = f(t, x(t)) . \tag{1}$$

This operator is of fundamental importance in both the theory and applications of nonlinear analysis, and has been studied between many function spaces X and Y. In particular, the following three cases are of interest: $X = Y = S$ is the space of all measurable functions over Ω, X and Y are two ideal spaces over Ω (see below), or $X = Y = C$ is the space of all continuous functions over Ω. A detailed exposition of this theory may be found in the book [6].

Suppose now that F is a *multivalued* real function (multifunction)

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on $\Omega \times \mathbf{R}$. Therefore, applying F to a (single-valued) function $x = x(t)$ on Ω, we get a multifunction

$$Y(t) = F(t, x(t)) \tag{2}$$

on Ω. In this case, the *multivalued superposition operator* N_F generated by F is defined by

$$N_F(x)(t) = \{y(t)\colon y(s) \in Y(s) \text{ for all } s \in \Omega\}, \tag{3}$$

i.e. N_F is the set of all *selections* of $F(\cdot, x(\cdot))$. In contrast to the operator (1), the multivalued superposition operator (3) has not yet been studied systematically, although multivalued superposition operators occur quite frequently in many fields of applied mathematics, such as in control theory, in the mechanics of hysteresis and relay phenomena, in convex analysis, in game theory, in dynamical systems without uniqueness, and in some parts of mathematical economics. It is the purpose of this paper to give a brief systematic account of some important properties of the operator (3), such as boundedness and continuity, in the function spaces mentioned above.

The paper consists of two parts. In the first part, we study the operator (3) from the viewpoint of *boundedness* and *continuity* in the cases $X = Y = S$, X and Y are ideal spaces, and $X = Y = C$. In contrast to the operator (1), here the case of measurable functions is much easier than that of continuous functions; this is due to the fact that, loosely speaking, *measurable selections* of a measurable multifunction are easily found, while *continuous selections* of a continuous multifunction exist only under additional hypotheses, as Michael's classical selection theorem shows. In the second part, we shall be concerned with certain *closure* and *convexification* procedures for the function F and the corresponding operator N_F. These procedures are classical tools in applications to integral or differential equations with discontinuous data; roughly speaking, they serve for «filling the gaps» of a given discontinuous non-linearity. For details, we refer to the recent book [14] on differential equations with discontinuous right-hand side.

1. The superposition operator in the spaces S.

Let Ω be a bounded domain in the Euclidean space. By $S = S(\Omega)$ we denote the set of all (classes of) Lebesgue-measurable real functions on

Ω, equipped with the metric

$$d(x, y) = \inf_{0<h} \{h + \text{mes}\,\{t\colon |x(t) - y(t)| > h\}\}\,; \tag{4}$$

this set is a complete metric linear space (see e.g. [12]).

Let $F\colon \Omega \times \mathbb{R} \to 2^{\mathbb{R}}$ be some multifunction. (In the following we shall write 2^T (resp. $\mathrm{Bd}(T)$, $\mathrm{Cl}(T)$, $\mathrm{Cp}(T)$, $\mathrm{Cv}(T)$) for the system of all (resp. all bounded, closed, compact, convex) nonempty subsets of a topological linear space T.) We call F *superpositionally measurable* (or *sup-measurable*, for short) if, for any $x \in S$, the multifunction (2) is measurable in the usual sense (see e.g. [8], [9], or [15]); if the multifunction (2) admits only a measurable (single-valued) selection y, we call F *weakly sup-measurable.* (In case of measurable selections y, we require that (3) holds only for almost all $s \in \Omega$, of course.) In either case, we may define the superposition operator (3) in the space S, because we have then $S \cap N_F(x) \neq \emptyset$ for all $x \in S$.

The problem arises to characterize the (weak) sup-measurability of a multifunction F in simpler terms. In general, this is a highly nontrivial problem even in the single-valued case (see e.g. [4]); *sufficient* conditions, however, are easily found. Recall that F is called a *Carathéodory multifunction* if $F(\cdot, u)\colon \Omega \to 2^{\mathbb{R}}$ is measurable for each $u \in \mathbb{R}$, and $F(t, \cdot)\colon \mathbb{R} \to 2^{\mathbb{R}}$ is continuous for (almost) each $t \in \Omega$. If the continuity of $F(t, \cdot)$ is replaced by upper (resp. lower) semi-continuity, F is called an *upper* (resp. *lower*) *Carathéodory multifunction.* One may show that every Carathéodory multifunction is sup-measurable [9]. Interestingly, if F is merely an upper Carathéodory multifunction, F need not be sup-measurable; a simple example is given in [23]. Nevertheless, if F is upper Carathéodory, then F is weakly sup-measurable [10]. Surprisingly, a lower Carathéodory multifunction need even not be weakly sup-measurable; this shows that also from the viewpoint of sup-measurability, there is an «unsymmetry» between upper and lower semi-continuity.

For example [24], let $\Omega = [0, 1]$, D a non-measurable subset of Ω, and $F\colon \Omega \times \mathbb{R} \to \mathrm{Cp}(\mathbb{R})$ defined by

$$F(t, u) = \begin{cases} \{0\} & \text{if } u = t \text{ and } t \in \Omega \setminus D\,, \\ \{1\} & \text{if } u = t \text{ and } t \in D\,, \\ [0, 1] & \text{if } u \neq t\,. \end{cases}$$

It is easy to see that F is lower Carathéodory. However, F is not weakly sup-measurable: in fact, putting $x(t) = t$ into (2) yields

$$Y(t) = \begin{cases} \{0\} & \text{if } t \in \Omega \setminus D, \\ \{1\} & \text{if } t \in D, \end{cases}$$

which does not admit a measurable selection!

We point out that conditions for (weak) sup-measurability may be obtained by combining semi-continuity properties of F in each variable separately. For example [11], if $F(\cdot, u)$: $\Omega \to \mathrm{Bd}(\mathbb{R})$ is upper semi-continuous, and $F(t, \cdot)$: $\mathbb{R} \to \mathrm{Bd}(\mathbb{R})$ is lower semi-continuous, then F is weakly sup-measurable. (It is in fact easy to see that the above multifunction F is not upper semi-continuous in t.)

Other sufficient conditions for sup-measurability which are related to the measurability of F an the product $\Omega \times \mathbb{R}$ may be found in [27].

Suppose now that F is a weakly sup-measurable multifunction satisfying

$$0 \in F(t, 0) \qquad (t \in \Omega); \tag{5}$$

the condition (5) may always be fulfilled by passing, if necessary, to the «shifted» multifunction $\hat{F}(t, u) = F(t, u) - F(t, 0)$. By (5), the relation

$$P_D N_F \subseteq N_F P_D \tag{6}$$

holds, where

$$P_D x(t) = \begin{cases} x(t) & \text{if } t \in D, \\ 0 & \text{if } t \notin D \end{cases} \tag{7}$$

denotes multiplication by the characteristic function of $D \subseteq \Omega$. Obviously, if (5) is replaced by the stronger condition

$$F(t, 0) = \{0\} \qquad (t \in \Omega), \tag{8}$$

equality holds in (6).

We point out that the superposition operator (3) always satisfies the following property of *disjoint additivity*: whenever $x_1, x_2, \dots, x_n$ are measurable functions with disjoint supports, then

$$N_F(x_1 + \dots + x_n) + (n-1) N_F(\theta) = N_F(x_1) + \dots + N_F(x_n), \tag{9}$$

where θ denotes the (almost everywhere) zero function. This property is sufficient for most purposes (see e.g. the proof of Theorem 3 below).

We begin now the investigation of the operator (3) in the space S.

The following useful result on Carathéodory multifunctions was proved in the single-valued case in [16].

LEMMA 1. *Let $F\colon \Omega \times \mathrm{R} \to \mathrm{Cp}(\mathrm{R})$ be a Cathéodory multifunction, and let v and w be two measurable functions on Ω. Further, let*

$$m(t) = \max \bigcup_{v(t) \leq u \leq w(t)} F(t, u) \qquad (t \in \Omega). \tag{10}$$

Then there exists a measurable function z such that $m \in N_F(z)$.

PROOF. First of all, we remark that the function (10) is well defined, since any upper semi-continuous multifunction $\Phi\colon \mathrm{R} \to \mathrm{Cp}(\mathrm{R})$ maps compact sets into compact sets [9]. Moreover, the function m is obviously measurable, as well as all functions

$$m_h(t) = \max \bigcup_{\substack{v(t) \leq u \leq w(t) \\ u \leq h}} F(t, u) \qquad (t \in \Omega)$$

$(0 < h < \infty)$. If we define

$$z(t) = \inf \{\xi\colon m(t) \in F(t, \xi)\}, \tag{11}$$

then z is measurable, since the relation $z(t) > h$ is equivalent to the relation $m(t) > m_h(t)$. ■

We remark that Lemma 1 may also be proved by means of Filippov's implicit function lemma for multifunctions [13].

We prove now a boundedness and continuity result for the operator (3) in S. Recall that a set $N \subset S$ is *bounded* if and only if, for any sequence $(y_n)_n$ in N and any sequence $(\delta_n)_n$ in R with $\delta_n \to 0$, the sequence $(\delta_n y_n)_n$ converges to θ in S.

THEOREM 1. *Let $F\colon \Omega \times \mathrm{R} \to \mathrm{Cp}(\mathrm{R})$ be a Carathéodory multifunction. Then the superposition operator* (3) *generated by F is bounded in the space S.*

PROOF. Let $M \subset S$ be a bounded set, $(x_n)_n$ an arbitrary sequence in M, and $(\delta_n)_n$ a sequence of positive real numbers such that $\delta_n \to 0$. We have to show that any sequence $y_n \in N_F(x_n)$ satisfies

$$\mu(\delta_n y_n, h) \to 0 \qquad (n \to \infty) \tag{12}$$

for $0 < h < \infty$, where

$$\mu(z, h) = \mathrm{mes}\, \{t\colon t \in \Omega,\ |z(t)| > h\}. \tag{13}$$

Given $\varepsilon > 0$, we may choose $h_\varepsilon > 0$ such that $\mu(x_n, h_\varepsilon) \leqslant \varepsilon$ uniformly in $n \in \mathbb{N}$, since M is bounded in S. Put

$$\tilde{x}_n(t) = \begin{cases} x_n(t) & \text{if } |x_n(t)| \leqslant h_\varepsilon, \\ h_\varepsilon \operatorname{sign} x_n(t) & \text{if } |x_n(t)| > h_\varepsilon, \end{cases} \tag{14}$$

and observe that $-h_\varepsilon \leqslant \tilde{x}_n(t) \leqslant h_\varepsilon$ for all $t \in \Omega$. By Lemma 1, the function

$$m_\varepsilon(t) = \max \bigcup_{|u| \leqslant h_\varepsilon} F(t, u) \qquad (t \in \Omega) \tag{15}$$

is well defined and measurable; moreover $|\eta_n| \leqslant m_\varepsilon(t)$ for any $\eta_n \in F(t, \tilde{x}_n(t))$. Denote by D_n the set of all $t \in \Omega$ for which $F(t, x_n(t)) \neq F(t, \tilde{x}_n(t))$; by construction, the set D_n has measure at most ε. On the other hand, for any $t \in \Omega \setminus D_n$ we have $y_n(t) \in F(t, \tilde{x}_n(t))$, hence

$$|\delta_n y_n(t)| \leqslant \delta_n m_\varepsilon(t) \to 0 \qquad (n \to \infty).$$

Since $\varepsilon > 0$ is arbitrary, this proves (12). ■

THEOREM 2. *Let $F: \Omega \times \mathbb{R} \to \mathrm{Cp}(\mathbb{R})$ be a Carathéodory multifunction. Then the superposition operator* (3) *generated by F is continuous in the space S.*

PROOF. Let $x_* \in S$ and $(x_n)_n$ be a sequence in S which converges in measure (i.e. in the metric (4)) to x_*. Without loss of generality, we may assume that $(x_n)_n$ converges almost everywhere on Ω to x_*. Since F is Carathéodory, we have then also $Y_n(t) = F(t, x_n(t)) \to F(t, x_*(t)) = Y_*(t)$ for almost all $t \in \Omega$. Denoting by $U_\varepsilon(M)$ the ε-neighbourhood of a set M, by Egorov's theorem we may find a set $D \subset \Omega$ with $\operatorname{mes} D < \varepsilon$ such that

$$Y_n(t) \subseteq U_\varepsilon(Y_*(t)), \qquad Y_*(t) \subseteq U_\varepsilon(Y_n(t)) \tag{16}$$

for $t \in \Omega \setminus D$ and n large enough. But this implies that also

$$N_F(x_n) \subseteq U_{2\varepsilon}(N_F(x_*)), \qquad N_F(x_*) \subseteq U_{2\varepsilon}(N_F(x_n)). \tag{17}$$

In fact, given $y \in N_F(x_n)$, by (16) we may choose a measurable selection z of $F(t, x_*(t))$ with $|y(t) - z(t)| \leqslant \varepsilon$ on $\Omega \setminus D$, and get

$$d(y, z) = \inf_h \{h + \mu(y - z, h)\} \leqslant \varepsilon + \mu(P_D(y - z), \varepsilon) \leqslant \varepsilon + \operatorname{mes} D < 2\varepsilon.$$

This proves the first inclusion in (17); the second inclusion is proved similarly. ■

2. The superposition operator in ideal spaces.

Recall [29] that an *ideal space* is a Banach space X of measurable functions over Ω such that the relations $x \in S$, $y \in X$ and $|x(t)| \leqslant |y(t)|$ almost everywhere on Ω imply that also $x \in X$ and $\|x\| \leqslant \|y\|$. Classical examples of ideal spaces are the Lebesgue space $L_p\,(1 \leqslant p \leqslant \infty)$, the Orlicz space L_M (see e.g. [18]), the Lorentz space Λ_φ, and the Marcinkiewicz space M_φ (see e.g. [20]).

An ideal space X is called *regular* [29], if

$$\lim_{\operatorname{mes} D \to 0} \|P_D x\| = 0$$

for each $x \in X$ (P_D as in (7)), i.e. every function $x \in X$ has absolutely continuous norm. Another important class of ideal spaces is that of *split spaces* [5]; this means that there exists a sequence $\sigma(n)$ of natural numbers with the following property: whenever $(x_n)_n$ is a sequence of disjoint functions in X with norm $\|x_n\| \leqslant 1$, one can construct disjoint functions $x_{n,1}, \ldots, x_{n,\sigma(n)}$ such that $x_n = x_{n,1} + \ldots + x_{n,\sigma(n)}$ and the function $x_s = \sum_{n=1}^{\infty} x_{n,s(n)}$ has norm $\|x_s\| \leqslant 1$ for each choice $s = (s(1), s(2), \ldots)$ of natural numbers $1 \leqslant s(n) \leqslant \sigma(n)$. For instance, every Orlicz space L_M is a split space, as may be seen by putting

$$\sigma(n) \geqslant \sup\left\{ \int_\Omega M[x(t)]\,dt \colon \|x\| \leqslant 2^n \right\};$$

in particular, in the Lebesgue space $L_p\,(1 \leqslant p < \infty)$ one may choose $\sigma(n) = 2^{np}$.

We point out that the notions of regular spaces and split spaces are independent; for example, the Orlicz space L_M is regular if and only if the corresponding Young function M satisfies a Δ_2 condition [18].

Another concept from the theory of ideal spaces will be used below. Given an ideal space X, we denote by X' the *associate space* of all measurable functions y for which the norm

$$\|y\| = \sup\{\langle x, y\rangle \colon \|x\| \leqslant 1\}$$

makes sense and it is finite; here

$$\langle x, y \rangle = \int_{\Omega} x(t)\, y(t)\, dt \tag{18}$$

as usual. For example, the associate space to the Orlicz space L_M is the Orlicz space $L_M' = L_{M'}$, generated by the Young function

$$M'(v) = \sup\{u|v| - M(u): u \geqslant 0\}$$

(see e.g. [1], [28]). The associate space X' is a closed (in general, strict) subspace of the usual *dual space* X^*; one may show that $X' = X^*$ if and only if X is regular.

Suppose now that $F: \Omega \times \mathrm{R} \to 2^{\mathrm{R}}$ is a given sup-measurable multifunction, and the superposition operator N_F generated by F acts between two ideal spaces X and Y. It is easy to see that certain properties of the images $F(t, u)$ of the function F carry over to the images $N_F(x)$ of the corresponding operator N_F. For instance, if $F: \Omega \times \mathrm{R} \to \mathrm{Cv}(\mathrm{R})$, then also $N_F: X \to \mathrm{Cv}(Y)$; if $F: \Omega \times \mathrm{R} \to \mathrm{Bd}(\mathrm{R})$, then also $N_F: X \to \mathrm{Bd}(Y)$ (since L_∞ is imbedded in Y); if $F: \Omega \times \mathrm{R} \to \mathrm{Cl}(\mathrm{R})$, then also $N_F: X \to \mathrm{Cl}(Y)$ (since Y is embedded in S). In order to apply the main principles of the theory of multivalued operators, however, one has to require more properties of the operator N_F, the most important ones being *boundedness* and *continuity*. It turns out that boundedness of N_F is guaranteed by some properties of the «source space» X, while continuity of N_F follows from properties of the «target space» Y.

THEOREM 3. *Let $F: \Omega \times \mathrm{R} \to \mathrm{Cp}(\mathrm{R})$ be a weakly sup-measurable multifunction, and suppose that the superposition operator* (3) *generated by F satisfies $N_F: X \to \mathrm{Cl}(Y)$, where X is a split space. Then N_F is bounded between X and Y.*

PROOF. Suppose that N_F is unbounded on the unit ball of X, without loss of generality. Then there exist two sequences $(x_n)_n$ and $(y_n)_n$ such that $\|x_n\| \leqslant 1$, $y_n \in N_F(x_n)$, and $\|y_n\| > n\sigma(n) + [\sigma(n) - 1]R$, where $\sigma(n)$ is the numerical sequence occuring in the definition of a split space, and $R = \sup\{\|z\|: z \in N_F(\theta)\}$. By modifying the supports of the functions
x_n, if necessary, one may assume that the functions x_n are mutually disjoint (see [2] or [5]). Since X is split, we may write x_n in the form $x_n = x_{n,1} + \ldots + x_{n,\sigma(n)}$. By (9) we find functions $y_{n,j} \in N_F(x_{n,j})$ $(j = 1, \ldots, \sigma(n))$ and $z_n \in N_F(\theta)$ such that

$$y_n + (\sigma(n) - 1)\, z_n = y_{n,1} + \ldots + y_{n,\sigma(n)}.$$

For at least one index $s(n) \in \{1, \dots, \sigma(n)\}$ we have then $\|y_{n, s(n)}\| > n$ (otherwise $\|y_n\| \leq n\sigma(n) + (\sigma(n) - 1)\|z_n\|$). But then the function $x_* = \sum_{n=1}^{\infty} x_{n, s(n)}$ belongs to X (since X is a split space), and the function $y_* = \sum_{n=1}^{\infty} y_{n, s(n)}$ does not belong to Y (since $\|y_*\| \geq \|y_{n, s(n)}\| > n$), contradicting the hypothesis $N_F(x_*) \in \mathrm{Cl}(Y)$. ■

We remark that Theorem 3 was proved for the single-valued superposition operator (1) in [5].

THEOREM 4. *Let $F: \Omega \times \mathrm{R} \to \mathrm{Cp}(\mathrm{R})$ be a Carathéodory multifunction, and suppose that the superposition operator* (3) *generated by F satisfies $N_F: X \to \mathrm{Cl}(Y)$, where Y is regular. Then N_F is continuous between X and Y.*

PROOF. Suppose that N_F is discontinuous at $x_0 \in X$. Then there exists a sequence $(x_n)_n$ in X such that $\|x_n - x_0\| \leq 2^{-n}$ and $N_F(x_n) \not\to N_F(x_0)$.

Let $x_*(t) = \sum_{n=1}^{\infty} |x_n(t) - x_0(t)|$. By Lemma 1, there exists a measurable function z such that

$$m(t) = \max \bigcup_{|u - x_0(t)| \leq x_*(t)} F(t, u) \in F(t, z(t)),$$

hence $m \in Y$. Since $|x_n - x_0| \leq x_*$, we have

$$\sup_n \{P_D y(t): y \in N_F(x_n)\} \leq P_D m(t),$$

and hence the family of sets $P_D F(t, x_n(t))$ tends to zero, uniformly in $n \in \mathrm{N}$, as $\mathrm{mes}\, D \to 0$. Moreover, the sequence $N_F(x_n)$ is compact in measure, by Theorem 2. By a classical compactness criterion in regular ideal spaces ([29], see also [3], [7]), the sequence $(N_F(y_n))_n$ is also compact in the norm of Y, a contradiction. ■

We remark that Theorem 4 was proved for the single-valued superposition operator (1) in [19], see also [4].

Observe that in both theorems of this section the acting condition $N_F: X \to 2^Y$ was *assumed* a priori, and the properties of the operator N_F were then deduced from properties of either the space X or the space Y. The problem of *characterizing* the acting condition $N_F: X \to 2^Y$, in terms of the generating multifunction F, is much harder. In fact, there are some partial results only for very specific classes of spaces and functions. For instance, in the book [17] the authors make the following obvious remark: if a multifunction F satisfies the growth estimate

$$\sup\{|v|: v \in F(t,u)\} \leqslant a(t) + b|u|^{p/q} \qquad (a \in L_q, b \geqslant 0)$$

then the corresponding superposition operator (3) acts between the Lebesgue spaces L_p and $L_p (1 \leqslant p, q < \infty)$. Moreover, the authors claim in [17] (without proof) that, if F is (upper) Carathéodory, then N_F is (upper semi-) continuous between L_p and L_q.

Throughout this section, by the *acting* of the operator N_F between two spaces X and Y we meant that $N_F(x) \subseteq Y$ for all $x \in X$. One could also study conditions for the *weak acting* of N_F, i.e. $N_F(x) \cap Y \neq \emptyset$ for all $x \in X$. If X and Y are ideal spaces and we define two (single-valued) operators N_F^+ and N_F^- on X by

$$N_F^+(x) = \sup N_F(x), \qquad N_F^-(x) = \inf N_F(x),$$

then the acting (resp. the weak acting) of N_F between X and Y means, roughly speaking, that the operator N_F^+ (resp. the operator N_F^-) maps X into Y. It would be interesting to characterize the classes of multifunctions F, for which

$$N_F^+ = N_{F^+}, \qquad N_F^- = N_{F^-},$$

where the (single-valued) functions F^+ and F^- are defined by

$$F^+(t,u) = \sup F(t,u), \qquad F^-(t,u) = \inf F(t,u).$$

Generally speaking, the possibility of «interchanging» set-theoretic or topological procedures of the multifunction F, on the one hand, and of the corresponding operator N_F, on the other, are important in various applications. Two such procedures will be described in the last sections (see formulas (23) and (33) below).

3. The superposition operator in the space C.

In this section we assume that Ω is a compact set without isolated points in the Euclidean space. We are going to study the superposition operator (3) in the Banach space $C = C(\Omega)$ of all continuous functions on Ω with the usual norm

$$\|x\| = \max\{|x(t)|: t \in \Omega\}. \tag{19}$$

In analogy to the concepts introduced in the first section, we call a multifunction $F: \Omega \times \mathrm{R} \to 2^{\mathrm{R}}$ *superpositionally continuous* (or *sup-contin-*

uous, for short) if, for any $x \in C$, the multifunction (2) is continuous (i.e. both upper semi-continuous and lower semi-continuous); if the multifunction (2) admits only a continuous (single-valued) selection y, we call F *weakly sup-continuous*. In either case, we may define the superposition operator (3) in the space C, because we have then $C \cap N_F(x) \neq \emptyset$ for all $x \in C$. Simple sufficient conditions for (weak) sup-continuity are given in the following.

LEMMA 2. *Every continuous multifunction $F\colon \Omega \times \mathbb{R} \to \mathrm{Cp}(\mathbb{R})$ is sup-continuous. Every lower semi-continuous multifunction $F\colon \Omega \times \mathbb{R} \to \mathrm{Cv}(\mathbb{R}) \cap \mathrm{Cl}(\mathbb{R})$ is weakly sup-continuous.*

PROOF. The first assertion follows from the fact the superposition of two continuous multifunctions with compact values is again continuous. The second assertion follows from the fact the lower semi-continuity of F implies that of the multifunction (2), and from Michael's selection theorem [21], [22]. ∎

We point out that the first part of Lemma 2 admits a converse. Indeed, if $(t_n, u_n) \in \Omega \times \mathbb{R}$ converges to (t_0, u_0), then, by the classical Tietze-Uryson lemma, we may find a continuous function x on Ω such that $x(t_n) = u_n$ and $x(t_0) = u_0$. Consequently, if $F\colon \Omega \times \mathbb{R} \to \mathrm{Cp}(\mathbb{R})$ is sup-continuous, we have

$$F(t_n, u_n) = Y(t_n) \to Y(t_0) = F(t_0, u_0),$$

which shows that F is continuous.

The second part of Lemma 2, however, is not invertible. To see this, consider the multifunction $F\colon \mathbb{R} \to \mathrm{Cv}(\mathbb{R}) \cap \mathrm{Cp}(\mathbb{R})$ defined by

$$F(u) = \begin{cases} [0,1] & \text{if } u \neq 0, \\ [0,2] & \text{if } u = 0. \end{cases}$$

Then F is certainly weakly sup-continuous (since, for any $x \in X$, the function $y(t) \equiv 0$ is a continuous selection of $Y(t) = F(x(t))$), but not lower semi-continuous. This example shows, in addition, that weak sup-continuity does not imply sup-continuity. The *lower* semi-continuity of F is crucial in the second statement of Lemma 2, in order to apply Michael's theorem. It is easy to see that the *upper* semi-continuity of F

does not imply its weak sup-continuity. For example, the multifunction

$$F(u)=\begin{cases}\{-1\} & \text{if } u<0,\\ [-1,+1] & \text{if } u=0,\\ \{+1\} & \text{if } u>0,\end{cases} \tag{20}$$

is upper semi-continuous on R, but maps the continuous function $x(t)=t$ into the multifunction $Y(t)=F(t)$ which does not admit a continuous selection.

The following two theorems show that the *continuity* of the multifunction F does not only imply the sup-continuity of F, but also ensures the boundedness and continuity of the corresponding operator N_F. Since the proofs are completely obvious, we drop them.

THEOREM 5. *Let* $F: \Omega\times \mathrm{R}\to \mathrm{Cp}(\mathrm{R})$ *be continuous. Then the superposition operator* (3) *generated by* F *is bounded in the space* C.

THEOREM 6. *Let* $F: \Omega\times \mathrm{R}\to \mathrm{Cl}(\mathrm{R})$ *be continuous. Then the superposition operator* (3) *generated by* F *is continuous in the space* C.

4. The closure of the superposition operator.

Let Ω be again a bounded domain in Euclidean space, and let $F: \Omega\times \mathrm{R}\to \mathrm{Cl}(\mathrm{R})$ be a weakly sup-measurable multifunction. Suppose that the superposition operator N_F generated by F acts between two ideal spaces X and Y. For fixed $t_0\in\Omega$, denote by

$$\overline{F}(t_0,u_0)=\bigcap_{\varepsilon,\delta>0}\{v\colon v\in F(t_0,u)+h,\ |u-u_0|\leqslant\varepsilon,\ |h|\leqslant\delta\} \tag{21}$$

the *closure of the function* $F(t_0,\cdot)\colon \mathrm{R}\to\mathrm{Cl}(\mathrm{R})$. The multifunction $\overline{F}$ is then also weakly sup-measurable and generates a superposition operator $N_{\overline{F}}$ according to the definition (3). On the other hand, we may consider the *closure of the operator* $N_F\colon X\to \mathrm{Cl}(Y)$ i.e.

$$\overline{N}_F(x_0)=\bigcap_{\varepsilon,\delta>0}\{y\colon y\in N_F(x)+z,\ \|x-x_0\|\leqslant\varepsilon,\ \|z\|\leqslant\delta\}\,. \tag{22}$$

The following theorem shows that this is essentially the same, since the closure of the superposition operator coincides with the superposition operator of the closure.

THEOREM 7. *Let* $F: \Omega \times \mathbb{R} \to \mathrm{Cl}(\mathbb{R})$ *be a weakly sup-measurable multifunction. With* $\overline{F}$ *given by* (21) *and* $\overline{N}_F$ *given by* (22), *the equality*

$$\overline{N}_F = N_{\overline{F}} \tag{23}$$

holds, where $N_{\overline{F}}$ *is considered as an operator between* X *and* Y.

PROOF. Fix $x_0 \in X$, and let $y_0 \in \overline{N}_F(x_0)$. By (22), we may find two sequences $(x_n)_n$ and $(y_n)_n$ such that $y_n \in N_F(x_n)$, $x_n \to x_0$, and $y_n \to y_0$. Since the spaces X and Y are imbedded in S, and each sequence which converges in measure admits an almost everywhere convergent subsequence, we have

$$x_{n_k}(t) \to x_0(t), \qquad y_{n_k}(t) \to y_0(t) \qquad (k \to \infty)$$

for $t \in \Omega \setminus N$, where $\mathrm{mes}\, N = 0$. But for $t_0 \in \Omega \setminus N$, $u_0 = x_0(t_0)$, and $u_k = x_{n_k}(t_0)$ we get then $y_{n_k}(t_0) \in F(t_0, u_k)$ and $y_0(t_0) \in \overline{F}(t_0, u_0)$ hence $y_0 \in N_{\overline{F}}$, by (21).

Conversely, let $y_0 \in N_{\overline{F}}(x_0)$, i.e. $y_0(t_0) \in \overline{F}(t_0, x_0(t_0))$ for $t_0 \in \Omega \setminus N$ with $\mathrm{mes}\, N = 0$. Define a sequence of multifunctions $\Phi_k: \Omega \to \mathrm{Cl}(\mathbb{R}^2)$ by

$$\Phi_k(t) = \left\{ (u, v) \colon v \in F(t, u), |u - x_0(t)| \leq \frac{1}{k}, |v - y_0(t)| \leq \frac{1}{k} \right\}. \tag{24}$$

By Sainte-Beuve's selection theorem [25], [26], we may find two sequences $(x_k)_k$ in X and $(y_k)_k$ in Y, respectively, such that $(x_k(t), y_k(t)) \in \Phi_k(t)$, i.e. $y_k \in N_F(x_k)$ and

$$|x_k(t) - x_0(t)| \leq \frac{1}{k}, \qquad |y_k(t) - y_0(t)| \leq \frac{1}{k}, \qquad (t \in \Omega \setminus N). \tag{25}$$

The estimates (25) imply, in particular, that $x_k \to x_0$ in X and $y_k \to y_0$ in Y. This means that for $z_k = y_0 - y_k$ we have $\|z_k\| \to 0$ and $y_0 \in N_F(x_k) + z_k$, hence $y_0 \in \overline{N}_F(x_0)$, by (22). ∎

We remark that Theorem 7 was proved in the case $X = L_p$ and $Y = L_q$ in [17]. The problem of describing the closure of the superposition operator (3) in the space C is somewhat more delicate. Let Ω be a compact set in Euclidean space, and $F: \Omega \times \mathbb{R} \to \mathrm{Cl}(\mathbb{R})$ be a continuous multifunction, Moreover, we assume that F is *quasi-concave* in the sense that

$$F(t, (1-\lambda) u_0 + \lambda u_1) \supseteq (1-\lambda) F(t, u_0) + \lambda F(t, u_1) \tag{26}$$

for $t \in \Omega$, $u_1, u_2 \in \mathbb{R}$, and $0 \leq \lambda \leq 1$. Choosing $u_0 = u_1$ in (26) one sees that every quasi-concave multifunction takes convex values; the converse is not true. For example, the convex-valued multifunction $F(u) = [0, f(u)]$ is quasi-concave only if the function f is concave on $\mathbb{R}$.

Before proving an analogue to Theorem 7 in the space C, we recall some facts about lower semi-continuous multifunctions. If Φ and Ψ are two lower semi-continuous multifunctions, the intersection $\Phi \cap \Psi$ need not be again lower semi-continuous, as simple examples show (see e.g. [9]). A multifunction Ψ is called *quasi-open* at a point t_0 if, roughly speaking, each interior point w_0 of $\Psi(t_0)$ admits a neigbourhood W which is contained in $\Psi(t)$ for each t sufficiently close to t_0. For example, every closed ball $\Psi(t) = \{w\colon |w - c(t)| \leq r\}$ in $\mathbb{R}^m$ defines a quasi-open multifunction, provided the centre $c = c(t)$ varies continuously with t. The proof of the following lemma may be found in [9].

LEMMA 3. *If* $\Phi\colon \Omega \to \mathrm{Cl}(\mathbb{R}^m)$ *is lower semi-continuous and* $\Psi\colon \Omega \to \mathrm{Cl}(\mathbb{R}^m)$ *is quasi-open, then* $\Phi \cap \Psi$ *is lower semi-continuous.*

We are now in a position to give a certain analogue to Theorem 7 in the space C.

THEOREM 8. *Let* $F\colon \Omega \times \mathbb{R} \to \mathrm{Cl}(\mathbb{R}) \cap \mathrm{Cv}(\mathbb{R})$ *be a continuous quasi-concave multifunction. With* $\overline{F}$ *given by* (21) *and* $\overline{N}_F$ *given by* (22), *the equality* (23) *holds, where* N_F *is considered as an operator in* C.

PROOF. The inclusion $\overline{N}_F(x_0) \subseteq N_{\overline{F}}(x_0)$ is proved in rather the same way as in Theorem 7. To prove the converse inclusion, fix $y_0 \in N_{\overline{F}}$, and let Φ_k denote again the multifunction (24). By the quasi-concavity of the multifunction F we have $\Phi_k\colon \Omega \to \mathrm{Cl}(\mathbb{R}^2) \cap \mathrm{Cv}(\mathbb{R}^2)$. To apply now Michael's theorem in the same way as we applied Sainte-Beuve's theorem in the proof of Theorem 7, it suffices to show that the multifunction Φ_k is lower semi-continuous. By Lemma 3, it suffices in turn to prove that the multifunction $\Phi(t) = \{(u, v)\colon v \in F(t, u)\}$ is lower semi-continuous. To this end, fix $t_0 \in \Omega$, and let $W \subseteq \mathbb{R}^2$ be open with $\Phi(t_0) \cap W \neq \emptyset$. In particular, we have then $v_0 \in F(t_0, u_0)$ for some $(u_0, v_0) \in W$. Since $F(\cdot, u_0)$ is lower semi-continuous, we may find a neighbourhood $\Omega_0 \subset \Omega$ of t_0 such that $F(t, u_0) \cap V \neq \emptyset$ for $t \in \Omega_0$, where V denotes the second projection of W on $\mathbb{R}$. For $t \in \Omega_0$ and $v \in F(t, u_0) \cap V$ we have then $(u_0, v) \in \{u_0\} \times V \subseteq W$, hence $\Phi(t) \cap W \neq \emptyset$. ■

The condition (26), which we needed as a technical assumption in order to guarantee the convexity of the values of the multifunctions (24), is of course very restrictive. It is very likely that Theorem 8 is true also without the assumption (26).

Theorems 7 and 8 show that equality (23) holds if we consider the operator (3) *either* between two ideal spaces X and Y, *or* in the space C. These two cases have to be distinguished carefully, since (23) may *fail* if (3) acts, say, from the space C into an ideal space Y. Consider, for example, the multifunction

$$F(u) = \begin{cases} \{\sin(1/u)\} & \text{if } u \neq 0, \\ \{0\} & \text{if } u = 0, \end{cases} \tag{27}$$

over $\Omega = [0, 1]$. Obviously, the multifunction (21) has the form

$$\overline{F}(u) = \begin{cases} \{\sin(1/u)\} & \text{if } u \neq 0, \\ [-1, 1] & \text{if } u = 0. \end{cases}$$

If we consider the operator (3) from L_1 into L_1, then, by Theorem 7, the sets $\overline{N_F}(\theta)$ and $N_{\overline{F}}(\theta)$ are equal; they coincide with the closed unit ball in the space L_∞. However, if we consider (3) from C into L_1, say, then $\overline{N_F}(\theta)$ is strictly smaller then $N_{\overline{F}}(\theta)$. In fact, any $y \in \overline{N_F}(\theta)$ may be represented in the form

$$y(t) = \sin\frac{1}{x(t)} + z(t) \qquad (x(t) \neq 0), \tag{28}$$

where x is continuous on Ω, and z has a small L_1-norm. Of course, not every function y in the unit ball of L_∞ is of this form (or may be approximated by functions of this form).

5. The convexification of the superposition operator.

Another way to extend a given multifunction F consists in the following.

Let $F: \Omega \times \mathbb{R} \to \mathrm{Cl}(\mathbb{R})$ be weakly sup-measurable, and suppose that the superposition operator N_F generated by F acts between two ideal spaces X and Y, where Y is *regular*. For fixed $t_0 \in \Omega$, denote by

$$F^{\square}(t_0, u_0) = \bigcap_{\varepsilon > 0} \overline{\mathrm{co}}\,\{v: v \in F(t_0, u), |u - u_0| \leq \varepsilon\} \tag{29}$$

the *convexification of the function* $F(t_0, \cdot): \mathbb{R} \to \mathrm{Cl}(\mathbb{R})$. Similarly, denote by

$$N_F^{\square}(x_0) = \bigcap_{\varepsilon > 0} \overline{\mathrm{co}}\,\{y: y \in N_F(x), \|x - x_0\| \leq \varepsilon\} \tag{30}$$

the *convexification of the operator* $N_F: X \to \mathrm{Cl}(Y)$. For example, if we define a multifunction F: $\mathrm{R} \to \mathrm{Cp}(\mathrm{R})$ by

$$F(u) = \{\operatorname{sign} u\} = \begin{cases} \{-1\} & \text{if } u < 0, \\ \{0\} & \text{if } u = 0, \\ \{+1\} & \text{if } u > 0, \end{cases}$$

then the closure $\overline{F}$ is given by

$$\overline{F}(u) = \begin{cases} \{-1\} & \text{if } u < 0, \\ \{-1, 0, +1\} & \text{if } u = 0, \\ \{+1\} & \text{if } u > 0, \end{cases}$$

while the convexification $F^{\square}$ is given by (20). At first glance, one might think that the convexification $F^{\square}(t_0, u_0)$ coincides with the closed convex hull of the closure $\overline{F}(t_0, u_0)$. The following example shows that this is false. Consider the multifunction F: $\mathrm{R} \to \mathrm{Cp}(\mathrm{R}) \cap \mathrm{Cv}(\mathrm{R})$ defined by

$$F(u) = \begin{cases} \left[-\dfrac{1}{|u|} - 1, -\dfrac{1}{|u|}\right] & \text{if } u \neq 0,\ u \notin \boldsymbol{Q}, \\ \left[\dfrac{1}{|u|}, \dfrac{1}{|u|} + 1\right] & \text{if } u \neq 0,\ u \in \boldsymbol{Q}, \\ \{0\} & \text{if } u = 0. \end{cases}$$

Then $\overline{F}(0) = \{0\}$, hence $\overline{\mathrm{co}}\,\overline{F}(0) = \{0\}$, but $F^{\square}(0) = \mathrm{R}$. In general, the following holds.

LEMMA 4. *For any multifunction* F: $\mathrm{R} \to \mathrm{Cv}(\mathrm{R}) \cap \mathrm{Cl}(\mathrm{R})$ *the inclusion*

$$\mathrm{co}\,\overline{F}(u_0) \subseteq F^{\square}(u_0) \tag{31}$$

is true. Moreover, if F is upper semi-continuous at u_0, then equality holds in (31).

PROOF. Without loss of generality, let $u_0 = 0$. Denote by $M(\varepsilon, \delta)$ the set of all $v \in F(u) + h$ for $|u| \leqslant \varepsilon$ and $|h| \leqslant \delta$. Then

$$\mathrm{co}\,\overline{F}(0) \subseteq \bigcap_{\varepsilon>0} \mathrm{co} \bigcap_{\delta>0} M(\varepsilon, \delta) = \bigcap_{\varepsilon>0} \mathrm{co}\,\overline{M(\varepsilon, 0)} = \bigcap_{\varepsilon>0} \mathrm{co}\,M(\varepsilon, 0) = F^{\square}(0),$$

and hence (31) holds. Now let F be upper semi-continuous at 0, and suppose that there is a $z \in F^{\square}(0)$ such that $z \notin \overline{\mathrm{co}}\,\overline{F}(0)$. Choose an open interval $V \supset \mathrm{co}\,\overline{F}(0)$ with $z \notin \overline{V}$. Since F is upper semi-continuous at 0, we may find an $\varepsilon > 0$ such that $F(u) \subseteq V$ for $|u| \leqslant \varepsilon$. Consequently,

$$z \in F^{\square}(0) \subseteq \mathrm{co} \bigcup_{|u| \leqslant \varepsilon} F(u) \subseteq \overline{\mathrm{co}}\, V = \overline{V}, \tag{32}$$

a contradiction.

The following theorem is parallel to Theorem 7. ∎

THEOREM 9. *Let $F: \Omega \times \mathbb{R} \to \mathrm{Cl}(\mathbb{R})$ be a weakly sup-measurable multifunction. With $F^{\square}$ given by* (29) *and $N_F^{\square}$ given by* (30), *the equality*

$$N_F^{\square} = N_{F^{\square}}, \tag{33}$$

holds, where N_F is considered as an operator between X and Y.

PROOF. One inclusion in (33) follows from the fact that, by Theorem 7,

$$N_F^{\square}(x_0) \subseteq \overline{\mathrm{co}} N_F(x_0) = \overline{\mathrm{co}} N_{\overline{F}}(x_0) \subseteq N_{F^{\square}}(x_0), \tag{34}$$

for any $x_0 \in X$. Conversely, to prove the inclusion

$$N_{F^{\square}}(x_0) \subseteq N_F^{\square}(x_0), \tag{35}$$

observe that both sets in (35) are bounded, closed, and convex subsets of Y, and hence (35) is equivalent to showing that, for each $w \in Y'$,

$$\sup\{\langle y, w\rangle : y \in N_{F^{\square}}(x_0)\} \leqslant \sup\{\langle z, w\rangle : z \in N_F^{\square}(x_0)\}, \tag{36}$$

by the classical Hahn-Banach theorem (observe that $Y' = Y^*$, since we assumed Y to be regular, see (18)). To show (36) for fixed $w \in Y' = Y^*$, it is in turn sufficient to find a function $z_0 \in N_F^{\square}(x_0)$ such that

$$\langle y, w\rangle \leqslant \langle z_0, w\rangle, \tag{37}$$

for all $y \in N_{F^{\square}}(x_0)$. Let

$$\varphi(t) = \begin{cases} \max F^{\square}(t, x_0(t)) & \text{if } w(t) \geqslant 0, \\ \min F^{\square}(t, x_0(t)) & \text{if } w(t) < 0, \end{cases}$$

and

$$Z_0(t) = \overline{F}(t, x_0(t)) \cap [\varphi(t), \infty). \tag{38}$$

If z_0 is any measurable selection of the multifunction Z_0, then z_0 belongs to $N_F^\square(x_0)$ and satisfies (37), and so we are done. ■

We remark that Theorem 9 was proved in the case $X = L_p$ and $Y = L_q$ in [17].

THEOREM 10. *Let* $F\colon \Omega \times \mathbb{R} \to \mathrm{Cl}(\mathbb{R}) \cap \mathrm{Cv}(\mathbb{R})$ *be a continuous quasi-concave multifunction. With* $F^\square$ *given by* (29) *and* $N_F^\square$ *given by* (30), *the equality* (33) *holds, where* N_F *is considered as an operator in* C.

PROOF. The proof follows essentially the same line as that of Theorem 9. Observe, in particular, that the continuity of F implies the lower semi-continuity of the multifunction (38), and thus we may again apply Michael's theorem. ■

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Manoscritto pervenuto in redazione il 19 gennaio 1990.